On Hollow – J–Lifting Modules

Ali Kabban, Wasan Khalid

Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq

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Abstract

In this paper, we introduce and study the concepts of hollow – J–lifting modules and FI – hollow – J–lifting modules as a proper generalization of both hollow–lifting and J–lifting modules. We call an R–module M as hollow – J– lifting if for every submodule N of M with \( \frac{M}{N} \) is hollow, there exists a submodule K of M such that M = K \( \oplus \) K and K \( \subseteq_{\text{coessential}} \) N in M. Several characterizations and properties of hollow – J–lifting modules are obtained. Modules related to hollow – J–lifting modules are given.


1. Introduction

Orhan , Keskin and Tribak introduced the concept of hollow–lifting modules; An R–module is hollow – lifting if for every submodule N of M with \( \frac{M}{N} \) is hollow, there exists a direct summand K of M, such that K is a coessential submodule of N in M [1]. Following Kabban and Khalid [2], an R–module M is J–lifting module if for every submodule N of M, there exists a submodule K of N, such that M = K \( \oplus \) K, K \( \subseteq \) N and N \( \cap \) K \( \ll \) J K.

Throughout this paper, R will denote arbitrary rings with identity and all R–modules are unitary left R–modules. Let M be an R–module and N is a submodule of M. N is called J–small submodule of M (denoted by N \( \ll \) J M), if whenever M = N + K, K \( \subseteq \) M , such that J(\( \frac{M}{K} \)) = \( \frac{M}{K} \), implies M = K [3]. Let K and N be submodules of M, such that K \( \subseteq \) N \( \subseteq \) M , then K is called J–coessential submodule of N in M (denoted by K \( \subseteq_{\text{coessential}} \) N in M ) if \( \frac{N}{K} \) \( \ll \) J \( \frac{M}{K} \) [2]. Recall that a submodule N of an R–module M

*Email: alikuban5@gmail.com
is J–supplement of K in M if N + K = M and N ∩ K ⪯ J N [3]. A submodule N of M is fully invariant if g(N) ⊆ N for all g ∈ End(M). An R–module M is called duo if every submodule of M is fully invariant [4]. In this paper, we introduce hollow–J–lifting . An R–module M is called hollow–J–lifting module if for every fully invariant submodule N of M with \( \frac{M}{N} \) is hollow, there exists a submodule K of M, such that M = K ⊕ \( \hat{K} \) and \( K \subseteq_{jce} N \) in M . Also, we introduce FI–hollow– J–lifting . Let M be an R–module . M is called FI–hollow– J–lifting module if for every fully invariant submodule N of M with \( \frac{M}{N} \) is hollow, there exists a submodule K of M, such that M = K ⊕ \( \hat{K} \) and \( K \subseteq_{jce} N \) in M . Several characterizations and properties of hollow –J– lifting modules and FI – hollow –J–lifting modules are obtained .

2. Hollow – J–Lifting modules

In this section, we define hollow J–lifting modules and some of their basic properties. Also we prove some new results.

**Definition (2.1)**: Let M be an R–module. M is called hollow–Jacobson–lifting module (for short hollow–J–lifting), if for every submodule N of M with \( \frac{M}{N} \) is hollow, there exists a submodule K of M, such that M = K ⊕ \( \hat{K} \) and \( K \subseteq_{jce} N \) in M.

**Examples and Remarks (2.2)**:

1) \( \mathbb{Z}_4 \) as \( Z \)–module is hollow–J–lifting.
2) \( \mathbb{Q} \) as \( Z \)–module is not hollow–J–lifting, by following Proposition (2.3).
3) Consider the module M = \( \mathbb{Z}_2 \oplus \mathbb{Z}_4 \). Clearly, \( \mathbb{Z}_2 \) and \( \mathbb{Z}_4 \) as \( Z \)–module are hollow modules. Since M = \( \mathbb{Z}_2 \oplus \mathbb{Z}_4 \) is a J–lifting module, then it is hollow–J–lifting, by following Proposition (2.3).
4) Every hollow–lifting is hollow–J–lifting. The converse is not true in general. For example, \( \mathbb{Z} \) as \( Z \)–module.

**Proposition (2.3)**: Let \( H_1 \) and \( H_2 \) be J–hollow modules. Then the following are equivalent for the module M = \( H_1 \oplus H_2 \).

