A Note on Separating Vectors for Operators on A HILBERT Space

Adil G. Naoum & Zainab A. H. Al-Shimri
Department of Mathematics, College of Science, University of Baghdad, Baghdad-Iraq.
Received: 8/1/2003 ***Accepted: 31/10/2005***

Abstract

Let H and K be separable complex Hilbert spaces, and let $B(H,K)$ be the space of bounded linear operators from H to K. If $H=K$ we write $B(H)$ for $B(H,H)$.In this note we study separating vectors for separators on $B(H\oplus K)$ that have representations of the form $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$, where $A\in B(H)$,$B\in B(K)$ and $C\in B(K,H)$.

الخلاصة

لتكن H و K فضاءات هلبرت العقدية قابلة للفصل وليكن $B(H,K)$ فضاء المؤثرات الخطية المقيدة من H الى K .أذا كان $H=K$ فنضع $B(H)$ عوضا عن $B(H,H)$. في هذا البحث ندرس المتجهات الفاصلة للمؤثرات الخطية على $H\oplus K$ التي لها تمثيل مصفوفي من الصيغة $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ حيث ان $A \ni B(H)$, $B \ni B(K)$ و $B(K,H) \ni C$.

Introduction

Let H be separable complex Hilbert space, and let $B(H)$ be the algebra of bounded (linear) operators on H. For $T \in B(H)$, we let $W(T)$ be the weak closure of the sub algebra generated by T in $B(H)$ .A vector $x\in H$ is called separating vector for T if for each $A\neq 0\in W(T)$, $Ax=0$ implies $A=0$,[12]. A vector $x\in H$ is called a cyclic vector for T if for each $A\neq 0\in W(T)$, $Ax=0$ implies $A=0$, [12]. A vector $x\in H$ is called cyclic vector for T if the span of the set $\{x,Tx,T^2x,\ldots\}$is norm dense in H ,[4]. It is known that every cyclic vector is a separating vector [2,p289]. However the converse is false in general.

In [11] an example is given of an operator that does not have a separating vector. Thus the question arises :which operators have separating vectors, and when is the set of separating vectors dense in $B(H)$. It is known that normal operators, algebraic operators and triangular operators have separating vectors, [3],[11],[13].

In this not we study separating vectors for operator matrices, operators represented by matrices where entries are operators. For the properties of such operators see[4]. For general references of operators, see [1],[2],[3],[4],[7].

Preliminaries

Let H be a separable complex Hilbert space and $B(H)$ is the algebra of bounded linear operators of H. Let A be a subalgebra of $B(H)$. A vector $x\in H$ is called a separating vector for A is the map $T\rightarrow Tx$ is injective, i.e. if $Tx=0$ for some $T\in A$, then $T=0$,[12]. A vector $x\in H$ is

This paper represents a portion of the Ph.D. thesis written by the second author under the supervision of the first .It was submitted to the college of Science ,University of Baghdad in the year 2001.

called a cyclic vector for A if the set $\{Tx\backslash T\in A\}$ is dense in H in the norm topology[4].

Let $T \in B(H)$, and let $W(T)$ be the closure in the weak operator topology of the subalgebra of $B(H)$ generated by T, [8]. We say that a vector $x\in H$ is a separating vector for T if x is a separating vector for $W(T)$, we some times write $x\in$ sep($w(T)$)=sep(T). And x is a cyclic vector for T if x is a cyclic vector for $W(T)$.

The following result is known:

Lemma 1.1 [5,p.289] Let $T\in B(H)$. If x is a cyclic vector for T, then x is a separating vector for T.

The following example shows that the converse of 1.1 is false.

Example 1.2. Let U be the unilateral shift operator on H where $H=\ell_2(C)$ by U is defined on the standard basis for $\ell_2(C)$ by $Ue_i=e_{i+1}$ $\forall i=1,2,\ldots$ It is easily seen that e_1 is cyclic vector for U because the span of the set $\{U^n e, n=0,1,2,\ldots\}$is dense in H. Thus by 1.1, e_1 is separating vector. on the other hand, one can check easily that e_n, n≥2 is a separating vector for U but are not cyclic vectors.

