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On Analytical Solution of Time-Fractional Type Model of the Fisher's Equation

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Abstract

In this paper, the time-fractional Fisher's equation (TFFE) is considered to exam the analytical solution using the Laplace q-Homotopy analysis method (Lq-HAM)". The Lq-HAM is a combined form of q-homotopy analysis method (q-HAM) and Laplace transform. The aim of utilizing the Laplace transform is to outdo the shortage that is mainly caused by unfulfilled conditions in the other analytical methods. The results show that the analytical solution converges very rapidly to the exact solution.

Keywords: Fisher's Equation; Laplace q-Homotopy Analysis Method; Fractional Derivative

حول الحل التحليلي لمعادلة فيشر الزمنية ذات الرتب الكسرية

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الخلاصة

قمنا في هذا البحث بدراسة الحل التحليلي لمعادلة فيشر الزمنية ذات الرتب الكسرية باستخدام طريقة تحليل لابلاس الكيوهوموتوبي. إن طريقة تحليل لابلاس الكيوهوموتوبي ناتجة عن تركيب طريقة التحليل الكيوهوموتوبي وتحويل لابلاس. إن الهدف من استخدام تحويل لابلاس هو تجاوز النقص الذي يحدث بشكل أساسي بسبب الشروط غير المحققة في الطرق التحليلية الأخرى. أظهرت النتائج أن الحل التحليلي باستخدام هذه الطريقة يتقارب وبشكل سريع جدا الى الحل المضبوط.

1. Introduction

Fractional differential equations (FDEs) represent an important area of study because of their applications in various fields of science and engineering. Several phenomena can be modeled by non-linear fractional differential equations. For example, electrochemistry [1], electrical circuits [2], signal processing [3], probability [4], and so on. In recent years, analytical and numerical techniques were created to obtain approximate solutions to the FDEs, such as the Adomian decomposition technique (ADM) [5, 6], variational iteration technique (VIM) [7], homotopy perturbation technique (HPM) [8] and the homotopy analysis technique (HAM) [9-12]. Recently, a new analytic method named q-homotopy analysis method (q-HAM) was introduced [13, 14]. The "q-HAM" has numerous applications in a wide range of problems [15-20].

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In this paper, we consider the TFFE

$$D_t^\alpha u(x,t) = D_{xx}u(x,t) + \lambda u(x,t)\big(1-u(x,t)\big), (x,t) \in [0,1]\times[0,1] \quad (1)$$

subject to the initial condition:

$$u(x,0) = u_0(x,t), \quad (2)$$

where $0<\alpha\le 1$, λ is a real parameter, D_t^α represents the Caputo fractional derivative in time [21], $u_0(x,t)$ is the given function and $D_{xx} = u_{xx}$ is the linear differential operator. This problem is considered a mathematical model for a wide scope of significant physical phenomena. It has become one of the most important classes of nonlinear equations due to its occurrence in many chemical and biological processes. The time-fractional Fisher's equation was solved by homotopy perturbation technique (HPM) [8] and homotopy analysis technique (HAM) [9].

The purpose of this paper is to apply the Lq-HAM, which is a combination of q-HAM and Laplace transform to provide an approximate solution for the TFFE. In section two, we state some necessary concepts of fractional calculus. In section three, we introduce the basic idea of Lq-HAM for TFFE. Finally, in section four, we solve two numerical examples.

2. Preliminaries

In this section, we state some necessary concepts of fractional calculus that will help us to achieve the aim of this paper [21, 22].

Definition 2.1 A real function $v(t), t>0$ is said to be in space $C_\vartheta(\vartheta\in\mathbb{R})$ if there exists a real number $>\vartheta$, such that $v(t)=t^\rho v_1(t)$, where $v_1(t)\in C(0,\infty)$, and it is said to be in the space C_ϑ^m if and only if $v^{(m)}\in C_\vartheta, m\in\mathbb{N}$.