1) M is hollow–J–lifting module.
2) M is J–lifting module.

**Proof**: (1)⇒(2) Let N be a submodule of M. Consider the two natural projection maps \( \pi_1: M \rightarrow H_1 \) and \( \pi_2: M \rightarrow H_2 \). If \( \pi_1(N) \neq H_1 \) and \( \pi_2(N) \neq H_2 \), then by our assumption, \( \pi_1(N) \ll_{J} H_1 \) and \( \pi_2(N) \ll_{J} H_2 \). So according to a previous work [3, Proposition (2.6.(6))], we get \( \pi_1(N) \oplus \pi_2(N) \ll_{J} H_1 \oplus H_2 \). Now, claim that N \( \subseteq \pi_1(N) \oplus \pi_2(N) \). To recognize that, let n ∈ N, then n ∈ M = \( H_1 \oplus H_2 \) and hence \( n = (h_1, h_2) \), where \( h_1 \in H_1 \), \( h_2 \in H_2 \). Now, \( \pi_1(n) = \pi_1((h_1, h_2)) = h_1 \) and \( \pi_2(n) = \pi_2((h_1, h_2)) = h_2 \). This implies that n = \( \pi_1(n) \oplus \pi_2(n) \), and we get N \( \subseteq \pi_1(N) \oplus \pi_2(N) \) and hence N \( \ll_{J} M \). Thus, M is J–lifting module. Now, assume that \( \pi_1(N) = H_1 \), then \( \pi_1(N) = \pi_1(M) \). So, it is easy to see that M = N + H_2. Now, by the second isomorphism theorem, \( \frac{N + H_2}{N} \cong \frac{H_2}{N \cap H_2} \). Since H_2 is J–hollow, then \( \frac{H_2}{N \cap H_2} \) is J–hollow, and hence \( \frac{M}{N} \) is J–hollow. But M is hollow–J–lifting, therefore there exists a J–coessential submodule of N in M which is a direct summand of M. Thus, M is J–lifting.

(2)⇒(1) It is clear.

**Proposition (2.4)**: Let M be an R–module. If M is a hollow–J–lifting module, then \( \frac{M}{N} \) is hollow–J–lifting for every fully invariant submodule N of M.

**Proof**: Let \( \frac{A}{N} \) be a submodule of \( \frac{M}{N} \) such that \( \frac{M}{A\cap N} \) is hollow. Then by the third isomorphism theorem, \( \frac{M}{A\cap N} \equiv \frac{M}{A} \). Since M is hollow–J–lifting module, then there exists a submodule K of M such that K \( \subseteq_{jce} A \) in M and M = K \( \oplus H \), for some H \( \subseteq M \). Now, clearly, K + N \( \subseteq A \) and hence \( \frac{K + N}{N} \subseteq \frac{A}{N} \). Let \( f: \frac{M}{K} \rightarrow \frac{M}{K + N} \) be a mapping defined by \( f(m + K) = m + (K + N) \), for all m ∈ M. One can easily check that f is an epimorphism. Since K \( \subseteq_{jce} A \) in M, then by a previous study [3, Proposition (2.6.(5))], \( f(\frac{A}{K}) \ll_{J} \frac{M}{K + N} \) and hence \( f(\frac{A}{K}) = \frac{A}{K + N} \ll_{J} \frac{M}{K + N} \). So K + N \( \subseteq_{jce} A \) in M. By
the third isomorphism theorem, we get \( \frac{K + N}{N} \subseteq_{\text{jce}} \frac{A}{N} \). Now, since \( N \) is fully invariant submodule of \( M \), then by an earlier study [5, lemma (5.4)].

Thus, there exists a submodule \( K \) of \( N \) such that \( K = \frac{K + N}{N} \). Hence, \( \frac{K + N}{N} \) is a direct summand of \( \frac{M}{N} \).

Corollary (2.5): Let \( M \) be a duo hollow–l–lifting module. Then every direct summand of \( M \) is hollow–l–lifting.

Proof: It is clear by Proposition (2.4).

Theorem (2.6): An \( R \)–module \( M \) is hollow–l–lifting, if and only if for every submodule \( N \) of \( M \) with \( \frac{M}{N} \) is hollow, there exists a submodule \( K \) of \( N \) such that \( M = K \oplus B \), where \( B \subseteq M \) and \( N \cap B \ll_j B \).

