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On Hollow – J–Lifting Modules

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Abstract
In this paper, we introduce and study the concepts of hollow – J–lifting modules and FI – hollow – J–lifting modules as a proper generalization of both hollow–lifting and J–lifting modules . We call an R–module M as hollow – J – lifting if for every submodule N of M with $\frac{M}{N}$ is hollow, there exists a submodule K of M such that $M = K \oplus \acute{K}$ and $K \subseteq_{Jce} N$ in M . Several characterizations and properties of hollow –J–lifting modules are obtained . Modules related to hollow – J–lifting modules are given .

Keywords: Hollow–Lifting Modules, J–Lifting Modules, Hollow–J–Lifting Modules, FI – Hollow –J–Lifting Modules.

حول مقاسات الرفع المجوفة من النمط –J

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الخلاصة
في هذا البحث , نقدم وندرس مفاهيم مقاسات الرفع المجوفة من النمط–J ومقاسات الرفع المجوفة من النمط–J – FI كتعميم مناسب لكل من مقاسات الرفع المجوفة ومقاسات الرفع من النمط–J . تدعى M مقاس رفع مجوف من النمط–J اذا كان لكل مقاس جزئي N من M بحيث ان $\frac{M}{N}$ مقاس مجوف , يوجد مقاس جزئي K من N بحيث ان $M = K \oplus \acute{K}$ و $K \subseteq_{Jce} N$ in M . يتم الحصول على العديد من خصائص مقاسات الرفع المجوفة من النمط–J . ويتم اعطاء المقاسات ذات الصلة بمقاسات الرفع المجوفة من النمط–J

1. Introduction

Orhan , Keskin and Tribak introduced the concept of hollow–lifting modules; An R–module is hollow – lifting if for every submodule N of M with $\frac{M}{N}$ is hollow , there exists a direct summand K of M, such that K is a coessential submodule of N in M [1]. Following Kabban and Khalid [2] , an R–module M is J–lifting module if for every submodule N of M , there exists a submodule K of N, such that $M = K \oplus \acute{K}$, $\acute{K} \subseteq M$ and $N \cap \acute{K} \ll_J \acute{K}$.

Throughout this paper, R will denote arbitrary rings with identity and all R–modules are unitary left R–modules . Let M be an R–module and N is a submodule of M . N is called J–small submodule of M (denoted by $N \ll_J M$), if whenever $M = N + K$, $K \subseteq M$, such that $J(\frac{M}{K}) = \frac{M}{K}$, implies $M = K$ [3] . Let K and N be submodules of M , such that $K \subseteq N \subseteq M$, then K is called J–coessential submodule of N in M (denoted by $K \subseteq_{Jce} N$ in M) if $\frac{N}{K} \ll_J \frac{M}{K}$ [2] . Recall that a submodule N of an R–module M

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is J–supplement of K in M if N + K = M and $N \cap K \ll_J N$ [3] . A submodule N of M is fully invariant if $g(N) \subseteq N$ for all $g \in$ End (M). An R–module M is called duo if every submodule of M is fully invariant [4]. In this paper, we introduce hollow–J–lifting . An R–module M is called hollow–J–lifting module if for every submodule N of M with $\frac{M}{N}$ is hollow, there exists a submodule K of M, such that $M = K \oplus \acute{K}$ and $K \subseteq_{Jce} N$ in M . Also, we introduce FI–hollow– J–lifting . Let M be an R–module . M is called FI–hollow–J–lifting module if for every fully invariant submodule N of M with $\frac{M}{N}$ is hollow, there exists a submodule K of M, such that $M = K \oplus \acute{K}$ and $K \subseteq_{Jce} N$ in M . Several characterizations and properties of hollow –J– lifting modules and FI – hollow –J–lifting modules are obtained .

2. Hollow – J–Lifting modules

In this section , we define hollow J–lifting modules and some of their basic properties . Also we prove some new results .

Definition(2.1) : Let M be an R–module. M is called hollow–Jacobson–lifting module (for short hollow–J–lifting), if for every submodule N of M with $\frac{M}{N}$ is hollow, there exists a submodule K of M, such that $M = K \oplus \acute{K}$ and $K \subseteq_{Jce} N$ in M .