The following example shows even finite dimensional algebras may not have separating vectors.

Example 1.3. Let $A\subseteq B(C^3)$ such that

$$A=\left\{\begin{pmatrix} d_1 & a & b \\ 0 & d_2 & c \\ 0 & 0 & d_3 \end{pmatrix} : d_1+d_2+d_3=0, b,c\in C\right\}$$

Note first that dim A=5.Let T_1,T_2,T_3,T_4,T_5 be five linearly indepent elements in A. If x is a separating vector for A, then the vector T_1x,T_2x,T_3x,T_4x,T_5x would be linearly indepented in C^3. In fact if $\Sigma\ C_iT_i=0$, theΣ $C_iT_i=0$,and C_i $\forall i$.

The following is an example of an operator that has separating vectors.

Example 1.4. Let H be finite dimensional complex Hilbert space with dim H=n, n≥2. Let $\{e_1,e_2,\ldots,e_n\}$ be an orthonormal basis for H. The backward shift operator T on H is defined on the basis by $Te_1=0$, and $Te_i=e_{i-1}$ $i\geq 2$.since H is finite dimensional, then it can be seen easily using Cayley-Hamilton theorem that

$W(T) = \{a_0I + a_1T+\ldots\ldots+ a_{n-1}T^{n-1}, a_i \in C\}$

We claim that $x=(x_1,x_2,\ldots,x_n) \in H$ is a separating vector iff $x_n\neq 0$. To see this, $A= a_0I+ a_1T+\ldots+a_{n-1}T^{n-1}$ in $W(T)$ and Ax=0. This gives the system of equations

$$\begin{aligned} a_0x_1+ a_1x_2+\ldots+a_{n-1}x_n&=0\\ a_0x_2+\ldots+a_{n-2}x_n&=0\\ &\vdots\\ a_0x_{n-1}+ a_1x_n&=0\\ a_0x_n&=0 \end{aligned}$$

If $x_n\neq 0$, then $a_i=0$ $\forall i$, and $A = 0$. On the other hand, one can see that if $x_n\neq 0$, then $(x_1,x_2,\ldots,x_{n-1},0)$ is not separating vector for T. in fact $T^{n-1}(x_1,x_2,\ldots,x_{n-1},0)=0$ but $T^{n-1}\neq 0$(Note that $T^n=0$). In the following, we give another class of operators that have separating vectors.

Example 1.5. Let $H=\ell_2(C)$ and let D be a diagonal operator on H defined on the basis $\{e_n\}$ by $De_n=\alpha_n e_n$, $\alpha_n\in C$, n=1,2,3,… .It can be seen easily that $W(D)$ consist of all diagonal operators. Thus, if $A\in W(D)$ then $Ae_n=\beta_n\ e_n$, $\beta_n\in C$,n=1,2,3,…….It is clear now that if $x=(x_1,x_2,\ldots)\in H$ such that $x_i\neq 0$ $\forall i$, then x is a separating vector.

Remark 1.6. An example is given in [10] of an operator $T\in B(H)$ which does not have a separating vector.

Remark 1.7. If an operator T in $B(H)$ has a separating vector, then the set of separating vectors may not be dense in H. For this, see Ex.14.

Next we discuss separating vectors of similar operators. We first recall the following definitions [5].

Definition 1.8. Let H and K be Hilbert spaces. We say that an operator $A\in B(H)$ is quasiaffine of an operator $B\in B(H)$ if $BC=CA$ for some injective dense range operator $C\in B(H,K)$.

The operators A and Bquasisimilar if A is a qusiaffine to B and B is a quasiaffine to A.

It is clear that if A and B are similar ,then they are quasisimilar.

Proposition 1.9. Let H and K be Hilbert spaces. Assume $A\in B(H)$ is quasiaffint to $B\in B(K)$. If A has a separating vector, then B is a separating vector.