Proof: \( \implies \) Let \( N \) be a submodule of \( M \) with \( \frac{M}{N} \) is hollow. Since \( M \) is hollow–l–lifting, then there exists a direct summand \( K \) of \( M \) such that \( K \subseteq_{\text{jce}} N \) in \( M \) and \( M = K \oplus B \), where \( B \subseteq M \) and \( N \cap M = N \cap (K \oplus B) = K \oplus (N \cap B) \), by the modular law. We want to show that \( N \cap B \ll_j B \). Where \( K \subseteq B \), let \( (N \cap B) + X = B \), with \( J(\frac{B}{X}) = \frac{B}{N} \).

\( \therefore \) \( \exists \) \( \frac{M}{K} = \frac{N + X}{B} + \frac{X + K}{B} \), to prove that \( \frac{B}{X} = X \). Now, \( \frac{M}{K} = \frac{N + X}{K} + \frac{X + K}{K} \), such that \( \frac{M}{X + K} = \frac{X + K}{X + K} \), since \( \frac{M}{X + K} = \frac{X + K}{X + K} \).

\( \therefore \) \( \frac{M}{K} = \frac{X + K}{K} \), therefore \( \frac{M}{K} = \frac{X + K}{K} \), so \( M = X + K \). Since \( M = K \oplus B \) and \( X \subseteq B \), then \( B = X \). Thus, \( N \cap B \ll_j B \).

\( \Leftarrow \) Let \( N \) be a submodule of \( M \) with \( \frac{M}{N} \) is hollow, then by our assumption, there exists a submodule \( K \) of \( N \) such that \( M = K \oplus B \) (where \( B \subseteq M \)) and \( N \cap B \ll_j B \). Let \( \frac{N}{K} + \frac{X}{K} = \frac{M}{K} \), with \( J(\frac{M}{X}) = \frac{M}{X} \) and \( X \) is submodule of \( M \) containing \( K \), to prove that \( \frac{X}{K} = \frac{M}{K} \). Thus \( M = N + X \). By the modular law, we have \( N = N \cap M = N \cap (K \oplus B) = K \oplus (N \cap B) \), and hence \( M = N + X = K + (N \cap B) + X = (N \cap B) + X \). Now, since \( N \cap B \ll_j B \), as shown in an earlier study [3, Proposition (2.6(4))] \( N \cap B \ll_j N \cap K \) and \( J(\frac{M}{X}) = \frac{M}{K} \).

\( \therefore \) \( \frac{M}{K} \subseteq_{\text{jce}} N \) in \( M \). Thus, \( M \) is hollow–l–lifting.

Remark (2.7): An \( R \)–module \( M \) is hollow–l–lifting, if and only if, for every submodule \( N \) of \( M \) with \( \frac{M}{N} \) is hollow, there exists a submodule \( K \) of \( N \) such that \( M = K \oplus B \), where \( B \subseteq M \) and \( N \cap B \ll_j M \).

Proof: As clearly shown by a previous article [3, Proposition (2.6(4))].

Theorem (2.8): An \( R \)–module \( M \) is hollow–l–lifting, if and only if for every submodule \( N \) of \( M \) with \( \frac{M}{N} \) is hollow, \( N \) has \( J \)–supplement \( K \) in \( M \) such that \( K \cap N \) is a direct summand of \( N \).

Proof: \( \Leftarrow \) Suppose that \( M \) is hollow–l–lifting and let \( N \) be a submodule of \( M \) with \( \frac{M}{N} \) is hollow. Then there is a submodule \( K \) of \( N \) such that \( K \subseteq_{\text{jce}} N \) in \( M \) and \( M = K \oplus B \), for some \( B \subseteq M \). By the modular law, \( N = N \cap M = N \cap (K \oplus B) = K \oplus (N \cap B) \). Then \( N \cap B \ll_j B \) is a direct summand of \( N \) and \( M = N + B \). By the same argument of Theorem (2.6), we have \( N \cap B \ll_j B \). Therefore, \( B \) is \( J \)–supplement of \( N \) in \( M \).