Examples and Remarks (2.2) :

1) $\mathbb{Z}_4$ as $\mathbb{Z}$–module is hollow–J–lifting .

2) $\mathbb{Q}$ as $\mathbb{Z}$–module is not hollow–J–lifting, by following Proposition (2.3) .

3) Consider the module $M = \mathbb{Z}_2 \oplus \mathbb{Z}_4$. Clearly, $\mathbb{Z}_2$ and $\mathbb{Z}_4$ as $\mathbb{Z}$–module are hollow modules . Since $M = \mathbb{Z}_2 \oplus \mathbb{Z}_4$ is a J–lifting, then it is hollow–J–lifting, by following Proposition (2.3) .

4) Every hollow–lifting is hollow–J–lifting . The converse is not true in general . For example, $\mathbb{Z}$ as $\mathbb{Z}$–module .

Proposition (2.3) : Let H_1 and H_2 be J–hollow modules . Then the following are equivalent for the module $M = H_1 \oplus H_2$.

1) M is hollow–J–lifting module .

2) M is J–lifting module .

Proof : (1)⇒(2) Let N be a submodule of M . Consider the two natural projection maps $\pi_1: M \longrightarrow H_1$ and $\pi_2: M \longrightarrow H_2$. If $\pi_1(N) \neq H_1$ and $\pi_2(N) \neq H_2$, then by our assumption, $\pi_1(N) \ll_J H_1$ and $\pi_2(N) \ll_J H_2$. So according to a previous work [3 , Proposition (2.6.(6))] , we get , $\pi_1(N) \oplus \pi_2(N) \ll_J H_1 \oplus H_2$. Now, claim that $N \subseteq \pi_1(N) \oplus \pi_2(N)$. To recognize that , let $n \in N$, then $n \in M = H_1 \oplus H_2$ and hence $n = (h_1 , h_2)$, where $h_1 \in H_1$, $h_2 \in H_2$. Now , $\pi_1(n) = \pi_1((h_1 , h_2)) = h_1$ and $\pi_2(n) = \pi_2((h_1 , h_2)) = h_2$. This implies that $n = (\pi_1(n) , \pi_2(n))$, and we get $N \subseteq \pi_1(N) \oplus \pi_2(N)$ and hence $N \ll_J M$. Thus, M is J–lifting module. Now, assume that $\pi_1(N) = H_1$, then $\pi_1(N) = \pi_1(M)$. So, it is easy to see that $M = N + H_2$. Now, by the second isomorphism theorem, $\frac{N + H_2}{N} \cong \frac{H_2}{N \cap H_2}$. Since H_2 is J–hollow , then $\frac{H_2}{N \cap H_2}$ is J–hollow, and hence $\frac{M}{N}$ is J–hollow . But M is hollow–J–lifting , therefore there exists a J–coessential submodule of N in M which is a direct summand of M . Thus, M is J–lifting .

(2)⇒(1) It is clear .

Proposition (2.4) : Let M be an R–module . If M is a hollow–J–lifting module , then $\frac{M}{N}$ is hollow–J–lifting for every fully invariant submodule N of M .

