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The Resolution of Weyl Module for Two Rows in Special Case of the Skew-Shape

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Abstract
The aim of this work is to survey the two rows resolution of Weyl module and locate the terms and the exactness of the Weyl Resolution in the case of skew-shape (8,6)/(2,1).

Keywords: Resolution, resolution of Weyl module, place polarization, skew-shape, box map.

تحلل مقاس وايل في حالة خاصة لصفين من شبه الشكل المنحرف

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الخلاصة
الهدف من العمل هذا, هو مراجعة تطبيق لتحلل مقاس وايل لصفين في حالة شبه الشكل المنحرف (8,6)/(2,1) ولإيجاد حدود ذلك التحلل وبرهان انه تام.

Introduction

Let F be a free module over a commutative ring R with identity and $D_r F$ be divided power algebra of degree r that underlies the free module F.

A partition of length $n = l(\lambda)$ is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of non -negative integers in non-increasing order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$. A relative sequence is a pair (λ, μ) of sequences such that $\mu \leq \lambda$ means that $\mu_i \leq \lambda_i$ for all $i \geq 1$. We shall use the notation $\frac{\lambda}{\mu}$ to represent relative sequences. If both λ and μ are partitions, then the relative sequence $\frac{\lambda}{\mu}$ will be called a skew partition**.**

Author in an earlier work [1] started the use of letter place algebra and differential bar complex. While the authors in another study [2] described the Weyl module $K_{\lambda/\mu}F$ associated to the skew-shape, which has the image of

For $K_{\lambda/\mu}F = Im(d'_{\lambda/\mu})$ where $d'_{\lambda/\mu}: DF \to \wedge F$ (Weyl map), so we have

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$\sum D_{p+k}\otimes D_{q-k} \longrightarrow D_p\otimes D_q \xrightarrow{d'_{\lambda/\mu}} K_{\lambda/\mu} \longrightarrow 0$

And, by using letter place, the maps will be

$$\begin{pmatrix} w \\ w' \end{pmatrix}\!\left|\begin{matrix} 1^{(p+k)} \\ 2^{(q-k)} \end{matrix}\right. \xrightarrow{\partial_{21}^{(k)}} \begin{pmatrix} w \\ w' \end{pmatrix}\!\left|\begin{matrix} 1^{(p)}\ 2^{(k)} \\ 2^{(q-k)} \end{matrix}\right. \to \sum_w \begin{pmatrix} w_{(1)} \\ w' w_{(2)} \end{pmatrix}\!\left|\begin{matrix} (t+1)'(t+2)' \ldots (p+t)' \\ 1'2'3' \ldots q' \end{matrix}\right.;$$

where $w\otimes w' \in D_{p+k}\otimes D_{q-k}$, $\square = \sum_{k=t+1}^{q} \partial_{21}^{(k)}$, and $d'_{\lambda/\mu} = \partial_{q'2} \ldots \partial_{1'2}\partial_{(p+t)'1} \ldots \partial_{(t+1)'1}$, is the composition of place polarizations from positive places {1,2} to negative place {1', 2',…, $(p+t)$'}.

Specifically, $\square$ sends an element $x\otimes y$ of $D_{p+k}\otimes D_{q-k}$ to $\sum x_p\otimes x'_k y$; where $\sum x_p\otimes x'_k$ is the component of the diagonal of x in $D_p\otimes D_k$, [3].

The author in another article [4] introduced these notions as follows:

Let Z_{21} be the free generator of divided power algebra $D(Z_{21})$ in one generator. The divided power element $Z_{21}^{(k)}$ of degree k of the free generator Z_{21} acts on $D_{p+k}\otimes D_{q-k}$ by place polarization of degree k from place 1 to place 2.

