Coquasi – Invertible Submodule

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Abstract

Let R be a commutative ring with identity. We call the proper submodule N of M a coquasi - invertible submodule if $Hom(M,N)=0$ the main purpose of this work is the study of the properties of coquasi - invertible submodules. and give definithon of corational submodule. Then show that if M is self projective, then N is coquasi - invertible submodule of M if and only if N is corational in M.

الخلاصه

لتكن R حلقة ابدالية أحادية. نعرف المقاس الجزئي الفعلي N من M بأنه شبه عكوس مضاد إذا كان $Hom(M,N)=0$. أن الهدف الرئيسي من هذا العمل هو دراسة خواص المقاسات الجزئية شبه عكوسه المضادة .

من نتائجنا الأولية قمنا بالبرهنة على انه إذا كان N مقياساً جزئيا شبه عكوس مضاد فأن $ann(M)=ann(\frac{M}{N})$ و $T(M)\subseteq ann(N)$ كذلك قمنا بتعريف المقاس الأولي المضاد والمقاس الجزئي النسبي المضاد ودراسة خواصهما ومناقشة العلاقة بينهما وبين المقاسات الجزئية شبه العكوسه المضادة.

Introduction:

Let R be a commutative ring with identity. The concept of an invertible submodule of an R - module was introduced in [1]. It was shown in [1] that if N is an invertible submodule of M, then $Hom(\frac{M}{N},M)=0$. In [5] this result was adopted as a definition of quasi - invertible submodule, i.e. a submodule N of an R - module M is quasi - invertible if $Hom(\frac{M}{N},M)=0$ .In this paper we introduce a dual of this concept . Thus we call the proper submodule N of M coquasi - invertible submodule if $Hom(M,N)=0$ The main purpose of this work is the study of the properties of coquasi - invertible submodules Let N be a submodule of an R - module M, we say that N is corational in M if $Hom(M,\frac{N}{K})=0$ for all submodules K of M such that $K\subseteq N\subseteq M$. We show that if M is self projective, then N is coquasi - invertible submodule of M if and only if N is corational in M.

§1.Basic properties: we start this section by the following definition.

Definition 1-1:A proper submodule N of an R - module M is called coquasi - invertible submodule of M if $Hom(M,N)=0$.

Not that the zero submodule is a coquasi - invertible submodule of any nonzero R - module M.

It is clear that if N is a nonzero coquasi - invertible submodule of an R - module M, then N can not be a direct summand of M. In the following we give two examples, the first one is of a coquasi - invertible submodule and the other is not.

1- Consider Q, the set of rational number as a Z - module. The submodule Z of Q is a coquasi – invertible submodules, since $Hom(Q,Z)=0$.

2- Consider Z_4 as Z module. The submodule $\{\bar{0},\bar{2}\}$ of Z_4 is not a coquasi - invertible submodule of Z_4 since the homomorphism $f: Z_4 \to \{\bar{0},\bar{2}\}$ defined by $f(\bar{1})=\bar{2}$ is a nonzero homomorphism

Proposition 1-2: Let N be a coquasi - invertible submodule of an R - module M, then $ann(M)=ann(\frac{M}{N})$.

Proof: Let $r \in ann(\frac{M}{N})$,then $rM \subseteq N$ .Define $f: M \to N$ by $f(m)=rm$, for every $m \in M$. N is a coquasi - invertible submodule of M, therefore. $f=0$ Thus $rM=0$,this implies that. $r \in annM$ The other inclusion is clear.

The converse of proposition 1-2 is not true as is shown by the following example.

Consider $M = Z \oplus Z$ as a Z - module and let $N = Z \oplus \{0\}$. It is clear that $ann(Z \oplus Z) = ann(\frac{Z \oplus Z}{Z \oplus \{0\}})$, while $Hom(Z \oplus Z, Z \oplus \{0\}) \neq 0$ .i.e. $Z \oplus \{0\}$ is not coquasi - invertible submodule of $Z \oplus Z$.

Next we study the traces of coquasi – invertible submodule. But first we need the following.

Remark 1-3: Let N be a coquasi – invertible submodule of the R - module M. If $\alpha \in Hom(M,R)$,then $\alpha(M) \subseteq \bigcap_{\phi} \ker\phi$, $\phi \in Hom(R,N)$.

