Rasheed and Chlebik Iraqi Journal of Science, 2020, Vol. 61, No.1, pp: 147-152
DOI: 10.24996/ijs.2020.61.1.16

ISSN: 0067-2904

Blow-up Rate Estimates and Blow-up Set for a System of Two Heat Equations with Coupled Nonlinear Neumann Boundary Conditions

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Received: 13/7/ 2019 Accepted: 18/ 8/2019

Abstract

This paper deals with the blow-up properties of positive solutions to a parabolic system of two heat equations, defined on a ball in R^n,associatedwith coupled Neumann boundary conditions of exponential type. The upper bounds of blow-up rate estimates are derived. Moreover, it is proved that the blow-up in this problem can only occur on the boundary.

Keywords: Heat equation; Neumann boundary conditions; Blow-up set; Blow-up rate estimate; Green function.

1. Introduction

In this paper, we consider the following parabolic system of two heat equations associated with Neumann boundary conditions:

$$\left.\begin{array}{lll} u_t = \Delta u, & v_t = \Delta v, & (x,t) \in B_R \times (0,T), \\ \frac{\partial u}{\partial \eta} = \lambda_1 e^{v^p}, & \frac{\partial v}{\partial \eta} = \lambda_2 e^{u^q}, & (x,t) \in \partial B_R \times (0,T), \\ u(x,0) = u_0(x), & v(x,0) = v_0(x), & x \in B_R, \end{array}\right\} \quad (1)$$

where $p, q > 1; \lambda_1, \lambda_2 > 0$; B_R is a ball in $R^n; \eta$ is the outward normal; u_0, v_0 are both smooth functions, radially symmetric, nonzero, nonnegative and satisfy the condition:

$$\Delta u_0, \Delta u_0 \geq 0, \quad u_{0r}(|x|), v_{0r}(|x|) \geq 0, \text{for } x \in \overline{B}_R \ ,$$

$$\text{and} \quad \frac{\partial u_0}{\partial \eta} = \lambda_1 e^{v_0^p}, \quad \frac{\partial v_0}{\partial \eta} = \lambda_2 e^{u_0^q}, \quad x \in \partial B_R \quad (2)$$

Since the last decades, many authors have studied the blow-up properties to solutions of parabolic problems, defined on bounded domains [see for instance 1, 2]. One of the studied problems is the system of two heat equations defined in a ball, associated with coupled Neumann boundary conditions:

$$\left.\begin{array}{lll} u_t = \Delta u, & v_t = \Delta v, & (x,t) \in B_R \times (0,T), \\ \frac{\partial u}{\partial \eta} = f(v), & \frac{\partial v}{\partial \eta} = g(u), & (x,t) \in \partial B_R \times (0,T), \\ u(x,0) = u_0(x), & v(x,0) = v_0(x), & x \in B_R, \end{array}\right\} \quad (3)$$

This problem was previously studied [3-6] ; in case of the nonlinear functions f and g take one of the two forms:

$$f(v) = v^p, \quad g(u) = u^q, \quad p, q > 1. \quad (4)$$

$$f(v) = e^{pv}, \quad g(u) = e^{qu}, \quad p, q > 0. \quad (5)$$

For both cases, it was shown that if the initial data (u_0, v_0) are nonzero and nonnegative, then the blow-up can only occur on the boundary.

In addition to that, with case 4, it was proved that the blow-up rate estimates take the form:

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$$c \le \max_{x\in\overline{\Omega}} u(x,t)(T-t)^{\frac{p+1}{2(pq-1)}} \le C, \quad t\in(0,T),$$
$$c \le \max_{x\in\overline{\Omega}} v(x,t)(T-t)^{\frac{q+1}{2(pq-1)}} \le C, \quad t\in(0,T)$$

where c and C are positive constants.

While, with case 5, it was proved that the blow-up rate estimates take the form:

$$C_1 \le e^{qu(R,t)}(T-t)^{\frac{1}{2}} \le C_2,$$
$$C_3 \le e^{pv(R,t)}(T-t)^{1/2} \le C_4$$

where C_1, C_2, C_3 and C_4 are positive constants.