The graded algebra with identity $A = D(Z_{21})$ acts on the graded module $M = D_{p+k}\otimes D_{q-k} = \sum M_{q-k}$. Hence, M is a graded left A-module, where for $w = Z_{21}^{(k)} \in A$ and $v \in D_{\beta_1}\otimes D_{\beta_2}$, so we have:

$$w(v) = Z_{21}^{(k)}(v) = \partial_{21}^{(k)}(v)$$

If we take (t^{+}) graded strand of degree q

$\mathcal{M}_\bullet: 0 \longrightarrow \mathcal{M}_{q-t} \xrightarrow{\partial_S} \ldots \longrightarrow \mathcal{M}_e \xrightarrow{\partial_S} \mathcal{M}_1 \xrightarrow{\partial_S} \mathcal{M}_0,$

of the normalized Bar complex Bar$(M, A; S,\bullet)$, where $S = \{x\}$.

We illustrate some important standard concepts which are needed in our work.

The maps $\{S_i\}$ are defined as follows [2]:

$S_0: D_p\otimes D_q \rightarrow \sum_{k>0} Z^{(t+k)} x D_{p+t+k} \otimes D_{q-t-k}$

$$\begin{pmatrix} w \\ w' \end{pmatrix}\!\left|\begin{matrix} 1^{(p)} & 2^{(k)} \\ 2^{(q-k)} & \end{matrix}\right. \longrightarrow \begin{cases} 0 & ;\text{ if } k \le t \\ Z_{21}^{(k)} x \begin{pmatrix} w \\ w' \end{pmatrix}\!\left|\begin{matrix} 1^{(p+k)} \\ 2^{(q-k)} \end{matrix}\right. & ;\text{ if } k > t \end{cases}$$

And for the higher dimensions, they are defined as

$S_{l-1}: \sum_{k_i>0} Z_{21}^{(t+k_1)} x Z_{21}^{(k_2)} x \ldots Z_{21}^{(k_{l-1})} x\, D_{p+t+|k|} \otimes D_{q-t-|k|}$

$\rightarrow Z_{21}^{(t+k_1)} x Z_{21}^{(k_2)} x \ldots Z_{21}^{(k_{l-1})} x\, Z_{21}^{(k_l)} x\, D_{p+t+|k|} \otimes D_{q-t-|k|}$

$$Z_{21}^{(t+k_1)} x Z_{21}^{(k_2)} x \ldots Z_{21}^{(k_{\ell-1})} x \begin{pmatrix} w \\ w' \end{pmatrix}\!\left|\begin{matrix} 1^{(p+t+|k|)} & 2^{(m)} \\ 2^{(q-t-|k|-m)} & \end{matrix}\right.$$

$$\longrightarrow \begin{cases} 0 & ;\text{if } m = 0 \\ Z_{21}^{(t+k_1)} x Z_{21}^{(k_2)} x \ldots Z_{21}^{(k_{\ell-1})} x\, Z_{21}^{(m)} x \begin{pmatrix} w \\ w' \end{pmatrix}\!\left|\begin{matrix} 1^{(p+t+|k|+m)} \\ 2^{(q-t-|k|-m)} \end{matrix}\right. & ;\text{if } m > 0 \end{cases}$$

While the modules of the resolution were written as [2]: M_i for i = 0,1,…, $q-t$, with $M_0 = D_p\otimes D_q$, and $M_i = Z_{21}^{(t+k_1)} x Z_{21}^{(k_2)} x \ldots Z_{21}^{(k_i)} x\, D_{p+t+|k|} \otimes D_{q-t-|k|}$, for $i \ge 1$.

Hassan [5] studied the resolution of Weyl module in the case of two-rowed skew-shape $(p+t, q)/(t, 0)$. While another study [6] exhibited the terms and the exactness of the Weyl resolution in the case of partition (8,7). In this work, we locate the terms and the exactness of the Weyl Resolution in the case of skew-shape (8,6)/(2,1).

2. Results of the case (8,6)/(2,0)

In this section, we find the term and the exactness for the resolution of Weyl module in the case of the skew-shape (8,6)/(2,0).