Proof: Suppose that $\alpha(M) \not\subset \bigcap_{\phi} \ker\phi$, $\phi \in Hom(R,N)$, then there exists $\phi_0 : R \to N$ such that . $\phi_0 \circ \alpha \neq 0$ This is a contradiction therefore $\alpha(M) \subseteq \bigcap_{\phi} \ker\phi$.

Recall that the trace of an R - module M, denoted by T(M) is $T(M) = \sum_{\phi} \phi(M)$ where $\phi \in Hom(M,R)$ Proposition 1-4: N is a coquasi – invertible submodule of the R - module M, then $T(M) \subseteq ann(N)$.

Proof: If $T(M)=0$,the result is clear . Suppose that $T(M) \neq 0$, then there exists $\phi \neq 0, \phi \in Hom(M,R)$. By Remark 1-3 $\phi(M) \subseteq \bigcap_{\psi} \ker\psi, \psi \in Hom(R,N)$ .For every $x \in N$ define $h_x : R \to N$ by $h_x(r) = rx$,then $h_x \circ \phi = 0$,i. e $(h_x \circ \phi)(M) = \phi(M)x = 0$,thus $\phi(M)N = 0$ and therefore $\phi(M) \subseteq ann(N)$, this implies that $T(M) \subseteq ann(N)$.

The converse of proposition 1.-4 is not true. Consider the following example.

Example1-5: Consider Z_4 as Z - module ,then $\{\bar{0},\bar{2}\}$ is a submodule of Z_4 But $Hom(Z_4, Z) = 0$, thus $T(Z_4) = 0$ while $\{\bar{0},\bar{2}\}$ is not a coquasi - invertible submodule of Z_4.

Recall that an R - module M is called torsionless module if $\bigcap_{\phi} \ker\phi = 0$ where $\phi \in Hom(M,R)$

In the following proposition we give a condition under which a torsionless submodule becomes a coquasi - invertible submodule.

Proposition1-6: Let N be a torsionless submodule of the R - module M If $T(M)=0$,then N is a coquasi - invertible submodule of M.

Proof: Suppose that $Hom(M,N) \neq 0$ Then there exists a nonzero homomorphism $f: M \to N$ .Therefore there exists $m \in M$ such that $f(m) \neq 0$. N is torsionless submodule, then $f(m) \notin \bigcap_{\phi} \ker\phi$ where $\phi \in Hom(N,R)$ .Thus, there exists $\phi_o : N \to R$ such that $f(m) \notin \ker\phi_0$. Hence $\phi_0 \circ f \neq 0$. This implies that $T(M) \neq 0$ a contradiction.

Corollary1-7: Let N be a torsionless submodule of the R - module M If $ann(N)=0$,then N is a coquasi - invertible submodule of M if and only if $T(M)=0$.

The proof is clear from Proposition 1-4 and Proposition 1-6 Recall that an R - module M is called a multiplication module, if every submodule N of M is of the form *IM* for some ideal I of R [7]. It is known that every faithful multiplication module is torsionless [8], thus we have.

Corollary 1-8: If N is a faithful multiplication submodule of an R - module M, then N is a coquasi - invertible submodule of M if and only if $T(M)=0$

Corollary1-9: Let M be an R - module. If M contains a faithful cyclic submodule which is a coquasi - invertible submodule, then every faithful cyclic submodule has this property.

Proof: Let N be a faithful cyclic submodule which is a coquasi - invertible. And let K be a faithful cyclic submodule then K is multiplication [2]. By Prop. 1-4 $T(M)\subseteq ann(N)$,then $T(M)=0$ .By corollary 1-8 K is coquasi - invertible submodule.

Recall that, for an integral domain R, an R - module M is called torsion - free if every $m\in M, m\neq 0$ and for every $r\in R, r\neq 0$ then $rm\neq 0$

Corollary1-10: Let M be a torsion - free R - module with $T(M)=0$. Then every cyclic submodule of M is coquasi - invertible submodule.

Let J be a proper ideal in the ring R. we say that J ***is*** coquasi - invertible ideal of R, if J is a coquasi - invertible R - submodule of R as R - module. The following Proposition shows that the ideal J is coquasi - invertible if and only if $J=0$.

Proposition 1-11: Let J be a Proper ideal of R , then J is coquasi - invertible ideal if and only if $ann(\frac{R}{J})=0$.