\( \Leftarrow \) Let \( N \) be a submodule of \( M \) with \( \frac{M}{N} \) is hollow, then by our assumption there is \( M = N + K, N \cap K \ll_j K, \) and \( N = (N \cap K) \oplus L \), where \( L \subseteq N \). Now, \( M = N + K = (N \cap K) + L + K = L + K \). It is clear that \( L \cap K = 0 \), so \( M = L \oplus K \). Let \( \frac{N}{L} + \frac{X}{L} = \frac{M}{L} \), with \( J(\frac{M}{L}) = \frac{M}{L} \), where \( X \subseteq M \) containing \( L \). Then \( M = N + X \). So \( M = (N \cap K) \oplus L + X = (N \cap K) + X \). Now, since \( N \cap K \ll_j K \), and by a previous study [3, Proposition (2.6(4))], \( N \cap K \ll_j K \) and \( J(\frac{M}{L}) = \frac{M}{L} \). Then \( M = X \) and \( \frac{X}{L} = \frac{M}{L} \), thus \( L \subseteq_{\text{jce}} \frac{M}{L} \), therefore \( L \subseteq_{\text{jce}} N \) in \( M \). Then \( M \) is hollow–l–lifting.

Theorem (2.9): Let \( M \) be an \( R \)–module. Then the following statements are equivalent.

1) \( M \) is hollow–l–lifting.
2) Every submodule \( N \) of \( M \) with \( \frac{M}{N} \) is hollow can be written as \( N = K \oplus L \), with \( K \) is a direct summand of \( M \) and \( L \) \( \ll \) \( M \).

3) Every submodule \( N \) of \( M \) with \( \frac{M}{N} \) is hollow can be written as \( N = K + L \), with \( K \) is a direct summand of \( M \) and \( L \) \( \ll \) \( M \).

**Proof**: (1)\( \Rightarrow \) (2) Let \( N \) be a submodule of \( M \), with \( \frac{M}{N} \) is hollow. Since \( M \) is hollow–J-lifting, then there exists a submodule \( K \) of \( M \), such that \( K \subseteq \text{Hom}_R(N, M) \) and \( M = K \oplus B \), where \( B \subseteq M \). By the modular law, \( N = N \cap M = N \cap (K \oplus B) = K \oplus (N \cap B) \). By the same argument of Theorem (2.6), we have \( N \cap B \ll B \). Let \( L = N \cap B \), so \( N = K \oplus L \), where \( K \) is a direct summand of \( M \) and \( L \) \( \ll \) \( M \).

(2)\( \Rightarrow \) (3) It is clear.

(3)\( \Rightarrow \) (1) Let \( N \) be a submodule of \( M \) with \( \frac{M}{N} \) is hollow. By (3), \( N \) can be written as \( N = K + L \), with \( K \) is a direct summand of \( M \) and \( L \) \( \ll \) \( M \). We want to show that \( K \subseteq \text{Hom}_R(N, M) \) in \( M \). Let \( K \subseteq X \) and \( \frac{N}{K} + \frac{X}{K} = \frac{M}{K} \), with \( J(\frac{M}{X}) = \frac{M}{K} \), to prove that \( \frac{X}{K} = \frac{M}{K} \). Then \( M = N + X = K + L + X = L + X \). Since \( L \ll M \) and \( J(\frac{M}{X}) = \frac{M}{X} \), then \( M = X \) and \( \frac{X}{K} = \frac{M}{K} \). Thus, \( \frac{N}{K} \ll \frac{M}{K} \), therefore, \( K \subseteq \text{Hom}_R(N, M) \) in \( M \) and \( M \) is hollow–J-lifting.

**Proposition (2.10)**: Let \( M \) be hollow–J-lifting. If \( M = K + N \), where \( N \) is a direct summand of \( M \) and \( \frac{M}{K} \) is hollow, then \( N \) contains a \( J \)-supplement of \( K \) in \( M \).

**Proof**: Since \( M \) is hollow–J-lifting and \( \frac{M}{K} \) is a hollow module, then by Theorem (2.9), \( K \cap N = B \oplus L \), where \( B \) is a direct summand of \( M \) and \( L \ll M \). But \( N \) is a direct summand of \( M \) and \( L \subseteq N \), therefore by the same study [3, Proposition (2.7)] \( L \ll N \). Let \( M = B \oplus H \), where \( H \subseteq M \). By the modular law, \( N = N \cap M = N \cap (B \oplus H) = B \oplus (N \cap H) \). Let \( C = N \cap H \), so \( M = K + B + C = K + C \).