The terms of the Resolution Weyl module are:

$M_0 = D_6\otimes D_6$

$M_1 = Z_{21}^{(3)} x\, D_9\otimes D_3 \oplus Z_{21}^{(4)} x\, D_{10}\otimes D_2 \oplus Z_{21}^{(5)} x\, D_{11}\otimes D_1 \oplus Z_{21}^{(6)} x\, D_{12}\otimes D_0$

$M_2 = Z_{21}^{(3)} x Z_{21}^{(1)} x\, D_{10}\otimes D_2 \oplus Z_{21}^{(4)} x Z_{21}^{(1)} x\, D_{11}\otimes D_1 \oplus Z_{21}^{(3)} x Z_{21}^{(2)} x\, D_{11}\otimes D_1 \oplus Z_{21}^{(5)} x Z_{21}^{(1)} x\, D_{12}\otimes D_0$
$\oplus Z_{21}^{(4)} x Z_{21}^{(2)} x\, D_{12}\otimes D_0 \oplus Z_{21}^{(3)} x Z_{21}^{(3)} x\, D_{12}\otimes D_0$

$M_3 = Z_{21}^{(3)} x Z_{21}^{(1)} x Z_{21}^{(1)} x\, D_{11}\otimes D_1 \oplus Z_{21}^{(4)} x Z_{21}^{(1)} x Z_{21}^{(1)} x\, D_{12}\otimes D_0 \oplus$

$Z_{21}^{(3)} x Z_{21}^{(2)} x Z_{21}^{(1)} x\, D_{12} \otimes D_0 \oplus Z_{21}^{(3)} x Z_{21}^{(1)} x Z_{21}^{(2)} x\, D_{12} \otimes D_0 \oplus$
$M_4 = Z_{21}^{(3)} x Z_{21}^{(1)} x Z_{21}^{(1)} x Z_{21}^{(1)} x\, D_{12} \otimes D_0.$

Thus we have the complex

$$\begin{tikzcd}
\mathcal{M}_4 \arrow[r, "\partial_x"] \arrow[d, "\mathrm{id}"] & \mathcal{M}_3 \arrow[r, "\partial_x"] \arrow[d, "\mathrm{id}"] \arrow[dl, "\mathcal{S}_3"'] & \mathcal{M}_2 \arrow[r, "\partial_x"] \arrow[d, "\mathrm{id}"] \arrow[dl, "\mathcal{S}_2"'] & \mathcal{M}_1 \arrow[r, "\partial_x"] \arrow[d, "\mathrm{id}"] \arrow[dl, "\mathcal{S}_1"'] & \mathcal{M}_0 \arrow[d, "\mathrm{id}"] \arrow[dl, "\mathcal{S}_0"'] \\
\mathcal{M}_4 \arrow[r, "\partial_x"] & \mathcal{M}_3 \arrow[r, "\partial_x"] & \mathcal{M}_2 \arrow[r, "\partial_x"] & \mathcal{M}_1 \arrow[r, "\partial_x"] & \mathcal{M}_0
\end{tikzcd}$$

The constructions of a contracting homotopies { S_i }, i = 1, 2, 3 are:

$S_0 : D_6 \otimes D_6 \;\; \rightarrow \sum_{k>0} Z_{21}^{(k+2)} x\, D_{6+k} \otimes D_{6-k}$ such that

$$\mathcal{S}_0\left(\left(\begin{array}{c|cc} w & 1^{(6+k)} & 2^{(k)} \\ w' & 2^{(6-k)} & \end{array}\right)\right) = \begin{cases} 0 & ;\text{ if } \; k \le 2 \\ Z_{21}^{(k+2)} x \left(\begin{array}{c|c} w & 1^{(6+k)} \\ w' & 2^{(6-k)} \end{array}\right) & ;\text{ if } \; k = 3,4,5 \end{cases}$$