Proof: Suppose that J is coquasi - invertible ideal of R. By proposition1-2 , $ann(\frac{R}{J})=ann(R)$, But $ann(R)=0$, therefore $ann(\frac{R}{J})=0$. The converse, let $f\in Hom(R,J)$ and let $r\in R$, $f(1)(r+J)=f(1)r+J\subseteq J$, thus $f(1)\in ann(\frac{R}{J})$. But $ann(\frac{R}{J})=0$ therefore $f(1)=0$ and hence $f=0$.

§2 Characterization for coquasi - invertible submodule.

The following theorems gives some characterizations for coquasi - invertible submodule .

Theorem2-1: Let N be a nonzero proper submodule of the R - module M. Then N is a coquasi – invertible submodule if and only if for every $\phi\in End(M)$ such that $\pi\circ\phi=\pi$, we have ϕ is the identity homomorphism, where $\pi: M\to\frac{M}{N}$ is the natural epimorphism.

Proof: Let $\phi\in End(M)$ and let $m\in M$,then $(\pi\circ\phi)(m)=\pi(m)$ and hence $\phi(m)-m\in N$ i.e. $(\phi-I)m\in N$. so $\phi-I$ is a homomorphism from M into N, and N is coquasi - invertible submodule of M, therefore $\phi=I$.

For the converse, let $f\in Hom(M,N)$ and let $i: N\to M$ be the inclusion homomorphism. For each $m\in M$,We have

$[\pi\circ(I-i\circ f)](m)=\pi(m-f(m))=$
$m-f(m)+N=m+N=\pi(m)$

This implies that $\pi\circ(I-i\circ f)=\pi$ and thus $I-i\circ f=I$, i.e. $f=0$.

Theorem 2-2-: Let N be a submodule of the R - module M. then N is a coquasi – invertible submodule of M if and only if for every $\phi: M\to\frac{M}{N}$ if there exists $\psi: M\to M$ such that $\pi\circ\psi=\phi$, then ψ is unique.

Proof: Let $\phi: M\to\frac{M}{N}$ and let $\pi: M\to\frac{M}{N}$ be the natural homomorphism. If $\psi: M\to M$ and $\psi': M\to M$ are such that $\pi\circ\psi=\phi$ and $\pi\circ\psi'=\phi$,then for every $m\in M$,we have $\psi(m)-\psi'(m)\in N$. But $\psi-\psi'$ is a homomorphism, and N is coquasi - invertible submodule of M, therefore $\psi-\psi'=0$ and hence $\psi=\psi'$.

For the converse, let $f: M\to N$ and let $i: N\to M$ be the inclusion homomorphism, then $i\circ f\in End(M)$ and

$\pi\circ(i\circ f)(m)=\pi\circ f(m)=f(m)+N=0$.

Since $\pi \circ (i \circ f) = \pi \circ 0$. this implies that $f = 0$.

§3. Coquasi - invertible submodule and corational submodule.

It is well – known that every R - module can be embedded in an injective R - module $\hat{M}$ with M essential in $\hat{M}$. $\hat{M}$ is called the injective hull of M [4, P.128].A submodule U of an R - module M is called rational in M if $Hom(\frac{M}{U}, \hat{M}) = 0$ where $\hat{M}$ is the injective hull of M. [3, P.33].The following is a useful characterization of these kind of submodules.

Proposition 3-1: Let U be a submodule of an R - module M, the following are equivalent;

1-U is rational in M.

2-For any $U \subseteq V \subseteq M, Hom(\frac{V}{U}, M) = 0$.

Proof: Let U be a rational submodule in M and let $V \subseteq M$ such that $U \subseteq V \subseteq M$. Suppose there exists a nonzero homomorphism $g \in Hom(\frac{V}{U}, M)$., Where $i : \frac{V}{U} \to \frac{M}{U}$ is the inclusion homomorphism and $J : M \to \hat{M}$ is the inclusion homomorphism from M to $\hat{M}$ the injective hull of M. Since $\hat{M}$ is injective module, there exists a homomorphism $h : \frac{M}{U} \to \hat{M}$ such that $h \circ i = j \circ g$. But $g \neq 0$ hence $h \neq 0$ a contradiction. Suppose there exists a nonzero homomorphism $f \in Hom(\frac{M}{U}, \hat{M})$ let $f^{-1}(M) = \frac{V}{U}$ for some $U \subseteq V \subseteq M$.