Proof : Let $\frac{A}{N}$ be a submodule of $\frac{M}{N}$ such that $\frac{\frac{M}{N}}{\frac{A}{N}}$ is hollow . Then by the third isomorphism theorem , $\frac{\frac{M}{N}}{\frac{A}{N}} \cong \frac{M}{N}$ is hollow . Since M is hollow–J–lifting module , then there exists a submodule K of M such that $K \subseteq_{Jce} A$ in M and $M = K \oplus H$, for some $H \subseteq M$. Now, clearly, $K + N \subseteq A$ and hence $\frac{K+N}{N} \subseteq \frac{A}{N}$. Let $f : \frac{M}{K} \longrightarrow \frac{M}{K+N}$ be a mapping defined by $f(m + K) = m + (K + N)$, for all $m \in$ M . One can easily check that f is an epimorphism . Since $K \subseteq_{Jce} A$ in M , then by a previous study [3, Proposition (2.6.(5))] , $f(\frac{A}{K}) \ll_J \frac{M}{K+N}$ and hence $f(\frac{A}{K}) = \frac{A}{K+N} \ll_J \frac{M}{K+N}$. So $K + N \subseteq_{Jce} A$ in M . By the third isomorphism theorem , we get $\frac{K+N}{N} \subseteq_{Jce} \frac{A}{N}$ in $\frac{M}{N}$. Now , since N is fully invariant submodule of M , then by an earlier study [5 , lemma (5.4)] , $\frac{M}{N} = \frac{K+N}{N} \oplus \frac{H+N}{N}$. Hence, $\frac{K+N}{N}$ is a direct summand of $\frac{M}{N}$. Thus, $\frac{M}{N}$ is hollow–J–lifting .

Corollary (2.5) : Let M be a duo hollow–J–lifting module . Then every direct summand of M is a hollow–J–lifting .

Proof : It is clear by Proposition (2.4) .

Theorem (2.6) : An R–module M is hollow–J–lifting , if and only if for every submodule N of M with $\frac{M}{N}$ is hollow, there exists a submodule K of N, such that M = K ⊕ B ,where B ⊆ M and $N \cap B \ll_J B$.

Proof : ⟹ Let N be a submodule of M with $\frac{M}{N}$ is hollow . Since M is hollow–J–lifting, then there exists a direct summand K of M such that $K \subseteq_{Jce} N$ in M and M = K ⊕ B , where B ⊆ M N = N ∩ M = N ∩ (K ⊕ B) = K ⊕ (N ∩ B) , by the modular law . We want to show that $N \cap B \ll_J B$. Where X ⊆ B, let (N ∩ B) + X = B , with $J(\frac{B}{X}) = \frac{B}{X}$, to prove that B = X. Then M = N + X. Now , $\frac{M}{K} = \frac{N+X}{K} = \frac{N}{K} + \frac{X+K}{K}$, to prove that $J(\frac{M}{X+K}) = \frac{M}{X+K}$, since $\frac{M}{X+K} = \frac{K+B}{X+K} = \frac{(X+K)+B}{X+K} \cong \frac{B}{B \cap (X+K)} = \frac{B}{X+(K \cap B)} = \frac{B}{X}$, by the second isomorphism and modular law . Since $J(\frac{B}{X}) = \frac{B}{X}$, then $J(\frac{M}{X+K}) = \frac{M}{X+K}$ and $\frac{N}{K} \ll_J \frac{M}{K}$, therefore $\frac{M}{K} = \frac{X+K}{K}$, so M = X + K . Since M = K ⊕ B and X ⊆ B, then B = X . Thus, $N \cap B \ll_J B$.

⟸) Let N be a submodule of M with $\frac{M}{N}$ is hollow , then by our assumption , there exists a submodule K of N such that M = K ⊕ B (where B ⊆ M) and $N \cap B \ll_J B$. Let $\frac{N}{K} + \frac{X}{K} = \frac{M}{K}$, with $J(\frac{M}{X}) = \frac{M}{X}$ and X is submodule of M containing K, to prove that $\frac{X}{K} = \frac{M}{K}$. Thus M = N + X. By the modular law, we have N = N ∩ M = N ∩ (K ⊕ B) = K ⊕ (N ∩ B), and hence M = N + X = K + (N ∩ B) + X = (N ∩ B) + X . Now , since $N \cap B \ll_J B$, as shown in an earlier study [3 , Proposition (2.6.(4))] , $N \cap B \ll_J M$ and $J(\frac{M}{X}) = \frac{M}{X}$. So M = X and $\frac{M}{K} = \frac{X}{K}$. Then $\frac{N}{K} \ll_J \frac{M}{K}$, therefore $K \subseteq_{Jce} N$ in M . Thus, M is hollow–J–lifting .

Remark (2.7) : An R–module M is hollow–J–lifting , if and only if, for every submodule N of M with $\frac{M}{N}$ is hollow, there exists a submodule K of N, such that M = K ⊕ B, where B ⊆ M and $N \cap B \ll_J M$.