Definition 2.2 The Riemann–Liouville fractional integral operator J^α of order $\alpha\ge 0$, of a function $v(t)\in C_\vartheta, \vartheta\ge -1$, is defined as

$J^\alpha v(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}v(\tau)d\tau \ \ (\alpha>0),$

$J^0 v(t) = v(t).$

Some of the basic properties of the operator J^α, which are required here, are introduced.

For $v\in C_\vartheta, \vartheta\ge -1, \alpha,\beta\ge 0, \gamma\ge -1$

(i) $J^\alpha J^\beta v(t) = J^{\alpha+\beta}v(t),$

(ii) $J^\alpha J^\beta v(t) = J^\beta J^\alpha v(t),$

(iii) $J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}t^{\alpha+\gamma}.$

Definition 2.3. The fractional derivative $v(t)$ in the Caputo's sense is defined as

$D_t{}^\alpha v(t) = J^{m-\alpha}D_t{}^m v(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}v^{(m)}d\tau,$

for $m-1<\alpha\le m,\ m\in\mathbb{N},\ t>0,\ v\in C_{-1}^m.$

Moreover, some of the most important properties are needed here.

(i) $D_t{}^\alpha J^\alpha v(t) = v(t),$

(ii) $J^\alpha D_t{}^\alpha v(t) = v(t) - \sum_{k=0}^{m-1} v^{(k)}(0^+)\frac{t^k}{k!},\ t>0,$

(iii) $D_t{}^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}t^{\gamma-\alpha},$

for $m-1<\alpha\le m,\ m\in\mathbb{N}, \vartheta\ge -1,\ v\in C_\vartheta^m.$

Lemma 2.1 [23]: If $m-1<\alpha\le m, m\in\mathbb{N}$, then the Laplace transform of the fractional derivative $D_t^\alpha v(t)$ is $\mathcal{L}[D_t^\alpha v(t)] = s^\alpha V(s) - \sum_{k=0}^{m-1} v^{(k)}(0^+)s^{\alpha-k-1},\ t>0$

where $V(s)$ is the Laplace transform of $v(t)$.

3. The Lq-HAM for TFFE

Consider the time-fractional Fisher's equation (1) subject to the initial condition (2).

Taking the Laplace transform of both sides of equation (1) and utilizing equation (2), gives

$$\mathcal{L}[D_t^\alpha u(x,t)] = \mathcal{L}[D_{xx}u(x,t)] + \lambda\mathcal{L}[u(x,t)] - \lambda\mathcal{L}[u^2(x,t)].$$

By using lemma (2.1), we obtain

$$\mathcal{L}[u(x,t)] = \frac{u_0(x,t)}{s} + \frac{1}{s^\alpha}\big[\mathcal{L}[D_{xx}u(x,t)] + \lambda\mathcal{L}[u(x,t)] - \lambda\mathcal{L}[u^2(x,t)]\big]. \quad (3)$$

Let the zero-order deformation equation

$$(1-nq)\mathcal{L}[\emptyset(x,t;q) - u_0(x,t,s)] = qhN[\emptyset(x,t;q)], \quad (4)$$

where $n \geq 1$, $0 \leq q \leq \frac{1}{n}$ denotes the embedded parameter, $h \neq 0$ is an auxiliary parameter. It is obvious that when $q = 0$ and $q = \frac{1}{n}$, equation (4) becomes:

$$\emptyset(x,t;0) = u_0(x,t)\,, \qquad \emptyset\left(x,t;\frac{1}{n}\right) = u(x,t). \tag{5}$$

Thus, as q increases from 0 to $\frac{1}{n}$, the solution $\emptyset(x,t;q)$ varies from the initial $u_0(x,t)$ to the solution $u(x,t)$.

By expanding $\emptyset(x,t;q)$ in Taylor series with respect to q, we get

$$\emptyset(x,t;q) = u_0(x,t) + \sum_{m=1}^{+\infty} u_m(x,t)q^m\,, \tag{6}$$

where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \emptyset(x,t;q)}{\partial q^m}\Big|_{q=0}\,. \tag{7}$$

Assume that h, $u_0(x,t)$ are chosen such that the series (6) converges at $=\frac{1}{n}$, then under these conditions the series solutions give

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{+\infty} u_m(x,t)\left(\frac{1}{n}\right)^m. \tag{8}$$

Defining the vector $u_r^{\rightarrow}(x,t) = \{u_0(x,t), u_1(x,t), u_2(x,t), \ldots, u_r(x,t)\}$.