Also \( K \cap N = K \cap (B \oplus C) = B \oplus (K \cap C) \). Let \( \pi_1 : B \oplus C \\rightarrow C \) be the natural projection map. So we have \( K \cap C = \pi_1(0 \oplus (K \cap C)) = \pi_1(K \cap N) = \pi_1(B \oplus L) = \pi_1(L) \). Then \( L \ll N \) and \( B \oplus C \), then by the same study [3, Proposition (2.6,5)], \( \pi_1(L) \ll C \), and hence \( K \cap N \ll C \). Thus \( C \) is a \( J \)-supplement of \( K \) in \( M \) and \( M \) is contained in \( N \).

**Proposition (2.11)**: Let \( M = M_1 \oplus M_2 \) be a duo module. Then \( M \) is hollow–J-lifting if and only if \( M_1 \) and \( M_2 \) are hollow–J-lifting.

**Proof**: (\( \Rightarrow \)) It is clear by Corollary (2.5).

(\( \Leftarrow \)) Let \( N \) be a submodule of \( M \) with \( \frac{M}{N} \) is hollow. By a previous study [5, Lemma (5.4)], \( \frac{M}{N} = \frac{N + M_1}{N} \oplus \frac{N + M_2}{N} \). Since \( \frac{M}{N} \) is hollow, we can assume that \( \frac{N + M_1}{N} = \frac{M_1}{N} \), then \( M_2 \subseteq N \). Since \( \frac{M}{N} \) is hollow, by the second isomorphism theorem, and \( M_2 \) is hollow–J-lifting, then there exists a direct summand \( K \) of \( M_1 \) such that \( \frac{N + M_2}{N} \ll \frac{M_1}{K} \). Since \( N = N \cap M = N \cap (M_1 \oplus M_2) \), then \( N = (N \cap M_1) \oplus (N \cap M_2) \), we get \( \frac{N}{K \oplus M_2} \ll \frac{M_1}{K} \). Moreover, it is easily seen that \( K \oplus M_2 \) is a direct summand of \( M \). Thus \( M \) is hollow–J-lifting.

**Proposition (2.12)**: Let \( M \) be an \( R \)-module. Then \( M \) is hollow–J-lifting module if and only if for every submodule \( N \) of \( M \) with \( \frac{M}{N} \) is hollow, there exists an idempotent \( f \in \text{End}(M) \) with \( f(M) \subseteq N \) and \( (I - f)(N) \ll (I - f)(M) \).

**Proof**: (\( \Rightarrow \)) Let \( N \) be a submodule of \( M \) with \( \frac{M}{N} \) is hollow. Since \( M \) is hollow–J-lifting, then by Theorem (2.8), \( N \) has a \( J \)– supplement \( K \) in \( M \) such that \( N \cap K \) is a direct summand of \( K \). Then \( M = N + K \), \( N \cap K \ll K \) and \( N = (N \cap K) \oplus X \), \( X \subseteq \subseteq N \). Then \( M = N + K = (N \cap K) \oplus X \subseteq \subseteq X + K \) and \( N \cap K \cap X = K \cap X = \{0\} \), and hence \( M = K \oplus X \). Now we define that the following map \( f : M \rightarrow X \) is the natural projection map. One can easily show that \( f \) is idempotent and \( f(M) \subseteq X \).

Since \( X \subseteq \subseteq N \), then \( f(M) \subseteq \subseteq N \). Now, \( (I - f)(M) = \{(I - f)(m), m \in M\} = \{(I - f)(a + b), a \in X, b \in K\} \). Let \( x \in (I - f)(N) \), then there is \( n \in N \), such that \( x = (I - f)(n) = n - f(n) \). Thus \( x \in N \) and \( x \in (I - f)(M) \). So \( x \in N \cap (I - f)(M) \). Hence, \( (I - f)(N) \subseteq N \cap (I - f)(M) \). Let \( d \in N \cap (I - f)(M) \),
then \( d \in N \) and \( d \in (I - f)(M) \). There is \( y \in M \) such that \( d = (I - f)(y) = y - f(y) \). Thus \( d + f(y) = y \in N \), then \( d \in (I - f)(N) \). So \((I - f)(N) = N \cap (I - f)(M) = N \cap K \ll j K \). Hence \((I - f)(N) \ll j (I - f)(M) \).