$S_1 : \sum_{k>0} Z_{21}^{(k+2)} x\, D_{8+k} \otimes D_{4-k} \;\; \rightarrow Z_{21}^{(k_1+2)} x Z_{21}^{(k_2)} x\, D_{8+k} \otimes D_{4-k}$ such that

$$\mathcal{S}_1\left(Z_{21}^{(k+2)} x \left(\begin{array}{c|cc} w & 1^{(8+k)} & 2^{(m)} \\ w' & 2^{(4-k-m)} & \end{array}\right)\right) = \begin{cases} 0 & ;\; if \;\; m = 0 \\ Z_{21}^{(k+2)} x Z_{21}^{(m)} x \left(\begin{array}{c|c} w & 1^{(8+k+m)} \\ w' & 2^{(4-k-m)} \end{array}\right) & ;\; if \;\; m = 1,2,3 \end{cases}$$

$S_2 : \sum_{k_i>0} Z_{21}^{(k_1+2)} x Z_{21}^{(k_2)} x\, D_{8+|k|} \otimes D_{4-|k|} \;\; \rightarrow \;\; Z_{21}^{(k_1+2)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x\, D_{8+|k|} \otimes D_{4-|k|}$ such that

$$\mathcal{S}_2\left(Z_{21}^{(k_1+2)} x Z_{21}^{(k_2)} x \left(\begin{array}{c|cc} w & 1^{(8+|k|)} & 2^{(m)} \\ w' & 2^{(4-|k|-m)} & \end{array}\right)\right)$$

$$= \begin{cases} 0 & ;\text{ if } \; m = 0 \\ Z_{21}^{(k_1+2)} x Z_{21}^{(k_2)} x Z_{21}^{(m)} x \left(\begin{array}{c|c} w & 1^{(8+|k|+m)} \\ w' & 2^{(4-|k|-m)} \end{array}\right) & ;\text{ if } \; m = 1,2, \end{cases} ;\text{ where } |k| = k_1 + k_2.$$

.

$S_3 : \sum_{k_i>0} Z_{21}^{(k_1+2)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x\, D_{8+|k|} \otimes D_{4-|k|} \;\; \rightarrow \;\; Z_{21}^{(k_1+2)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4)} x\, D_{8+|k|} \otimes D_{4-|k|}$

$$\mathcal{S}_3\left(Z_{21}^{(k_1+2)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x \left(\begin{array}{c|cc} w & 1^{(8+|k|)} & 2^{(m)} \\ w' & 2^{(4-|k|-m)} & \end{array}\right)\right)$$

$$= \begin{cases} 0 & ;\text{ if } \; m = 0 \\ Z_{21}^{(k_1+2)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(m)} x \left(\begin{array}{c|c} w & 1^{(8+|k|+m)} \\ w' & 2^{(4-|k|-m)} \end{array}\right) & ;\text{ if } \; m = 1 \end{cases} ;\text{ where } |k| = k_1 + k_2 + k_3.$$

Now, we have

$$\mathcal{S}_0 \partial_x \left(Z_{21}^{(k+2)} x \left(\begin{array}{c|cc} w & 1^{(8+k)} & 2^{(m)} \\ w' & 2^{(4-k-m)} & \end{array}\right)\right) = \mathcal{S}_0 \partial_{12}^{(k+2)} \left(\begin{array}{c|cc} w & 1^{(8+k)} & 2^{(m)} \\ w' & 2^{(4-k-m)} & \end{array}\right)$$
$$= \binom{k+2+m}{m} Z_{21}^{(k+2+m)} x \left(\begin{array}{c|c} w & 1^{(8+k+m)} \\ w' & 2^{(4-k-m)} \end{array}\right),$$

and

$$\partial_x \mathcal{S}_1 \left(Z_{21}^{(k+2)} x \left(\begin{array}{c|cc} w & 1^{(8+k)} & 2^{(m)} \\ w' & 2^{(4-k-m)} & \end{array}\right)\right) = \partial_x \left(Z_{21}^{(k+2)} x Z_{21}^{(m)} x \left(\begin{array}{c|c} w & 1^{(8+k+m)} \\ w' & 2^{(4-k-m)} \end{array}\right)\right)$$
$$= -\binom{k+2+m}{m} Z_{21}^{(k+2+m)} x \left(\begin{array}{c|c} w & 1^{(8+k+m)} \\ w' & 2^{(4-k-m)} \end{array}\right) + Z_{21}^{(k+2)} x \left(\begin{array}{c|cc} w & 1^{(8+k)} & 2^{(m)} \\ w' & 2^{(4-k-m)} & \end{array}\right)$$