Proof. Let $x\in H$ be a separating vector for A. The exist an operator $Q\in B(H,K)$such that Q is injective with dense range and $BQ=QB$. we claim that Qx is a separating vector for B. It is easily that $P(B)Q=QP(A)$ for each polynomail P. This implies that $W(B)Q=QW(A)$ .i.e. $T_2Q=QT_1$ for all $T_1\in W(A)$ and $T_2\in W(B)$. Assume $T_2\neq 0$, then

$T_2Qx\neq0$. In fact if $T_2Qx=0$, then $Q\,T_1x=0$, which implies that $T_1x=0$ and hence $T_1=0$, and $T_2=0$.Since Q has dense range, then $T_2=0$.Hence Qx is a separating vector for B.

Using a similar argument, one can prove the following

Proposition 1.10. If A and B in the last proposition, and if the set of separating vectors for a is dense in H, then the set of separating vectors of B is dense in K.

2. Separating Vectors of Operators Matrices

In this section we study separating vectors of operator matrices [4]. We consider first the operator $A\oplus B$ where $A\in B(H)$, $B\in B(K)$ and H,K are Hilbert spaces.

Proposition 2.1. For Hilbert space H, K, let $A\in B(H)$, $B\in B(K)$, and let $S=\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ in $B(H\oplus K)$.

Then for each $x\in H, y\in K, z=x\oplus y\in \text{Sep}(S)$ iff $x\in\text{Sep}(A)$ and $y\in\text{Sep}(B)$.

Proof. Let $x\in\text{Sep}(A)$ and $y\in\text{Sep}(B)$ and $z=x\oplus y$.

Assume $T\in W(S)$, then $T=\begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix}$ for $C\in W(A), D\in W(B)$.So

$$Tz=\begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} Cx \\ Dy \end{pmatrix}$$

If $Tz=0$, then $Cx=0$, $Dy=0$ which imply $C=0$ and $D=0$, and hence $T=0$. Thus $z\in\text{Sep}(W(S))$.

On the other hand, if $z\in\text{Sep}(W(S))$, then $z=x_1\oplus y_1$ where $x_1\in H$, and $y_1\in K$.

Let $C\in W(A)$ and $Cx_1=0$.

Let $T_1=\begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}$, thus $T_1\in W(S)$. Therefore

$T_1z=\begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}=\begin{pmatrix} Cx_1 \\ 0 \end{pmatrix}$ and $T_1z=0$. So, $T_1=0$ and hence $C=0$, therefore $x_1\in\text{Sep}(W(A))$. Similarly if $D\in W(B)$ and $Dy_1=0$, we have $y_1\in\text{Sep}(W(B))$ if $T_1=\begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}$

The proof of the following is similar to the proof of the last proposition.

Proposition 2.2. For Hilbert spaces H_i, $1\leq i\leq n$, let $A_i\in B(H_i)$. If we define $S\in B(\oplus\Sigma^n_{i=1} H_i)$ by

$$S=\begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_n \end{pmatrix},$$

then $W(S)$has a separating vector if each $W(A_i)$ has a separating vector.

The proof of the following is simple and hence is omitted.

Proposition 2.3. For Hilbert spaces H, K, let $A\in B(H), B\in B(K)$. If the sets $\text{Sep}(W(S))$ and $\text{Sep}(W(B))$ are dense in H and K respectively, then $\text{Sep}(W(S))$ is dense in $H\oplus K$, where $S=\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ in $B(H\oplus K)$.

Let us turn to separating vectors of operators matrices of the form $\begin{pmatrix} A & E \\ 0 & B \end{pmatrix}$ in $B(H\oplus C^n)$.We start by a special case.

Proposition 2.4. Let $A\in B(H)$ that have a separating vector. Let $y\in H$, $\lambda\in C$ such that $y\in\text{Ran}(A-\lambda I)$, where I is the identity operator on H. Define the operator $S\in B(H\oplus C)$ by $S=\begin{pmatrix} A & y \\ 0 & \lambda \end{pmatrix}$. Then the Algebra $W(S)$ has a separating vector. Moreover, if $\text{Sep}(A)$ is dense in H, then $\text{Sep}(S)$ is dense in $H\oplus C$.

Proof. Let $x\in H$ such that $y=(A-\lambda I)x$. Define $R\in B(H\oplus C)$ as $R=\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$. Thus $RSR^{-1}=\begin{pmatrix} A & 0 \\ 0 & \lambda \end{pmatrix}$.