\[ \iff \] Let \( N \) be a submodule of \( M \) with \( \frac{M}{N} \) is hollow. By our assumption, there exists an idempotent \( f \in \text{End}(M) \) with \( f(M) \subseteq N \) and \((I - f)(N) \ll j (I - f)(M) \). Claim that \( M = f(M) \oplus (I - f)(M) \). To show that, let \( m \in M \), then \( m = m + f(m) - f(m) = f(m) + m - f(m) = f(m) + (I - f)(m) \). Thus, \( M = f(M) + (I - f)(m) \). Now, let \( w \in f(M) \cap (I - f)(M) \), then \( w = f(m_1) \) and \( w = (I - f)(m_2) \), for some \( m_1, m_2 \in M \). So \( f(w) = f(m_1) = f((I - f)(m_2)) = f(m_2) - f(m_2) = 0 \). Then \( f(m_1) = f(m_2) = 0 \), hence \( w = 0 \). Thus, \( M = f(M) \oplus (I - f)(M) \). Clearly, \( N \cap (I - f)(M) = (I - f)(N) \). Since \((I - f)(N) \ll j (I - f)(M) \), then \( N \cap (I - f)(M) \ll j (I - f)(M) \). Thus \( M \) is hollow–J–lifting.

3.  \( \text{FI–Hollow–J–Lifting modules} \)

In this section, we introduce the concept of fully invariant hollow J–lifting modules and we illustrate it by some examples. We also give some basic properties.

Recall that an \( R \)--module \( M \) is called \textbf{FI–hollow–lifting} if, for every fully invariant submodule \( N \) of \( M \) with \( \frac{M}{N} \) is hollow, there exists a direct summand \( K \) of \( M \), such that \( K \subseteq N \) in \( M \) [6].

\textbf{Definition (3.1)}: Let \( M \) be an \( R \)--module . \( M \) is called \( \text{FI–hollow–Jacobson–lifting module} \) (for short \( \text{FI–hollow–J–lifting} \) ), if for every fully invariant submodule \( N \) of \( M \) with \( \frac{M}{N} \) is hollow, there exists a submodule \( K \) of \( M \), such that \( M = K \oplus K \) and \( K \subseteq \text{Jce} N \) in \( M \).

\textbf{Examples and Remarks (3.2)}:
1) It is clear that \( Z \) as \( Z \)--module is \( \text{FI–hollow–J–lifting} \).
2) \( Q \) as \( Z \)--module is not \( \text{FI–hollow–J–lifting} \).
3) Every \( \text{FI–hollow–J–lifting} \) is \( \text{FI–hollow–J–lifting} \).
4) Every \( \text{FI–hollow–lifting} \) is \( \text{FI–hollow–J–lifting} \). But the converse is not true in general. For an example of \( Z \)--module, assume that \( Z \) as \( Z \)--module is \( \text{FI–hollow–lifting} \). Since \( 2Z \) is fully invariant submodule of \( Z \), such that \( \frac{Z}{2Z} \) is hollow, there is a direct summand \( K \) of \( Z \), such that \( K \subseteq 2Z \) in \( Z \). But \( Z \) is indecomposable \( Z \)--module, so \( K = 0 \). Hence \( 2Z \ll Z \), which is a contradiction, since \( 2Z + 3Z = Z \), but \( 3Z \neq Z \).

\textbf{Proposition (3.3)}: An \( R \)--module \( M \) is \( \text{FI–hollow–J–lifting} \), if and only if every fully invariant submodule \( N \) of \( M \) with \( \frac{M}{N} \) is hollow, there exists a submodule \( K \) of \( N \), such that \( M = K \oplus B \), where \( B \subseteq M \) and \( N \cap B \ll j B \).