It is clear that $\mathcal{S}_0\partial_x + \partial_x\mathcal{S}_1 = \mathrm{id}_{\mathcal{M}_1}$.

$$\mathcal{S}_1\partial_x\left(\mathcal{Z}_{21}^{(k_1+2)} x \mathcal{Z}_{21}^{(k_2)} x \begin{pmatrix} w \\ w' \end{pmatrix}\Bigg| \begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(4-|k|-m)} & \end{matrix}\right)\right)$$
$$= \mathcal{S}_1\left[-\binom{|k|+2}{k_2}\mathcal{Z}_{21}^{(|k|+2)} x \left(\begin{matrix} w \\ w' \end{matrix}\Bigg| \begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(4-|k|-m)} & \end{matrix}\right) + \mathcal{Z}_{21}^{(k_1+2)} x \partial_{21}^{(k_2)}\left(\begin{matrix} w \\ w' \end{matrix}\Bigg| \begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(4-|k|-m)} & \end{matrix}\right)\right]$$
$$= -\binom{|k|+2}{k_2}\mathcal{Z}_{21}^{(|k|+2)} x \mathcal{Z}_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix}\Bigg| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(4-|k|-m)} \end{matrix}\right) + \binom{k_2+m}{m}\mathcal{Z}_{21}^{(k_1+2)} x \mathcal{Z}_{21}^{(k_2+m)} x \left(\begin{matrix} w \\ w' \end{matrix}\Bigg| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(4-|k|-m)} \end{matrix}\right),$$

and

$$\partial_x\mathcal{S}_2\left(\mathcal{Z}_{21}^{(k_1+2)} x \mathcal{Z}_{21}^{(k_2)} x \left(\begin{matrix} w \\ w' \end{matrix}\Bigg| \begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(4-|k|-m)} & \end{matrix}\right)\right) = \partial_x\left(\mathcal{Z}_{21}^{(k_1+2)} x \mathcal{Z}_{21}^{(k_2)} x \mathcal{Z}_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix}\Bigg| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(4-|k|-m)} \end{matrix}\right)\right)$$
$$= \binom{|k|+2}{k_2}\mathcal{Z}_{21}^{(|k|+2)} x \mathcal{Z}_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix}\Bigg| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(4-|k|-m)} \end{matrix}\right) - \binom{k_2+m}{m}\mathcal{Z}_{21}^{(k_1+2)} x \mathcal{Z}_{21}^{(k_2+m)} x \left(\begin{matrix} w \\ w' \end{matrix}\Bigg| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(4-|k|-m)} \end{matrix}\right)$$
$$+ \mathcal{Z}_{21}^{(k_1+2)} x \mathcal{Z}_{21}^{(k_2)} x \left(\begin{matrix} w \\ w' \end{matrix}\Bigg| \begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(4-|k|-m)} & \end{matrix}\right)$$
; where $|k| = k_1 + k_2$.

It is clear that $\mathcal{S}_1\partial_x + \partial_x\mathcal{S}_2 = \mathrm{id}_{\mathcal{M}_2}$.