In this paper, firstly, we show that the upper blow-up rate estimates for problem 1are as follows

$$\max_{\overline{B}_R} u(x,t) \le \log C_1 - \frac{\alpha}{2}\log(T-t), \quad 0<t<T,$$
$$\max_{\overline{B}_R} v(x,t) \le \log C_2 - \frac{\beta}{2}\log(T-t), \quad 0<t<T,$$

where $\alpha = \frac{p+1}{pq-1}, \beta = \frac{q+1}{pq-1}$.

Secondly, we prove that the blow-up in problem 1 can only occur on the boundary.

2. Preliminaries

It is well known that with any smooth initial functions (u_0, v_0), satisfying the compatibility condition 2, there exists a unique local classical solution to problem 1 [7]. On the other hand, it is easy to show that every nontrivial solution blows up simultaneously in finite time and that due to the known blow-up results of problem 3 with 4 and the comparison principle [2,3].

The next lemma, which was previously proved [2], states some properties of the classical solutions of problem1.

For simplicity, we denote $u(r,t) = u(x,t)$.

Lemma 2.1 Let (u, v) be a classical solution to problem 1. Then

1. u, v are positive, radial. Moreover, $u_r, v_r \ge 0$ in $[0,R]\times(0,T)$.
2. $u_t, v_t > 0$ in $\overline{B}_R \times (0,T)$.

3. Blow –up Upper Rate Estimates

The next Lemmas and theorem, proved in other articles [5,8], will be used in this section to derive the upper blow-up rate estimates for problem 1.

Lemma 3.1 [5]: Let A and B be positive and differentiable functions in $[0,\mathrm{T})$,such that they satisfy the two inequalities:

$$A'(t) \ge c\frac{B^p(t)}{\sqrt{T-t}}, \quad B'(t) \ge c\frac{A^q(t)}{\sqrt{T-t}}$$
$$fort \in [0,T),$$
$$A(t)\to+\infty \quad orB(t)\to+\infty \quad ast\to T^-,$$

where $p,q>0, c>0$ and $pq>1$.

Then there exists $C>0$ such that

$$A(t) \le C(T-t)^{-\frac{\alpha}{2}}, \quad B(t) \le C(T-t)^{-\frac{\beta}{2}}, \quad t\in[0,T),$$

where $\alpha = \frac{p+1}{pq-1}, \beta = \frac{q+1}{pq-1}$.

Lemma 3.2 [6]: Let $\mathrm{x}\in\overline{\mathrm{B}}_\mathrm{R}$. If $0\le a<n-1$. Then there exists$\mathrm{C}>0$ such that

$$\int_{S_R}\frac{ds_y}{|x-y|^a} \le C.$$

Theorem 3.3 (Jump relation, [8]) Let $\Gamma(\mathrm{x},\mathrm{t})$ be the fundamental solution of heat equation, namely

$$\Gamma(\mathrm{x},\mathrm{t}) = \frac{1}{(4\pi \mathrm{t})^{(\mathrm{n}/2)}}\exp[-\frac{|\mathrm{x}|^2}{4\mathrm{t}}]. \quad (6)$$

Let φ be a continuous function on $S_R\times[0,T]$. Then for any $x\in B_R, x^0\in S_R, 0<t_1<t_2\le T$, for some $T>0$, the function

$$U(x,t) = \int_{t_1}^{t_2}\int_{S_R}\Gamma(x-y,t-z)\varphi(y,z)ds_y d\tau$$

satisfies the jump relation

$$\frac{\partial}{\partial \eta}U(x,t) \to -\frac{1}{2}\varphi(x^0,t) + \frac{\partial}{\partial \eta}U(x^0,t)$$
$$as x \to x^0.$$

Theorem 3.4 Let (u,v) be a blow-up solution to problem1, and $T > 0$is the blow-up time. Then there exist two positive constants C_1, C_2 such that

$$\max_{\overline{B}_R} u(x,t) \le \log C_1 - \frac{\alpha}{2}\log(T-t), \quad 0 < t < T,$$
$$\max_{\overline{B}_R} v(x,t) \le \log C_2 - \frac{\beta}{2}\log(T-t), \quad 0 < t < T.$$

Proof: In order to prove this theorem, we follow the technique used in a previous work [5].