Proof : As clearly shown by a previous article [3 , Proposition (2.6.(4))] .

Theorem (2.8) : An R–module M is hollow–J–lifting , if and only if for every submodule N of M with $\frac{M}{N}$ is hollow, N has J–supplement K in M such that K ∩ N is a direct summand of N.

Proof:⟹) Suppose that M is hollow–J–lifting and let N be a submodule of M with $\frac{M}{N}$ is hollow. Then there is a submodule K of N such that $K \subseteq_{Jce} N$ in M and M = K ⊕ B , for some B ⊆ M. By the modular law , N = N ∩ M = N ∩ (K ⊕ B) = K ⊕ (N ∩ B) . Then (N ∩ B) is a direct summand of N and M = N + B . By the same argument of Theorem (2.6), we have $N \cap B \ll_J B$. Therefore, B is J–supplement of N in M .

⟸)Let N be a submodule of M with $\frac{M}{N}$ is hollow, then by our assumption there is M = N +K, $N \cap K \ll_J K$, and N = (N ∩ K) ⊕ L , where L ⊆ N . Now , M = N + K = (N ∩ K) + L + K = L + K . It is clear that L ∩ K = 0, so M = L ⊕ K . Let $\frac{N}{L} + \frac{X}{L} = \frac{M}{L}$, with $J(\frac{M}{X}) = \frac{M}{X}$, where X ⊆ M containing L . Then M = N + X . So M = (N ∩ K) ⊕ L + X = (N ∩ K) + X . Now , since $N \cap K \ll_J K$, and by a previous study [3 , Proposition (2.6.(4))], $N \cap K \ll_J M$, and $J(\frac{M}{X}) = \frac{M}{X}$. Then M = X and $\frac{X}{L} = \frac{M}{L}$, thus $\frac{N}{L} \ll_J \frac{M}{L}$, therefore $L \subseteq_{Jce} N$ in M . Then M is hollow–J–lifting .

Theorem (2.9) : Let M be an R–module . Then the following statements are equivalent .

1) M is hollow–J–lifting .

2) Every submodule N of M with $\frac{M}{N}$ is hollow can be written as N = K ⊕ L , with K is a direct summand of M and L $\ll_J$ M .

3) Every submodule N of M with $\frac{M}{N}$ is hollow can be written as N = K + L , with K is a direct summand of M and L $\ll_J$ M .

Proof : (1)⇒(2) Let N be a submodule of M, with $\frac{M}{N}$ is hollow. Since M is hollow–J–lifting, then there exists a submodule K of M, such that K $\subseteq_{Jce}$ N in M and M = K ⊕ B, where B ⊆ M. By the modular law, N = N ∩ M = N ∩ (K ⊕ B) = K ⊕ (N ∩ B) . By the same argument of Theorem (2.6) , we have N ∩ B $\ll_J$ B. Let L = N ∩ B , so N = K ⊕ L, where K is a direct summand of M and L $\ll_J$ M.

(2)⇒(3) It is clear.

(3)⇒(1)Let N be a submodule of M with $\frac{M}{N}$ is hollow. By (3), N can be written as N = K + L, with K is a direct summand of M and L $\ll_J$ M . We want to show that K $\subseteq_{Jce}$ N in M . Let K ⊆ X and $\frac{N}{K}+\frac{X}{K}=\frac{M}{K}$, with $J(\frac{M}{X})=\frac{M}{X}$, to prove that $\frac{X}{K}=\frac{M}{K}$. Then M = N + X = K + L + X = L + X. Since L $\ll_J$ M and $J(\frac{M}{X})=\frac{M}{X}$, then M = X and $\frac{X}{K}=\frac{M}{K}$. Thus, $\frac{N}{K}\ll_J\frac{M}{K}$, therefore, K $\subseteq_{Jce}$ N in M and M is hollow–J–lifting .