Differentiating equation (4) m times with respect to q then setting $q = 0$ and finally dividing them by $m!$ yields the so-called m^{th} order deformation equation

$$u_m(x,t) = z_m u_{m-1}(x,t) + h\mathcal{L}^{-1}\left[R_m(\vec{u}_{m-1}(x,t))\right], \tag{9}$$

where

$$R_m\left(\vec{u}_{m-1}(x,t)\right) = \mathcal{L}[u_{m-1}(x,t)] - \frac{1}{s^\alpha}\left[\mathcal{L}[D_{xx}u_{m-1}(x,t)] + \lambda\mathcal{L}[u_{m-1}(x,t)] - \lambda\mathcal{L}\left[\sum_{i=0}^{m-1} u_i\, u_{m-1-i}\right]\right] - \left(1 - \frac{1}{n}z_m\right)\frac{u_0(x,t)}{s}, \tag{10}$$

and

$$z_m = \begin{cases} 0 & m \leq 1 \\ n & m > 1 \end{cases} \tag{11}$$

4. Numerical Results

In this section, we apply the Lq-HAM for two examples.

Example 1 consider the following time-fractional problem

$$D_t^\alpha u(x,t) = D_{xx}u(x,t) + u(x,t)\big(1 - u(x,t)\big), (x,t) \in [0,1] \times [0,1]\,; \tag{12}$$

subject to the initial condition:

$$u(x,0) = \beta. \tag{13}$$

When $\alpha = 1$, the problem (12)-(13) gives the exact solution

$$u(x,t) = \frac{\beta e^t}{1-\beta+\beta e^t}\,. \tag{14}$$

By using the analysis in the previous section equation (9), we obtain

$$u_m(x,t) = z_m u_{m-1}(x,t) + h\mathcal{L}^{-1}\left[R_m(\vec{u}_{m-1}(x,t))\right],$$

where

$$R_m\left(\vec{u}_{m-1}(x,t)\right) = \mathcal{L}[u_{m-1}(x,t)] - \frac{1}{s^\alpha}\left[\mathcal{L}[D_{xx}u_{m-1}(x,t)] + \mathcal{L}[u_{m-1}(x,t)] - \mathcal{L}\left[\sum_{i=0}^{m-1} u_i\, u_{m-1-i}\right]\right] - \left(1 - \frac{1}{n}z_m\right)\frac{\beta}{s},$$

and z_m is defined as in equation (11).

By using Mathematica software, the following results are obtained

$$u_1(x,t) = \frac{ht^\alpha(-1+\beta)\beta}{\Gamma(\alpha+1)},$$

$$u_2(x,t) = \frac{ht^\alpha(\beta-1)\beta(ht^\alpha(2\beta-1)\Gamma(\alpha+1) + (h+n)\Gamma(2\alpha+1))}{\Gamma(\alpha+1)\Gamma(2\alpha+1)},$$

$\vdots$

Then the m^{th} order series solution of Lq- HAM is as follows

$$\emptyset_M(x,t) = \sum_{i=0}^{M} u_i(x,t)\left(\frac{1}{n}\right)^i \quad (15)$$

We notice that when $h = -1, n = 1$, we obtain the same solution given by HPM [8].

Figure-1 shows the 4^{th} -order approximate solution for different values of α with $\beta = \frac{2}{3}, n = 1, h = -1.05$ with the exact solution of problem (12)-(13).

Figure-2 shows the absolute error obtained by the 4^{th} -order approximation of problem (12-13) for α = 1.