Define the functions M and M_b as follows:

$$M(t) = \max_{\overline{B}_R} u(x,t), \quad and M_b(t) = \max_{S_R} u(x,t).$$

Similarly,

$$N(t) = \max_{\overline{B}_R} v(x,t), \quad and N_b(t) = \max_{S_R} v(x,t).$$

Depending on Lemma 2.1, both of M, M_b are monotone increasing functions.

Since u is a solution of heat equation, it cannot attain interior maximum without being constant. Therefore,

$$M(t) = M_b(t). \quad \text{Similarly} \quad N(t) = N_b(t).$$

Moreover, since u, v blow up simultaneously, we have

$M(t) \to +\infty, \quad N(t) \to +\infty \quad as t \to T^-$(7)

According to the second Green's identity [5,7, 9], with considering the Green function:

$G(x,y;z_1,t) = \Gamma(x-y,t-z_1)$,for $0 < z_1 < t < T$ and $x \in B_R$,

where Γ is defined in 6, the integral equation to problem 1, with respect to u, takes the form:

$$u(x,t) = \int_{B_R} \Gamma(x-y,t-z_1)u(y,z_1)dy + \int_{z_1}^{t}\int_{S_R} \lambda_1 e^{v^p(y,\tau)}\Gamma(x-y,t-\tau)ds_y d_\tau$$
$$-\int_{z_1}^{t}\int_{S_R} u(y,\tau)\frac{\partial\Gamma}{\partial\eta_y}(x-y,t-\tau)ds_y d\tau,$$

By applying Theorem 3.3 on the third term in the right-hand side of the last equation and with letting $x \to S_R$, we obtain

$$\frac{1}{2}u(x,t) = \int_{B_R} \Gamma(x-y,t-z_1)u(y,z_1)dy + \int_{z_1}^{t}\int_{S_R} \lambda_1 e^{v^p(y,\tau)}\Gamma(x-y,t-\tau)ds_y d_\tau$$
$$-\int_{z_1}^{t}\int_{S_R} u(y,\tau)\frac{\partial\Gamma}{\partial\eta_y}(x-y,t-\tau)ds_y d\tau,$$

for $x \in S_R, 0 < z_1 < t < T$.

Depending on Lemma 2.1, u, v are both radial and positive functions.

Therefore,

$$\int_{B_R} \Gamma(x-y,t-z_1)u(y,z_1)dy > 0,$$
$$\int_{z_1}^{t}\int_{S_R} \lambda_1 e^{v^p(y,\tau)}\Gamma(x-y,t-\tau)ds_y d_\tau = \int_{z_1}^{t} \lambda_1 e^{v^p(R,\tau)}[\int_{S_R} \Gamma(x-y,t-\tau)ds_y]d\tau.$$

This leads to

$$\frac{1}{2}M(t) \ge \int_{z_1}^{t} \lambda_1 e^{N^p(\tau)}[\int_{S_R} \Gamma(x-y,t-\tau)ds_y]d\tau - \int_{z_1}^{t} M(\tau)\left[\int_{S_R}\left|\frac{\partial\Gamma}{\partial\eta_y}(x-y,t-\tau)\right|ds_y\right]d\tau,$$
$$x \in S_R, 0 < z_1 < t < T.$$

It is known that (see [8]) there exists$C_0 > 0$, such that Γ satisfies

$$\left|\frac{\partial\Gamma}{\partial\eta_y}(x-y,t-\tau)\right| \le \frac{C_0}{(t-\tau)^{\mu}}\cdot\frac{1}{|x-y|^{(n+1-2\mu-\sigma)}},$$
$$x, y \in S_R, \sigma \in (0,1).$$

Choose $1 - \frac{\sigma}{2} < \mu < 1$, from Lemma 3.2, there exists$C^* > 0$ such that

$$\int_{S_R} \frac{ds_y}{|x-y|^{(n+1-2\mu-\sigma)}} < C^*.$$

Moreover, for $0 < t_1 < t_2$ and t_1 is closed tot_2, there exists $c > 0$, such that

$$\int_{S_R} \Gamma(x-y, t_2-t_1) ds_y \geq \frac{c}{\sqrt{t_2-t_1}},$$

Thus

$$\frac{1}{2}M(t) \geq c\int_{z_1}^{t} \frac{\lambda_1 e^{N^p(\tau)}}{\sqrt{t-\tau}} d\tau - C\int_{z_1}^{t} \frac{M(\tau)}{|t-\tau|^{\mu}} d\tau.$$