Hence $RSR^{-1}=A\oplus\lambda$. It is easy tosee that for operator $\lambda:C\to C$, ever --y non zero element in C is a separating vector. Thus by 2.1.$A\oplus\lambda$ has a separating vector, and thus by 1.9, S has a separating vector. The last assertion follows from 1.10, and 2.3.

Corollary 2.5. Let $A\in B(H)$ such that $\text{Sep}(A)\neq\phi$, and let $\lambda\in C$ such that $\lambda\notin\sigma(A)$. Then for each $y\in H$, the operator $S=\begin{pmatrix} A & y \\ 0 & \lambda \end{pmatrix}$ has a separating vector.

Proof. Since $\lambda\notin\sigma(A)$, then $A-\lambda I$ is invariable, hence is surjective. The result now follows from 2.4.

Remark 2.6. The condition $y\in\text{Ran}(A-\lambda I)$in proposition 2.5 is sufficient but is not necessary, as in seen by the following example.

Example: let $x\in H$ such that $y=(A-\lambda I)x$. Define $R\in B(H\oplus C)$ as $R=\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}$.

Thus $RSR^{-1}=\begin{pmatrix}A & 0\\ 0 & \lambda\end{pmatrix}$. Hence $RSR^{-1}=A\oplus\lambda$ i.e. the operator S is similar to the operator $A\oplus\lambda$. It is easy that for the operator $\lambda:C\to C$,every nonzero element in C is a separating vector. Thus by 2.1 $A\oplus\lambda$ has a separating vector, and thus by 1.9, S has a separating vector. The last assertion follows from 1.10 and 2.3.

Corollary 2.5. Let $A\in B(H)$ such that Sep(A) $\neq\phi$, and let $\lambda\in C$ such that $\lambda\notin\sigma(A)$. Then for each $y\in H$, the operator $S=\begin{pmatrix}A & y\\ 0 & \lambda\end{pmatrix}$ has a separating vector.

Proof. Since $\lambda\notin\sigma(A)$, then $A-\lambda I$ is invariable, hence is surjective. The result now follows from 2.4 .

Remark 2.6. The condition $y\in$Ran($A-\lambda I$) in proposition 2.5 is sufficient but is not necessary ,as is seen by the following example.

Example 2.7. Let D be a diagonal operator on $H=\ell_2(C)$ with diagonal elements $(1,\frac{1}{2},\ \frac{1}{3},\ldots)$. Let $x=(1,\frac{1}{2},\ \frac{1}{3},\ldots)$ and $S=\begin{pmatrix}D & X\\ 0 & 0\end{pmatrix}$ in $B(H\oplus C)$. It is clear that $x\notin$Ran(D).

On the other hand, It easy to show that $W(S)$ has a cyclic vector and hence has a separating vector by 1.1 . In fact, let $Z=0\oplus 1\in H\oplus C$. For n≥1,

$$S^{n+1}=\begin{pmatrix}D^{n+1} & D^n x\\ 0 & 0\end{pmatrix}$$

Thus, $S^{n+1}z=\begin{pmatrix}D^n x\\ 0\end{pmatrix}$. By [4,p86], D is a cyclic operator and hence span$\{D^n x/n=0,1,2,\ldots\}$ is dense in H. This implies that Z is cyclic vector for S.

More generally. We prove the following.

Proposition 2.8. Let $A\in B(H)$ such that the Sep($W(A)$)$\neq\phi$. Let $y_i\in H$, and $\lambda_i\in C$,i=1,2,…,n, and let

$$S=\begin{pmatrix}A & y_1 & y_2 & \cdots & \cdots & y_n\\ 0 & \lambda_1 & 0 & & & 0\\ 0 & 0 & \lambda_2 & 0 & & 0\\ & & & \ddots & & \\ & & & & \ddots & 0\\ 0 & & & & & \lambda_n\end{pmatrix}$$

Where $S\in B(H\oplus C^n)$. If $\lambda_i\notin\sigma(A)$ for all I,then S has a separating vector.