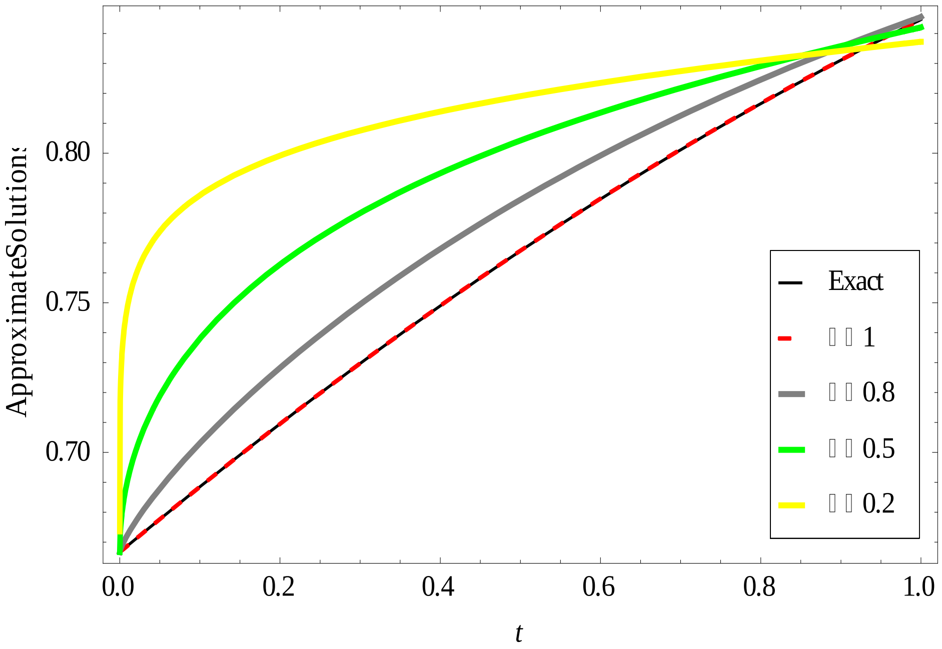

Figure 1-The 4^{th} -order approximate solution for different values of α with $\beta = \frac{2}{3}, n = 1, h = -1.05$ with the exact solution of problem (12)-(13)

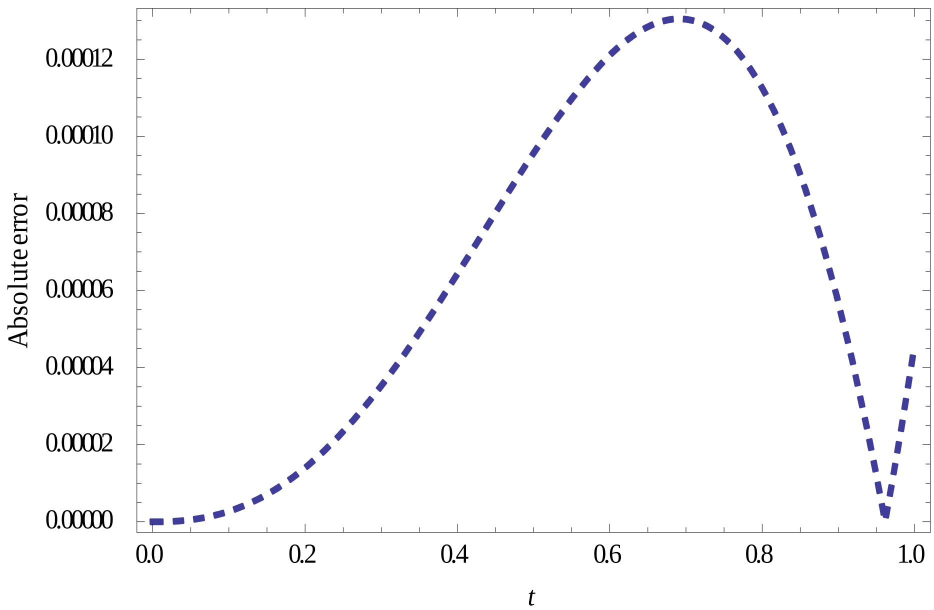

Figure 2- The absolute error obtained by 4th-order approximation of problem (12-13) for α = 1.