Since for $0 < z_1 < t_0 < t < T$, it follows that $M(t_0) \leq M(t)$, thus the last equation becomes

$$\frac{1}{2}M(t) \geq c\int_{z_1}^{t} \frac{\lambda_1 e^{N^p(\tau)}}{\sqrt{T-\tau}} d\tau - C_1^* M(t)|T-z_1|^{1-\mu}.$$

Similarly, for $0 < z_2 < t < T$, we have

$$\frac{1}{2}N(t) \geq c\int_{z_2}^{t} \frac{\lambda_2 e^{M^q(\tau)}}{\sqrt{T-\tau}} d\tau - C_2^* N(t)|T-z_2|^{1-\mu}.$$

Taking z_1, z_2 so that

$$C_1^*|T-z_1|^{1-\mu} \leq 1/2, \quad C_2^*|T-z_2|^{1-\mu} \leq 1/2,$$

it follows

$M(t) \geq c\int_{z_1}^{t} \frac{\lambda_1 e^{N^p(\tau)}}{\sqrt{T-\tau}} d\tau, \quad N(t) \geq c\int_{z_2}^{t} \frac{\lambda_2 e^{M^q(\tau)}}{\sqrt{T-\tau}} d\tau.$(8)

Since $M, N$are both increasing functions and by7, we can find $T_1 \in (0, T)$, such that

$M(t) \geq q^{\frac{1}{(q-1)}}, \quad N(t) \geq p^{\frac{1}{(p-1)}}, \quad$ for $T_1 \leq t < T$.

Thus

$$e^{M^q(t)} \geq e^{qM(t)}, \quad e^{N^p(t)} \geq e^{pN(t)}, \quad T^* \leq t < T.$$

Therefore, if we choose z_1, z_2 in (T^*, T), then 8 becomes

$$e^{M(t)} \geq c\int_{z_1}^{t} \frac{\lambda_1 e^{pN(\tau)}}{\sqrt{T-\tau}} d\tau \equiv I_1(t),$$

$$e^{N(t)} \geq c\int_{z_2}^{t} \frac{\lambda_2 e^{qM(\tau)}}{\sqrt{T-\tau}} d\tau \equiv I_2(t).$$

Clearly,

$$I_1'(t) = c\frac{\lambda_1 e^{pN(t)}}{\sqrt{T-t}} \geq \frac{c\lambda_1 I_2^p}{\sqrt{T-t}}, \quad I_2'(t) = c\frac{\lambda_2 e^{qM(t)}}{\sqrt{T-t}} \geq \frac{c\lambda_2 I_1^q}{\sqrt{T-t}}.$$

By Lemma 3.1, it follows that

$$I_1(t) \leq \frac{C\lambda_1}{(T-t)^{\frac{\alpha}{2}}}, \quad I_2(t) \leq \frac{C\lambda_2}{(T-t)^{\frac{\beta}{2}}}, \tag{9}$$

$t \in (\max\{z_1, z_2\}, T)$.

On the other hand, with assuming that t is close to T, we have

$$I_1(t) \geq c\int_{t^*}^{t} \frac{\lambda_1 e^{pN(\tau)}}{\sqrt{T-\tau}} d\tau \geq c\lambda_1 e^{pN(t^*)} \int_{2t-T}^{t} \frac{1}{\sqrt{T-\tau}} d\tau = 2c\lambda_1(\sqrt{2}-1)\sqrt{T-t}e^{pN(t^*)}$$

where $t^* = 2t - T$

Combining the last inequality with 9 yields

$$e^{N(t^*)} \leq \frac{C}{2c(\sqrt{2}-1)(T-t)^{\frac{p+1}{2p(pq-1)}+\frac{1}{2p}}} = \frac{2^{\frac{q+1}{2(pq-1)}}C}{2c(\sqrt{2}-1)(T-t^*)^{\frac{q+1}{2(pq-1)}}}.$$