Proposition (2.10) : Let M be hollow–J–lifting . If M = K + N , where N is a direct summand of M and $\frac{M}{K\cap N}$ is hollow , then N contains a J–supplement of K in M .

Proof : Since M is hollow–J–lifting and $\frac{M}{K\cap N}$ is a hollow module , then by Theorem (2.9) , K ∩ N = B ⊕ L , where B is a direct summand of M and L $\ll_J$ M . But N is a direct summand of M and L ⊆ N , therefore by the same study [3 , Proposition (2.7)] L $\ll_J$ N . Let M = B ⊕ H, where H ⊆ M. By the modular law, N = N ∩ M = N ∩ (B ⊕ H) = B ⊕ (N ∩ H) . Let C = N ∩ H, so M = K + B + C = K + C. Also K ∩ N = K ∩ (B ⊕ C) = B ⊕ (K ∩ C). Let π_1: B ⊕ C ⟶ C be the natural projection map. So we have K ∩ C = π_1(B ⊕ (K ∩ C)) = π_1(K ∩ N) = π_1(B ⊕ L) = π_1(L). Since L $\ll_J$ N = B ⊕ C, then by the same study [3, Proposition (2.6.(5))], π_1(L) $\ll_J$ C, and hence K∩C $\ll_J$ C. Thus C is a J–supplement of K in M and C is contained in N.

Proposition (2.11): Let M = $M_1 \oplus M_2$ be a duo module . Then M is hollow–J–lifting if and only if M_1 and M_2 are hollow–J–lifting.

Proof : ⟹) It is clear by Corollary (2.5) .

⟸) Let N be a submodule of M with $\frac{M}{N}$ is hollow. By a previous study [5 , Lemma (5.4)] , $\frac{M}{N}=\frac{N+M_1}{N}\oplus\frac{N+M_2}{N}$. Since $\frac{M}{N}$ is hollow , we can assume that $\frac{N+M_1}{N}=\frac{M}{N}$, then M_2 ⊆ N . Since $\frac{N+M_1}{N}\cong\frac{M_1}{N\cap M_1}$, by the second isomorphism theorem, and M_1 is hollow–J–lifting, then there exists a direct summand K of M_1 such that $\frac{N\cap M_1}{K}\ll_J\frac{M_1}{K}$. Since N = N ∩ M = N ∩ ($M_1 \oplus M_2$) , then N = (N ∩ M_1) ⊕ (N ∩ M_2) , we get $\frac{N}{K\oplus M_2}\ll_J\frac{M}{K\oplus M_2}$. Moreover, it is easily seen that K ⊕ M_2 is a direct summand of M . Thus M is hollow–J–lifting.

Proposition (2.12) : Let M be an R–module . Then M is hollow–J–lifting module if and only if for every submodule N of M with $\frac{M}{N}$ is hollow, there exists an idempotent *f* ∈ End (M) with *f* (M) ⊆ N and (I – *f*)(N) $\ll_J$ (I – *f*)(M) .

Proof : ⟹) Let N be a submodule of M with $\frac{M}{N}$ is hollow . Since M is hollow–J–lifting, then by Theorem (2.8) , N has a J – supplement K in M such that N ∩ K is a direct summand of K. Then M = N + K, N ∩ K $\ll_J$ K and N = (N ∩ K) ⊕ X , X ⊆ N . Then M = N + K = (N ∩ K) + X + K = X + K and N ∩ K ∩ X = K ∩ X = {0} , and hence M = K ⊕ X . Now we define that the following map *f* : M ⟶ X is the natural projection map. One can easily show that *f* is idempotent and *f* (M) ⊆ X . Since X ⊆ N , then *f* (M) ⊆ N . Now , (I – *f*)(M) = {(I – *f*)(*m*) , *m* ∈ M } = {(I – *f*)(*a* + *b*) , where *a* ∈ X, *b* ∈ K} = {(I – *f*)(*a* + *b*) = *a* + *b* – *a* = *b*} = K. We want to show that (I – *f*)(N) = N ∩ (I – *f*)(M) . Let *x* ∈ (I – *f*)(N) , then there is *n* ∈ N , such that *x* = (I – *f*)(*n*) = *n* – *f* (*n*) . Thus *x* ∈ N and *x* ∈ (I – *f*)(M) .So *x* ∈ N ∩ (I – *f*)(M) . Hence, (I – *f*)(N) ⊆ N ∩ (I – *f*)(M). Let *d* ∈ N ∩ (I – *f*)(M), then $d \in$ N and $d \in$ (I – f)(M). There is $y \in$ M such that d = (I – f)(y) = y – f (y) . Thus d + f (y) = $y \in$ N, then $d \in$ (I – f)(N). So (I – f)(N) = N ∩ (I – f)(M) = N ∩ K $\ll_J$ K. Hence (I – f)(N) $\ll_J$ (I – f)(M) .