Example 2 Considers the following time-fractional problem

$$D_t^{\alpha} u(x,t) = D_{xx} u(x,t) + 6u(x,t)\big(1 - u(x,t)\big), (x,t) \in [0,1] \times [0,1]\,; \quad (16)$$

subject to the initial condition:

$$u(x,0) = \frac{1}{(1+e^x)^2}. \quad (17)$$

When $\alpha = 1$, the problem (16-17) gives the exact solution

$$u(x,t) = \frac{1}{(1+e^{x-5t})^2}. \quad (18)$$

Using the analysis in the previous section, we obtain

$$u_m(x,t) = z_m u_{m-1}(x,t) + h\mathcal{L}^{-1}\left[R_m(\vec{u}_{m-1}(x,t))\right],$$

where

$$R_m\left(\vec{u}_{m-1}(x,t)\right) = \mathcal{L}[u_{m-1}(x,t)] - \frac{1}{s^\alpha}\left[\mathcal{L}[D_{xx}u_{m-1}(x,t)] + 6\mathcal{L}[u_{m-1}(x,t)] - 6\mathcal{L}\left[\sum_{i=0}^{m-1} u_i\, u_{m-1-i}\right]\right] - \left(1 - \frac{1}{n}z_m\right)\frac{1}{s(1+e^x)^2},$$

and z_m as defined in equation (11).

Using Mathematica software, the following result are obtained

$$u_1(x,t) = \frac{10e^x h t^\alpha}{(1+e^x)^3\Gamma(\alpha+1)},$$

$$u_2(x,t) = \frac{10e^x h t^\alpha(5(-1+2e^x)ht^\alpha\Gamma(\alpha+1) - (1+e^x)(h+n)\Gamma(2\alpha+1))}{(1+e^x)^4\Gamma(\alpha+1)\Gamma(2\alpha+1)}$$

$\vdots$

Then the m^{th} order series solution of Lq- HAM is as follows

$$\emptyset_M(x,t) = \sum_{i=0}^{M} u_i(x,t)\left(\frac{1}{n}\right)^i. \quad (19)$$

Figure-3 shows the 5th -order approximate solution for different values of α with $n = 5, h = -4.934$ when $x = 1$ with the exact solution of problem (16-17).

Figure-4 shows the absolute error obtained by the 5th -order approximation of problem (16-17) for $\alpha = 1$, when $x = 1$.

Table-1 shows the 5th order Lq-HAM approximate solutions for (16)-(17) when $\alpha = 0.7, 0.8, 0.9$ and 1 with $h = -4.934$ and $n = 5$. We notice that when $= 1$, then the approximate solution of Lq-HAM is consistent with the exact solution.

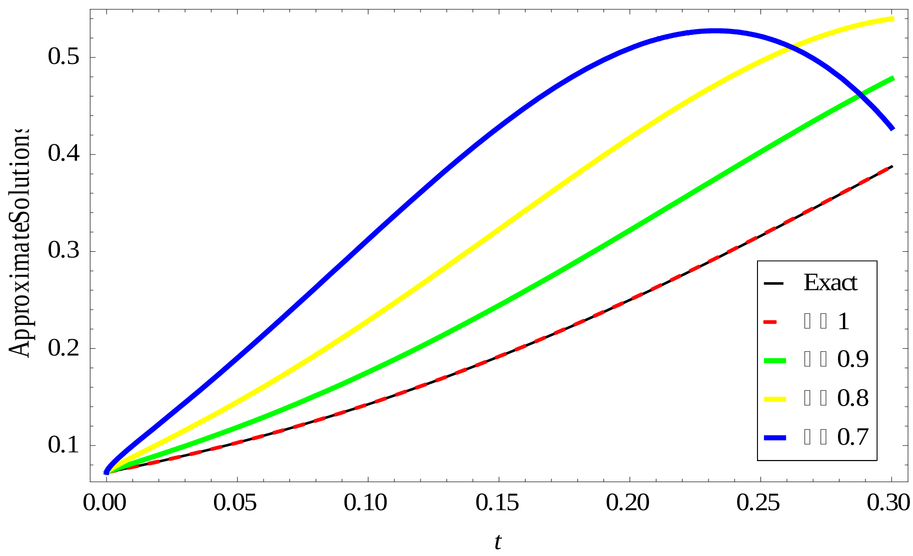

Figure 3-The 5th -order approximate solution for different values of α with $n = 5, h = -4.934$ when $x = 1$ with the exact solution of problem (16-17)