$$\mathcal{S}_2\partial_x\left(\mathcal{Z}_{21}^{(k_1+2)} x \mathcal{Z}_{21}^{(k_2)} x \mathcal{Z}_{21}^{(k_3)} x \left(\begin{matrix} w \\ w' \end{matrix}\Bigg| \begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(4-|k|-m)} & \end{matrix}\right)\right)$$
$$= \mathcal{S}_2\Bigg[\binom{k_1+k_2+2}{k_2}\mathcal{Z}_{21}^{(k_1+k_2+2)} x \mathcal{Z}_{21}^{(k_3)} x \left(\begin{matrix} w \\ w' \end{matrix}\Bigg| \begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(4-|k|-m)} & \end{matrix}\right) -$$
$$\binom{k_2+k_3}{k_3}\mathcal{Z}_{21}^{(k_1+2)} x \mathcal{Z}_{21}^{(k_2+k_3)} x \left(\begin{matrix} w \\ w' \end{matrix}\Bigg| \begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(4-|k|-m)} & \end{matrix}\right)$$
$$+ \mathcal{Z}_{21}^{(k_1+2)} x \mathcal{Z}_{21}^{(k_2)} x \partial_{21}^{(k_3)}\left(\begin{matrix} w \\ w' \end{matrix}\Bigg| \begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(4-|k|-m)} & \end{matrix}\right)\Bigg]$$
$$= \binom{k_1+k_2+2}{k_2}\mathcal{Z}_{21}^{(k_1+k_2+2)} x \mathcal{Z}_{21}^{(k_3)} x \mathcal{Z}_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix}\Bigg| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(4-|k|-m)} \end{matrix}\right) -$$
$$\binom{k_2+k_3}{k_3}\mathcal{Z}_{21}^{(k_1+2)} x \mathcal{Z}_{21}^{(k_2+k_3)} x \mathcal{Z}_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix}\Bigg| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(4-|k|-m)} \end{matrix}\right) +$$
$$\binom{k_3+m}{m}\mathcal{Z}_{21}^{(k_1+2)} x \mathcal{Z}_{21}^{(k_2)} x \mathcal{Z}_{21}^{(k_3+m)} x \left(\begin{matrix} w \\ w' \end{matrix}\Bigg| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(4-|k|-m)} \end{matrix}\right),$$

and

$$\partial_x\mathcal{S}_3\left(\mathcal{Z}_{21}^{(k_1+2)} x \mathcal{Z}_{21}^{(k_2)} x \mathcal{Z}_{21}^{(k_3)} x \left(\begin{matrix} w \\ w' \end{matrix}\Bigg| \begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(4-|k|-m)} & \end{matrix}\right)\right) =$$
$$\partial_x\left(\mathcal{Z}_{21}^{(k_1+2)} x \mathcal{Z}_{21}^{(k_2)} x \mathcal{Z}_{21}^{(k_3)} x \mathcal{Z}_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix}\Bigg| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(4-|k|-m)} \end{matrix}\right)\right)$$
$$= -\binom{k_1+k_2+2}{k_2}\mathcal{Z}_{21}^{(k_1+k_2+2)} x \mathcal{Z}_{21}^{(k_3)} x \mathcal{Z}_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix}\Bigg| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(4-|k|-m)} \end{matrix}\right) +$$
$$\binom{k_2+k_3}{k_3}\mathcal{Z}_{21}^{(k_1+2)} x \mathcal{Z}_{21}^{(k_2+k_3)} x \mathcal{Z}_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix}\Bigg| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(4-|k|-m)} \end{matrix}\right) -$$
$$\binom{k_3+m}{m}\mathcal{Z}_{21}^{(k_1+2)} x \mathcal{Z}_{21}^{(k_2)} x \mathcal{Z}_{21}^{(k_3+m)} x \left(\begin{matrix} w \\ w' \end{matrix}\Bigg| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(4-|k|-m)} \end{matrix}\right) +$$
$$\mathcal{Z}_{21}^{(k_1+2)} x \mathcal{Z}_{21}^{(k_2)} x \mathcal{Z}_{21}^{(k_3)} x \partial_{21}^{(m)}\left(\begin{matrix} w \\ w' \end{matrix}\Bigg| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(4-|k|-m)} \end{matrix}\right)$$
$$= -\binom{k_1+k_2+2}{k_2}\mathcal{Z}_{21}^{(k_1+k_2+2)} x \mathcal{Z}_{21}^{(k_3)} x \mathcal{Z}_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix}\Bigg| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(4-|k|-m)} \end{matrix}\right) +$$
$$\binom{k_2+k_3}{k_3}\mathcal{Z}_{21}^{(k_1+2)} x \mathcal{Z}_{21}^{(k_2+k_3)} x \mathcal{Z}_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix}\Bigg| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(4-|k|-m)} \end{matrix}\right) -$$

$$\binom{k_3+m}{m} Z_{21}^{(k_1+2)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3+m)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(4-|k|-m)} \end{matrix}\right) +$$
$$Z_{21}^{(k_1+2)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(4-|k|-m)} & \end{matrix}\right); \text{ where } |k| = k_1 + k_2 + k_3.$$