\textbf{Proof}: \( \Rightarrow \) Let \( N \) be a fully invariant submodule of \( M \) with \( \frac{M}{N} \) is hollow. Since \( M \) is \( \text{FI–hollow–J–lifting} \), then there is a submodule \( K \) of \( M \) such that \( K \subseteq \text{Jce} N \) in \( M \) and \( M = K \oplus B \), where \( B \subseteq M \). Let \( \varphi : \frac{M}{K} \rightarrow B \), be a mapping defined by \( \varphi(m + K) = b \) with \( m = k + b \), where \( k \in K \) and \( b \in B \). One can easily observe that \( \varphi \) is an isomorphism. Since \( \frac{N}{K} \ll j \frac{M}{K} \), then \( \varphi \left( \frac{N}{K} \right) \ll j \frac{B}{K} \) [3, Proposition (2.6.5)].

Also \( \varphi \left( \frac{N}{K} \right) = \{ \varphi(x + y + K) \mid x \in K \text{ and } y \in (N \cap B) \} = \{ y \mid y \in (N \cap B) \} = N \cap B \), so \( N \cap B \ll j B \).

\( \Leftarrow \) Let \( N \) be a fully invariant submodule of \( M \) with \( \frac{M}{N} \) hollow, then by our assumption, there exists a submodule \( K \) of \( N \), such that \( M = K \oplus B \), and \( N \cap B \ll j B \). Now, we want to show that \( K \subseteq \text{Jce} N \) in \( M \). Let \( \frac{N}{K} = \frac{X}{K} = \frac{M}{K} \), with \( J\left( \frac{M}{X} \right) = \frac{M}{X} \), to prove that \( \frac{X}{K} = \frac{M}{K} \), where \( X \) is a submodule of \( M \) containing \( K \). By the modular law, \( N = N \cap M = N \cap (K \oplus B) = K \oplus (N \cap B) \). Then \( M = N + X = K + (N \cap B) + X = (N \cap B) + K + X \), since \( J\left( \frac{M}{X} \right) = \frac{M}{X} \), by [3, Corollary(2.3)] \( J\left( \frac{M}{X+K} \right) = \frac{M}{X+K} \), and \( N \cap B \ll j B \). Also, by the above cited study [3, Proposition (2.6.4)] \( N \cap B \ll j M \). So \( M = K + X \). But \( K \subseteq X \), therefore \( M = X \), hence \( \frac{X}{K} = \frac{M}{K} \) and \( N \cap B \ll j \frac{M}{K} \). Thus \( K \subseteq \text{Jce} N \) in \( M \). So \( M \) is \( \text{FI–hollow–J–lifting} \).
Corollary (3.4) : An R–module M is FI–hollow–J–lifting, if and only if for every fully invariant submodule N of M with $\frac{M}{N}$ is hollow, there exists a submodule K of N, such that $M = K \oplus B$, where $B \subseteq M$ and $N \cap B \ll J M$.

Proof : It is clear [3, Proposition (2.6(4))].

Theorem (3.5) : An R–module M is FI–hollow–J–lifting, if and only if for every fully invariant submodule N of M with $\frac{M}{N}$ is hollow, there exists a J–supplement K in M such that $K \cap N$ is a direct summand of N.

Proof : By the same argument of the proof of the Theorem (2.8).

Theorem (3.6) : Let M be an R–module. Then the following statements are equivalent.
1) M is FI–hollow–J–lifting.
2) Every fully invariant submodule N of M with $\frac{M}{N}$ is hollow can be written as N = K \oplus L, with K is a direct summand of M and L \ll J M.
3) Every fully invariant submodule N of M with $\frac{M}{N}$ is hollow can be written as N = K + L, with K is a direct summand of M and L \ll J M.

Proof : By the same argument of the proof of the Theorem (2.9).

Proposition (3.7) : Let M be an R–module. If M is FI–hollow–J–lifting module, then $\frac{M}{N}$ is FI–hollow–J–lifting module, for every fully invariant submodule N of M.

Proof : By the same argument of the proof of the Proposition (2.4).

Corollary (3.8) : Let M be a duo FI–hollow–J–lifting module. Then every direct summand of M is FI–hollow–J–lifting.

Proof : It is clear by Proposition (3.7).

Proposition (3.9) : Let M_1 and M_2 are FI–hollow–J–lifting modules if and only if M = M_1 \oplus M_2 is FI–hollow–J–lifting.

Proof : \(\Rightarrow\) Let N be a fully invariant submodule of M with $\frac{M_{1}}{N}$ is hollow, then $M = M_{1} \oplus N$ or $M = M_{2} \oplus N$. Suppose that $M = M_{1} \oplus N$ (the case $M = M_{2} \oplus N$ being analogous), where $N \subseteq M_{1}$, $M = M_{1} \oplus N$, then $\frac{M}{N} = \frac{M_{1} \oplus N}{M_{1} \cap N}$ is hollow. Then $M_{1} \cap N$ is a fully invariant submodule of M.