⟸) Let N be a submodule of M with $\frac{M}{N}$ is hollow. By our assumption, there exists an idempotent $f \in$ End (M) with f (M) ⊆ N and (I – f)(N) $\ll_J$ (I – f)(M) . Claim that M = f (M) ⊕ (I – f)(M) . To show that , let $m \in$ M , then m = m + f (m) – f (m) = f (m) + m – f (m) = f (m) + (I – f)(m) . Thus, M = f (m) + (I – f)(m) . Now , let $w \in f$ (M) ∩ (I – f)(M) , then w = f (m_1) and w = (I – f)(m_2) , for some m_1 , m_2 $\in$ M . So f (w) = f (m_1) = f ((I – f)(m_2)) = f (m_2) – f (m_2) = 0. Then f (f (m_1) = f (m_1) = 0 , hence w = 0 . Thus, M = f (M) ⊕ (I – f)(M) . Clearly, N ∩ (I – f)(M) = (I – f)(N) . Since (I – f)(N) $\ll_J$ (I – f)(M) , then N ∩ (I – f)(M) $\ll_J$ (I – f)(M) . Thus M is hollow–J–lifting .

3. FI–Hollow–J–Lifting modules

In this section , we introduce the concept of fully invariant hollow J–Lifting modules and we illustrate it by some examples . We also give some basic properties .

Recall that an R–module M is called **FI–hollow–lifting** if, for every fully invariant submodule N of M with $\frac{M}{N}$ is hollow, there exists a direct summand K of M, such that K $\subseteq_{ce}$ N in M [6] .

Definition(3.1) : Let M be an R–module . M is called FI–hollow–Jacobson–lifting module (for short FI–hollow–J–lifting), if for every fully invariant submodule N of M with $\frac{M}{N}$ is hollow, there exists a submodule K of M, such that M = K ⊕ $\acute{K}$ and K $\subseteq_{Jce}$ N in M .

Examples and Remarks (3.2) :

1) It is clear that ℤ as ℤ–module is FI–hollow–J–lifting.
2) ℚ as ℤ–module is not FI–hollow–J–lifting .
3) Every hollow–J–lifting is FI–hollow–J–lifting.
4) Every FI–hollow–lifting is FI–hollow–J–lifting . But the converse is not true in general . For an example of ℤ as ℤ–module, assume that ℤ as ℤ–module is FI–hollow–lifting . Since 2ℤ is fully invariant submodule of $\mathbb{Z}_z$, such that $\frac{\mathbb{Z}}{2\mathbb{Z}} \cong \mathbb{Z}_2$ is hollow , there is a direct summand K of $\mathbb{Z}_z$, such that K $\subseteq_{ce}$ 2ℤ in ℤ . But ℤ is indecomposable ℤ–module, so K = 0 . Hence 2ℤ ≪ ℤ, which is a contradiction, since 2ℤ + 3ℤ = ℤ, but 3ℤ ≠ ℤ .

Proposition (3.3) : An R–module M is FI–hollow–J–lifting , if and only if for every fully invariant submodule N of M with $\frac{M}{N}$ is hollow, there exists a submodule K of N, such that M = K ⊕ B , where B ⊆ M and N ∩ B $\ll_J$ B .