It is clear that $S_2\partial_x + \partial_x S_3 = id_{M_3}$.

From the above, we get that $\{S_0, S_1, S_2, S_3\}$ is a contracting homotopy [7] which means that our complex is exact.

3. Results of the case (8,6)/(2,1)

In this section, we find the resolution of Weyl module in the case of the skew-shape (8,6)/(2,1) when t = 1.

The terms of the Resolution Weyl module are:

$M_0 = D_6 \otimes D_5$

$M_1 = Z_{21}^{(2)} x\, D_8 \otimes D_3 \oplus Z_{21}^{(3)} x\, D_9 \otimes D_2 \oplus Z_{21}^{(4)} x\, D_{10} \otimes D_1 \oplus Z_{21}^{(5)} x\, D_{11} \otimes D_0$

$M_2 = Z_{21}^{(2)} x Z_{21}^{(1)} x\, D_9 \otimes D_2 \oplus Z_{21}^{(3)} x Z_{21}^{(1)} x\, D_{10} \otimes D_1 \oplus Z_{21}^{(2)} x Z_{21}^{(2)} x\, D_{10} \otimes D_1 \oplus$
$\quad Z_{21}^{(4)} x Z_{21}^{(1)} x\, D_{11} \otimes D_0 \oplus Z_{21}^{(3)} x Z_{21}^{(2)} x\, D_{11} \otimes D_0 \oplus Z_{21}^{(2)} x Z_{21}^{(3)} x\, D_{11} \otimes D_0$

$M_3 = Z_{21}^{(2)} x Z_{21}^{(1)} x Z_{21}^{(1)} x\, D_{10} \otimes D_1 \oplus Z_{21}^{(3)} x Z_{21}^{(1)} x Z_{21}^{(1)} x\, D_{11} \otimes D_0 \oplus$
$\quad Z_{21}^{(2)} x Z_{21}^{(2)} x Z_{21}^{(1)} x\, D_{11} \otimes D_0 \oplus Z_{21}^{(2)} x Z_{21}^{(1)} x Z_{21}^{(2)} x\, D_{11} \otimes D_0 \oplus$

$M_4 = Z_{21}^{(2)} x Z_{21}^{(1)} x Z_{21}^{(1)} x Z_{21}^{(1)} x\, D_{11} \otimes D_0.$

Thus we have the complex

$$\begin{tikzcd}
\mathcal{M}_4 \arrow[r, "\partial_x"] \arrow[d, "\mathrm{id}"] & \mathcal{M}_3 \arrow[r, "\partial_x"] \arrow[d, "\mathrm{id}"] \arrow[dl, "S_3"'] & \mathcal{M}_2 \arrow[r, "\partial_x"] \arrow[d, "\mathrm{id}"] \arrow[dl, "S_2"'] & \mathcal{M}_1 \arrow[r, "\partial_x"] \arrow[d, "\mathrm{id}"] \arrow[dl, "S_1"'] & \mathcal{M}_0 \arrow[d, "\mathrm{id}"] \arrow[dl, "S_0"'] \\
\mathcal{M}_4 \arrow[r, "\partial_x"] & \mathcal{M}_3 \arrow[r, "\partial_x"] & \mathcal{M}_2 \arrow[r, "\partial_x"] & \mathcal{M}_1 \arrow[r, "\partial_x"] & \mathcal{M}_0
\end{tikzcd}$$