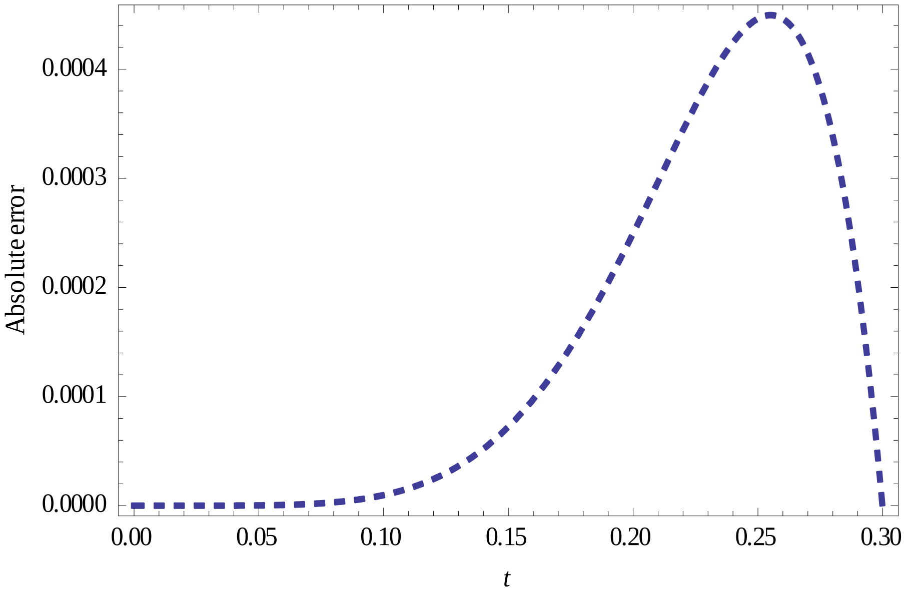

Figure 4-The absolute error obtained by 5th-order approximation of problem (16)-(17) for $\alpha = 1$ when $x = 1$.

Table 1-The 5th order Lq-HAM approximate solutions for (16)-(17) for different values of α with $h = -4.934$ and $n = 5$.

t	x	U_5Lq-HAM; $\alpha = 0.7$	U_5Lq-HAM; $\alpha = 0.8$	U_5Lq-HAM; $\alpha = 0.9$	U_5Lq-HAM; $\alpha = 1$	*Exact*
0.1	0.25	0.495340	0.423134	0.362072	0.316020	0.316042
	0.5	0.430680	0.353807	0.293206	0.249987	0.25000
	0.75	0.369801	0.288288	0.230424	0.191689	0.191689
	1.0	0.312428	0.228539	0.175669	0.142546	0.142536
0.2	0.25	0.560856	0.575416	0.523580	0.459829	0.461283
	0.5	0.495736	0.514935	0.454498	0.386355	0.387455
	0.75	0.486248	0.464094	0.387002	0.315635	0.316042
	1.0	0.509370	0.417091	0.321916	0.250249	0.25000
0.3	0.25	0.263329	0.552923	0.616961	0.588893	0.604195
	0.5	0.073206	0.483338	0.558501	0.520884	0.534446
	0.75	0.153317	0.490309	0.515845	0.454033	0.461283
	1.0	0.427485	0.539829	0.478075	0.387452	0.387455

5. Conclusions

The major concern of this paper is the demonstration of the successful use of Lq-HAM to obtain analytical solutions of TFFE. The Lq-HAM was used in a direct way that is the restrictive assumptions are avoided in Lq-HAM. These considerations give Lq-HAM a significant advantage in many problems. Our results confirm that the appropriate choices of the convergence control parameters h and n lead to the accuracy of Lq-HAM in the sense that , unlike other methods, just few terms are needed in our approximations to get close to the exact solution for $\alpha = 1$.

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