Proof : ⟹) Let N be a fully invariant submodule of M with $\frac{M}{N}$ is hollow . Since M is FI–hollow–J–lifting , then there is a submodule K of M such that K $\subseteq_{Jce}$ N in M and M = K ⊕ B, where B ⊆ M. Let φ: $\frac{M}{K}$ ⟶ B, be a mapping defined by φ(m + K) = b with m = k + b, where $k \in$ K and $b \in$ B . One can easily observe that φ is an isomorphism. Since $\frac{N}{K} \ll_J \frac{M}{K}$, then φ ($\frac{N}{K}$) $\ll_J$ B , [3 , Proposition (2.6.(5))]. Also φ ($\frac{N}{K}$) = { φ (x + y + K) | $x \in$ K and $y \in$ (N ∩ B) } = { y | $y \in$ (N ∩ B) } = N ∩ B , so N ∩ B $\ll_J$ B .

⟸) Let N be a fully invariant submodule of M with $\frac{M}{N}$ hollow , then by our assumption , there exists a submodule K of N, such that M = K ⊕ B , and N ∩ B $\ll_J$ B . Now , we want to show that K $\subseteq_{Jce}$ N in M . Let $\frac{N}{K}$ + $\frac{X}{K}$ = $\frac{M}{K}$, with J($\frac{M}{X}$) = $\frac{M}{X}$, to prove that $\frac{X}{K}$ = $\frac{M}{K}$, where X is a submodule of M containing K. By the modular law, N = N ∩ M = N ∩ (K ⊕ B) = K ⊕ (N∩B). Then M = N + X = K + (N ∩ B) + X = (N ∩ B) + K + X , since J($\frac{M}{X}$) = $\frac{M}{X}$, by [3, Corollary(2.3)] , J($\frac{M}{X+K}$) = $\frac{M}{X+K}$, and N ∩ B $\ll_J$ B .Also, by the above cited study [3, Proposition (2.6.(4))] , N ∩ B $\ll_J$ M . So M = K + X . But K ⊆ X , therefore M = X , hence $\frac{X}{K}$ = $\frac{M}{K}$, and $\frac{N}{K} \ll_J \frac{M}{K}$. Thus K $\subseteq_{Jce}$ N in M , so M is FI–hollow–J–lifting .

Corollary (3.4) : An R–module M is FI–hollow–J–lifting, if and only if for every fully invariant submodule N of M with $\frac{M}{N}$ is hollow , there exists a submodule K of N, such that M = K ⊕ B , where B ⊆ M and N ∩ B $\ll_J$ M .
Proof : It is clear [3 , Proposition (2.6(4))] .
Theorem (3.5) : An R–module M is FI–hollow–J–lifting, if and only if for every fully invariant submodule N of M with $\frac{M}{N}$ is hollow, there exists a J–supplement K in M such that K ∩ N is a direct summand of N .
Proof : By the same argument of the proof of the Theorem (2.8) .
Theorem (3.6) : Let M be an R–module . Then the following statements are equivalent .
1) M is FI–hollow–J–lifting .
2) Every fully invariant submodule N of M with $\frac{M}{N}$ is hollow can be written as N = K ⊕ L , with K is a direct summand of M and L $\ll_J$ M .
3) Every fully invariant submodule N of M with $\frac{M}{N}$ is hollow can be written as N = K + L , with K is a direct summand of M and L $\ll_J$ M .
Proof : By the same argument of the proof of the Theorem (2.9) .
Proposition (3.7) : Let M be an R–module . If M is FI–hollow–J–lifting module , then $\frac{M}{N}$ is FI–hollow–J–lifting module, for every fully invariant submodule N of M .
Proof : By the same argument of the proof of the Proposition (2.4) .
Corollary (3.8) : Let M be a duo FI–hollow–J–lifting module . Then every direct summand of M is FI–hollow–J–lifting .
Proof : It is clear by Proposition (3.7) .
Proposition (3.9) : Let M_1 and M_2 are FI–hollow–J–lifting modules if and only if M = $M_1 \oplus M_2$ is FI–hollow–J–lifting .
Proof : ⟹) Let N be a fully invariant submodule of M with $\frac{M}{N}$ is hollow , then M = M_1 ⊕ N or M = M_2 ⊕ N . Suppose that M = M_1 ⊕ N (the case M = M_2 ⊕ N being analogous), where N ⊆ M_1 , M = M_1 ⊕ N, then $\frac{M}{N} = \frac{M_1 \oplus N}{N} \cong \frac{M_1}{M_1 \cap N}$ is hollow . Then M_1 ∩ N is fully invariant submodule of M . Since M_1 is FI–hollow–J–lifting, we have $M_1 \cap N = L_1 \oplus S_1$, where L_1 is a direct summand of M_1 and $S_1 \ll_J M_1$. In a similar method , we have $M_2 \cap N = L_2 \oplus S_2$, where L_2 is a direct summand of M_2 and $S_2 \ll_J M_2$. Then N = L ⊕ S, where L = $L_1 \oplus L_2$ is a direct summand of M and S = $S_1 \oplus S_2 \ll_J$ M . Therefore , M = $M_1 \oplus M_2$ is FI–hollow–J–lifting (by Theorem (3.6)) .
⟸) It is clear by Corollary (3.8) .