Define $g : \frac{V}{U} \to M$ by $g(x+U) = f(x+U)$ for every $x \in V$. Since $f \neq 0$,then there exists $m + U \neq U \in \frac{M}{U}$ such that $0 \neq f(m+U) \in \hat{M}$. But M is essential in $\hat{M}$,So there exists $r \in R$ such that $0 \neq rf(m+U) = f(rm+U) \in M$. This implies that $rm + U \in \frac{V}{U}$ and hence $rf(m+U) = g(rm+U) \neq 0$ a contradiction.

Now, we introduce the dual of the concept of rational submodule as follows.

Definition 3-2: A submodule N of an R - module M is called corational in M if $Hom(M, \frac{N}{K}) = 0$ for all submodule K of M such that $K \subseteq N \subseteq M$.

It is clear that if N is corational in M, then N is proper submodule of M.

Example 3-3: Consider Z_{P^∞} as Z - module. It is known that every proper submodule of Z_{P^∞} is isomorphic to Z_{P^n} for some integer *n*. It is clear that $Hom(Z_{P^\infty}, \frac{Z_{P^n}}{Z_{P^k}}) = 0$. Then every proper submodule of Z_{P^∞} is corational in *M*.

It was shown [5, P.14] that every rational submodule is quasi - invertible. Similarly we have the following with clear proof.

Proposition 3-4Let N be a submodule of M. If N is corational in M then N is coquasi - invertible submodule of M.

In the following theorem we show that if M is a multiplication module, the converse of Prop.3-4 is true.

Theorem 3-5: Let M be a multiplication R - module, if N is a coquasi invertible submodule of M, then N is corational submodule in M.

Proof: Let M be a submodule of M, such that $K \subseteq N$ and suppose that $Hom(M, \frac{N}{K}) \neq 0$. There exists a nonzero homomorphism $f : M \to \frac{N}{K}$. Now let $m \in M$, then $f(m) = x + K \neq K$, for some $x \in N$. But $N = [N : M]M$,hence $x = \sum_{i=1}^{n} r_i m_i$ where $r_i \in [N : M]$, and $m_i \in M$,thus there exists $1 \leq i \leq n$ such that $r_i m_i \notin K$. Define $h : M \to N$ by $h(m) = r_i m$ for all $m \in M$.

In particular $h(m_i) = r_i m_i \notin K$,i. e $r_i m_i \neq 0$. This is a contradiction thus $f = 0$.

Recall that a submodule N of an R - module M is said to be small in M if for every submodule U of M with $N + U = M$ we have U=M The following proposition shows that every corational submodule is small.

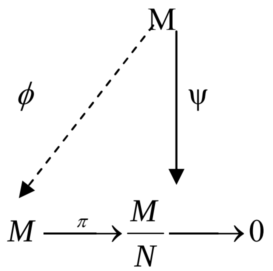
Proposition 3-6: Let N be a submodule of M. If N is corational in M then N is small in M.

Proof:Let K be a submodule of M such that $N + K = M$, then $\frac{M}{K} = \frac{N+K}{K}$. But $\frac{N+K}{K} \cong \frac{N}{N \cap K}$ and N is corational in M, thus $0 = Hom(M, \frac{N}{N \cap K}) \cong Hom(M, \frac{M}{K})$. In particular the natural epimorphism $\pi : M \to \frac{M}{K}$ must be zero. This implies that $k = M$.

The converse of proposition 3-6 is not true consider the following example.

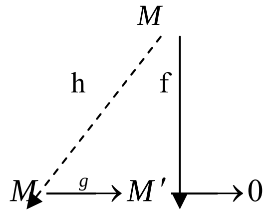
Example 3-7: Consider Z_4 as a Z - module. It is easily seen that the submodule $\{\bar{0}, \bar{2}\}$ is a small submodule of Z_4 . On other hand $\{\bar{0}, \bar{2}\}$ is not corational in Z_4 since $Hom(Z_4, \frac{\{\bar{0}, \bar{2}\}}{\{\bar{0}\}}) \neq 0$

Definition3-9: An R - module M is said to be self - projective if for every submodule N of M, any homomorphism $\phi : M \to \frac{M}{N}$ can be lifted to a homomorphism $\psi : M \to M$ i. e the following diagram is commutative.

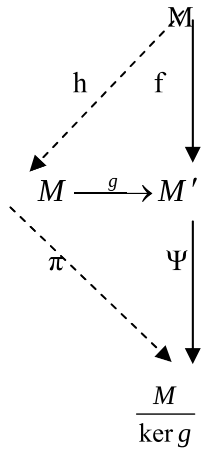


Where π is the natural epimorphism the following is a characterization of self - projective modules.