Proof. Since $\lambda_i\notin\sigma(A)$, i=1,2,…,n, then th operator $(A-\lambda_i I)$are invertible and hence $y_i\in$Ran$(A-\lambda_i I)$. Let $x_i\in H$ such that $y_i=(A-\lambda_i I)x_{i=1,2,\ldots,n.}$

Let

$$R=\begin{pmatrix}I & x_1 & x_2 & \cdots & \cdots & x_n\\ 0 & 1 & 0 & & & 0\\ 0 & 0 & 1 & 0 & & 0\\ & & & \ddots & & \vdots\\ & & & & \ddots & 0\\ 0 & & & & & 1\end{pmatrix}$$

Then $R\in B(H\oplus C^n)$. Note that

$$RSR^{-1}=\begin{pmatrix}A & & & & 0\\ & \lambda_1 & & & \\ & & \lambda_2 & & \\ & & & \ddots & \\ 0 & & & & \lambda_n\end{pmatrix}$$

Hence $RSR^{-1}=\begin{pmatrix}A & 0\\ 0 & D\end{pmatrix}$, when D is a diagonal operator on C^n with diagonal elements $\lambda_1,\lambda_2,\ldots,\lambda_n$.By 1.5, $W(D^n)$ has a separating vector, thus by 2.2, RSR^{-1} has a separating vector. The result follows from 1.9.

Next we turn to the operator matrix $\begin{pmatrix}A & E\\ 0 & B\end{pmatrix}$ in $B(H\oplus K)$. We need the following theorem.

Theorem 2.9. Let $A\in B(H)$ and $B\in(K)$. If $\sigma(A)\cap\sigma(B)=\phi$, then the operator $\begin{pmatrix}A & E\\ 0 & B\end{pmatrix}$ is similar to the operator $\begin{pmatrix}A & 0\\ 0 & B\end{pmatrix}$.

Proposition 2.10. Let $A\in B(H)$ and $B\in B(K)$ such that $\sigma(A)\cap\sigma(B)=\phi$. Assume that each of A and B has a separating vector in H and K respectively. Let $S=\begin{pmatrix}A & E\\ 0 & B\end{pmatrix}$ in $B(H\oplus K)$ where $E\in B(K,H)$. Then S has a separating vector .

Proof. By Th.2.9, the operator S is similar to the operator $S' = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in B(H \oplus K)$. The result follows from 1.9 and 1.4.

References

1. Al-Shaimary, Z.A.H, "*On separating vectors of linear operators on Hilbert space*", Ph.D. thesis, College of Science, University of Baghdad, (**2001**).
2. Conway, J.B., "*A course in funetional analysis*", Springer-Verlag, **1985**.
3. Douglas, R.G., "*Banach algebra techniques in operator theory*", Academic Press, **1972**.
4. Halmos, P.R., "*A Hilbert space problem book*", Springer Verlag, **1982**.
5. Radjabalipour, M and Radjawi, H., "*Operators with commutative commutants* ", Michigan Math. J.35, (**1988**).
6. Radjawi, H. and Rosenthal P., "*invariant subspaces*", Springer –Verlag, **1973**.
7. Taylor, A.E., "*Introduction to functional analysis*", Jhon Wiley, **1985**.
8. Shields, A.L. and Wallen, L.J., "*The commutatnts of sertain Hilbert space operator* ", Indiana U.Math.J. 20, (1971).
9. Wogen, W.R., "*On some operators with cyclic vectors* ", Indiana U. Math.J. 27, *(*1978*)*.
10. Wogen. W.R., "*Some counter examples in non-self alyerint algebra*", Ann. of Math. 126, (1987).
11. Wogen. W.R. and Larson, D.R., "*Extension of normal operators* ", Intg. Equat.oper.Th. 20, (1944).
12. Wogen, W.R., and Larson, D.R., "*Some problems of triangular and semi-traingular operators*", Contemporary Math., 120, (1991).
13. Wogen, W.R., and Larson, D.R., and Gong, W., "*Two results on separating vectors*", Indiana U.Math.J. 43, (1994). Both Authors are in department of mathematics, college of Science, University of Baghdad.