Since $M_{1}$ is FI–hollow–J–lifting, we have $M_{1} \cap N = L_{1} \oplus S_{1}$, where $L_{1}$ is a direct summand of $M_{1}$ and $S_{1} \ll J M_{1}$. In a similar method, we have $M_{2} \cap N = L_{2} \oplus S_{2}$, where $L_{2}$ is a direct summand of $M_{2}$ and $S_{2} \ll J M_{2}$.

Then $N = L \oplus S$, where $L = L_{2} \oplus T_{2}$ is a direct summand of M and $S = S_{1} \oplus S_{2} \ll J M$.

Therefore, $M = M_{1} \oplus M_{2}$ is FI–hollow–J–lifting (by Theorem (3.6)).

\(\Leftarrow\) It is clear by Corollary (3.8).

Recall that an R–module P is called projective cover of M, if P is projective and there exists an epimorphism $f : P \rightarrow M$ with Ker $f \ll P$ [7].

Proposition (3.10) : Let M be a projective module and J(M) = M. Then the following statements are equivalent.
1) M is FI–hollow–J–lifting module.
2) For every fully invariant submodule N of M with $\frac{M}{N}$ is hollow, then $\frac{M}{N}$ has projective cover.

Proof : (1)\(\Rightarrow\)(2) Let N be a fully invariant submodule of M with $\frac{M}{N}$ is hollow. Since M is FI–hollow–J–lifting module, then by Proposition (3.3), there exists a submodule K of N, such that $M = K \oplus \tilde{K}$, for some $\tilde{K} \subseteq M$ and $N \cap K \ll J \tilde{K}$. Now, consider the following two short exact sequences.

\[0 \rightarrow N \overset{\iota_{1}}{\rightarrow} N + \tilde{K} \overset{T_{1}}{\rightarrow} \frac{N + \tilde{K}}{N} \rightarrow 0\]

\[0 \rightarrow N \cap \tilde{K} \overset{\iota_{2}}{\rightarrow} \tilde{K} \overset{T_{2}}{\rightarrow} \frac{N \cap \tilde{K}}{N \cap \tilde{K}} \rightarrow 0\]

where $\iota_{1}$, $\iota_{2}$ are the inclusion maps and $T_{1}$, $T_{2}$ are the natural epimorphisms. By the second isomorphism theorem, $\frac{M}{N} = \frac{N + \tilde{K}}{N} \cong \frac{K}{N \cap \tilde{K}}$. Since M is a projective and $\tilde{K}$ is a direct summand of M, then $\tilde{K}$ is a projective. But Ker $T_{2} = N \cap \tilde{K} \ll J \tilde{K}$ and J(M) = M. By the above cited study [3,
Proposition (2.5)] , \( N \cap \hat{K} \ll \hat{K} \). Therefore, \( \hat{K} \) is projective cover of \( \frac{K}{N \cap \hat{K}} \). Since \( \frac{M}{N} \cong \frac{K}{N \cap \hat{K}} \), thus \( \frac{M}{N} \) has a projective cover.

(2)\( \Rightarrow \) (1) Let \( N \) be a fully invariant submodule of \( M \) with \( \frac{M}{N} \) is hollow, and let \( \varphi : M \rightarrow \frac{M}{N} \) be the natural epimorphism. By (2), \( \frac{M}{N} \) has projective cover. Thus by an earlier article [8, Lemma 17.17], there exists a decomposition \( M = M_1 \oplus M_2 \), such that \( \frac{\varphi}{M_2} : M_2 \rightarrow \frac{M}{N} \) is projective cover and \( M_1 \subseteq \text{Ker} \varphi \). This implies that \( M_1 \subseteq N \) and \( \text{Ker} \left( \frac{\varphi}{M_2} \right) = N \cap M_2 \ll M_2 \). But \( J(M) = M \), and by the above cited study [3, Proposition (2.5)] , \( N \cap M_2 \ll M_2 \). Thus \( M \) is Fl–hollow–J–lifting, by Proposition (3.3).

References