Proposition 3-10:An R - module M is self - projective if and only if for every epimorphism $g : M \to M'$ where M' is any R - module any homomorphism $f : M \to M'$ can be lifted to a homomorphism $h : M \to M$, i. e the following diagram is commutative.



Proof: Let M be a self - projective R - module and let $g : M \to M'$ be an epimorphism. Thus by first isomorphism theorem, there exists an isomorphism $\psi : M' \to \frac{M}{\ker g}$ make the diagram commutative.



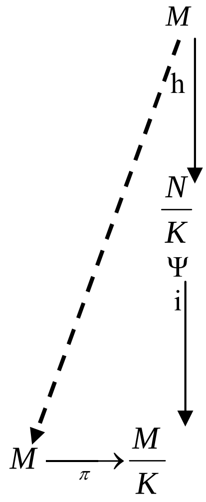
Where π is the natural epimorphism and for any $m' \in M', \psi(m') = x + \ker g$ where $g(x) = m'$. Since M is self - projective, then there exists a homomorphism $h : M \to M$ such that $\pi \circ h = \psi \circ f$. Let $m \in M$ then $h(m) + \ker g = x + \ker g$ where $g(x) = f(m)$. Now $h(m) - x \in \ker g$ i.e $g(h(m) - x) = 0$, then $g \circ h(m) - g(x) = 0$ thus $g \circ h(m) = f(m)$, therefore $g \circ h = f$ .The other direction is clear.

The following theorem gives a condition under which the converse of Proposition 3-4 is true.

Theorem 3-11: Let M be a self - projective R - module and let N be a submodule of M, then N is coquasi - invertible submodule of M if and only if N is corational in M.

Proof: Suppose N is a is coquasi - invertible submodule of the self - projective R - module M. Let K be a submodule of M, such that $K \subseteq N$

and let $h \in Hom(M, \frac{N}{K})$. Consider the following diagram.



Where $i : \frac{N}{K} \to \frac{M}{K}$ is the inclusion homomorphism. Since M is a self - projective module, there exists a homomorphism $\psi : M \to M$ such that $\pi \circ \psi = i \circ h$ .Let $m \in M$,then $h(m) = x + K$ for some $x \in N$ and hence $\pi \circ \psi(m) = i \circ h(m)$. That is $\psi(m) + K = x + K$,hence $\psi(m) - x \in K$ .But $x \in N$ and $K \subseteq N$, therefore $\psi(m) \in N$ .This implies that $\psi \in Hom(M, N)$. Now, N is a coquasi - invertible submodule, thus $\psi = 0$ and hence $h = 0$.

The converse follows from Prop. 3-4.

Recall that for any ring R, the Jacobson radical of R denoted by $J(R)$, is defined to be the intersection of all maximal right ideals of R. It is known that $J(R)$ is the sum of all small right ideal of R. Before we give the next proposition we need the following lemma. [3-4, P.187[

Lemma 3-12: Let M be a self - projective R - module and, then $J(S) = \{ f \in End(M) /$ Imf is small submodule of M}

In the next theorem we give a condition under which each small submodule of self - projective module is coquasi - invertible submodule.

Theorem 3-13: Let M be a self - projective R - module With $J(End(M)) = 0$ and let N be a submodule of M. then N is a small submodule in M if and only if N is coquasi - invertible submodule of M

Proof: Suppose that N is a small submodule in M. let $f \in Hom(M, N)$. Set $i \circ f = \phi$ where $i : N \to M$ is the inclusion homomorphism. Now , $\operatorname{Im}\phi = \operatorname{Im} f \subseteq N$,but N is a small submodule in M, thus $\operatorname{Im}\phi$ is small in M and therefore $\phi \in J(End(M))$ Lemma 3-12. This implies that $\phi = 0$ and hence $f = 0$ The converse, since N is a coquasi - invertible submodule of M, then by theorem 3-11, N is corational submodule of M. thus by proposition 3-6 N is small in M.

Remark3-14: The condition $J(End(M)) = 0$ is essential in the previous theorem. For $End(Z_4) \cong Z_4$ and $J(End(Z_4)) = Z_2$ and thus the condition of theorem 1.2.14 is not satisfied in the Z - module Z_4 .Note that the submodule $\{\bar{0}, \bar{2}\}$ is small submodule, which is not coquasi - invertible submodule.

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