Recall that an R–module *P* is called projective cover of M, if *P* is projective and there exists an epimorphism $f : P \longrightarrow M$ with Ker$f \ll P$ [7] .
Proposition (3.10) : Let M be a projective module and J(M) = M. Then the following statements are equivalent .
1) M is FI–hollow–J–lifting module .
2) For every fully invariant submodule N of M with $\frac{M}{N}$ is hollow, then $\frac{M}{N}$ has projective cover .
Proof : (1)⇒(2) Let N be a fully invariant submodule of M with $\frac{M}{N}$ is hollow. Since M is FI–hollow–J–lifting module , then by Proposition (3.3) , there exists a submodule K of N, such that M = K ⊕ Ḱ , for some Ḱ ⊆ M and N ∩ Ḱ $\ll_J$ Ḱ . Now , consider the following two short exact sequences .

$$0 \longrightarrow N \xrightarrow{i_1} N + \acute{K} \xrightarrow{T_1} \frac{N + \acute{K}}{N} \longrightarrow 0$$
$$0 \longrightarrow N \cap \acute{K} \xrightarrow{i_2} \acute{K} \xrightarrow{T_2} \frac{\acute{K}}{N \cap \acute{K}} \longrightarrow 0$$

where i_1 , i_2 are the inclusion maps and T_1 , T_2 are the natural epimorphisms . By the second isomorphism theorem , $\frac{M}{N} = \frac{N + \acute{K}}{N} \cong \frac{\acute{K}}{N \cap \acute{K}}$. Since M is a projective and Ḱ is a direct summand of M , then Ḱ is a projective . But Ker T_2 = N ∩ Ḱ $\ll_J$ Ḱ and J(M) = M. By the above cited study [3 , Proposition (2.5)] , $N \cap \acute{K} \ll \acute{K}$. Therefore, $\acute{K}$ is projective cover of $\frac{\acute{K}}{N \cap \acute{K}}$. Since $\frac{M}{N} \cong \frac{\acute{K}}{N \cap \acute{K}}$, thus $\frac{M}{N}$ has a projective cover.

(2)⇒(1) Let N be a fully invariant submodule of M with $\frac{M}{N}$ is hollow, and let $\varphi : M \longrightarrow \frac{M}{N}$ be the natural epimorphism . By (2) , $\frac{M}{N}$ has projective cover . Thus by an earlier article [8, Lemma 17.17] , there exists a decomposition $M = M_1 \oplus M_2$, such that ${}^{\varphi}/_{M_2}: M_2 \longrightarrow \frac{M}{N}$ is projective cover and $M_1 \subseteq$ Ker φ . This implies that $M_1 \subseteq N$ and Ker $({}^{\varphi}/_{M_2}) = N \cap M_2 \ll M_2$. But J(M) = M , and by the above cited study [3 , Proposition (2.5)] , $N \cap M_2 \ll_J M_2$. Thus M is FI–hollow–J–lifting, by Proposition (3.3).

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