It follows that, there exists a constant $c_1 > 0$ such that

$$e^{N(t^*)}(T-t^*)^{\frac{q+1}{2(pq-1)}} \leq c_1.$$

Similarly, we can find $c_2 > 0$ such that

$$e^{M(t^*)}(T-t^*)^{\frac{p+1}{2(pq-1)}} \leq c_2.$$

This leads to, there exist $C_1, C_2 > 0$ such that

$$\max_{\overline{B}_R} u(x,t) \leq \log C_1 - \frac{\alpha}{2}\log(T-t), \quad , \tag{10}$$

$$\max_{\overline{B}_R} v(x,t) \leq \log C_2 - \frac{\beta}{2}\log(T-t) \quad . \tag{11}$$

for $0 < t < T$

4. Blow-up Set

In this section, we study the blow-up set for problem 1, showing that the blow-up can only occur on the boundary. To prove this result, we recall the following lemma proved in a previous article [6].

Lemma 4.1. Let w be a continuous function on the domain $\overline{B}_R \times [0,T)$ and satisfies

$$\left.\begin{aligned} &w_t = \Delta w, && (x,t) \in B_R \times (0,T), \\ &w(x,t) \le \frac{C}{(T-t)^m}, && (x,t) \in S_R \times (0,T), \quad m > 0 \end{aligned}\right\}$$

Then for any $0 < a < R$,

$$\sup\{w(x,t): 0 \le |x| \le a, 0 \le t < T\} < \infty.$$

Proof: Set

$$h(x) = (R^2 - r^2)^2, r = |x|,$$
$$z(x,t) = \frac{C_1}{[h(x) + C_2(T-t)]^m}.$$

We can show that:

$$\Delta h - \frac{(m+1)|\nabla h|^2}{h} = 8r^2 - 4n(R^2 - r^2) - (m+1)16r^2 \ge -4nR^2 - 16R^2(m+1),$$

and

$$z_t - \Delta z = \frac{C_1 m}{[h(x)+C_2(T-t)]^{m+1}}(C_2 + \Delta h - \frac{(m+1)|\nabla h|^2}{h + C_2(T-t)}) \ge \frac{C_1 m}{[h(x)+C_2(T-t)]^{m+1}}(C_2 - 4nR^2 - 16R^2(m+1)).$$

Let $C_2 = 4nR^2 + 16R^2(m+1) + 1$, and take C_1 to be large such that

$$z(x,0) \ge w(x,0), \quad x \in B_R.$$

Let $C_1 \ge C(C_2)^m$, which implies that

$$z(x,t) \ge w(x,t) \quad on S_R \times [0,T).$$

Then from the maximum principle [10], it follows that

$$z(x,t) \ge w(x,t), \quad (x,t) \in \overline{B}_R \times (0,T)$$

and hence

$$\sup\{w(x,t): 0 \le |x| \le a, 0 \le t < T\} \le C_1(R^2 - a^2)^{-2m} < \infty,$$

for $0 \le a < R$.

Theorem 4.2 Let (u, v) be a blow-up solution to problem1, and $T > 0$ is the blow-up time. Then(u, vcan only blow-up on the boundary.

Proof: By using equations10 and11, we obtain

$$u(R,t) \le \frac{c_1}{(T-t)^{\frac{\alpha}{2}}}, \quad v(R,t) \le \frac{c_2}{(T-t)^{\frac{\beta}{2}}},$$

for $t \in (0,T)$.

From Lemma 4.1, it follows that

$$\sup\{u(x,t): (x,t) \in B_a \times [0,T)\} \le C_1(R^2 - a^2)^{-\alpha} < \infty,$$
$$\sup\{v(x,t): (x,t) \in B_a \times [0,T)\} \le C_1(R^2 - a^2)^{-\beta} < \infty,$$

for $a < R$. Therefore, if $x \in B_R$, it cannot be a blow-up point.

5. Conclusions

This paper is concerned with the blow-up properties of positive solutions to a system of two heat equations defined on a ball in R^n associated with coupled Neumann boundary conditions of exponential type. Firstly, the upper bounds of blow-up rate estimates are derived. Secondly, the blow-up set is considered. The main conclusion is that the blow-up in this problem only occurs on the boundary.

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