The constructions of a contracting homotopies $\{S_i\}$, i = 1,2,3 are:

$S_0 : D_6 \otimes D_5 \to \sum_{k>0} Z_{21}^{(k+1)} x\, D_{6+k} \otimes D_{5-k}$ such that

$$S_0\left(\left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(6+k)} & 2^{(k)} \\ 2^{(5-k)} & \end{matrix}\right)\right) = \begin{cases} 0 & ;\text{ if } k \le 1 \\ Z_{21}^{(k+1)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(6+k)} \\ 2^{(5-k)} \end{matrix}\right) & ;\text{ if } k = 3,4,5 \end{cases}$$

$S_1 : \sum_{k>0} Z_{21}^{(k+1)} x\, \mathcal{D}_{7+k} \otimes \mathcal{D}_{4-k} \longrightarrow Z_{21}^{(k_1+1)} x Z_{21}^{(k_2)} x\, \mathcal{D}_{7+k} \otimes \mathcal{D}_{4-k}$ such that

$$S_1\left(Z_{21}^{(k+1)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(7+k)} & 2^{(m)} \\ 2^{(4-k-m)} & \end{matrix}\right)\right) = \begin{cases} 0 & ;\text{ if } m = 0 \\ Z_{21}^{(k+1)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(7+k+m)} \\ 2^{(4-k-m)} \end{matrix}\right) & ;\text{ if } m = 1,2,3 \end{cases}$$

$S_2 : \sum_{k_i>0} Z_{21}^{(k_1+1)} x Z_{21}^{(k_2)} x\, \mathcal{D}_{7+|k|} \otimes \mathcal{D}_{4-|k|} \longrightarrow Z_{21}^{(k_1+1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x\, \mathcal{D}_{7+|k|} \otimes \mathcal{D}_{4-|k|}$ such that

$$S_2\left(Z_{21}^{(k_1+1)} x Z_{21}^{(k_2)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(7+|k|)} & 2^{(m)} \\ 2^{(4-|k|-m)} & \end{matrix}\right)\right)$$

$$= \begin{cases} 0 & ;\text{ if } m = 0 \\ Z_{21}^{(k_1+1)} x Z_{21}^{(k_2)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(7+|k|+m)} \\ 2^{(4-|k|-m)} \end{matrix}\right) & ;\text{ if } m = 1,2, \end{cases} ;\text{ where } |k| = k_1 + k_2.$$

$S_3 : \sum_{k_i>0} Z_{21}^{(k_1+1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x\, \mathcal{D}_{7+|k|} \otimes \mathcal{D}_{4-|k|} \longrightarrow Z_{21}^{(k_1+1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4)} x\, \mathcal{D}_{7+|k|} \otimes \mathcal{D}_{4-|k|}$

$$S_3\left(Z_{21}^{(k_1+1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(7+|k|)} & 2^{(m)} \\ 2^{(4-|k|-m)} & \end{matrix}\right)\right)$$

$$=\begin{cases} 0 & ;\text{ if } m=0 \\ \mathcal{Z}_{21}^{(k_1+1)}x\mathcal{Z}_{21}^{(k_2)}x\mathcal{Z}_{21}^{(k_3)}x\mathcal{Z}_{21}^{(m)}x\begin{pmatrix} w \\ w' \end{pmatrix}\!\left|\begin{matrix} 1^{(7+|k|+m)} \\ 2^{(4-|k|-m)} \end{matrix}\right) & ;\text{ if } m=1 \end{cases};\text{ where } |k| = k_1+k_2+k_3.$$

Now, we have

$$\mathcal{S}_0\partial_x\left(\mathcal{Z}_{21}^{(k+1)}x\left(\begin{matrix} w \\ w' \end{matrix}\middle|\begin{matrix} 1^{(7+k)} & 2^{(m)} \\ 2^{(4-k-m)} & \end{matrix}\right)\right) = \mathcal{S}_0\partial_{12}^{(k+1)}\left(\begin{matrix} w \\ w' \end{matrix}\middle|\begin{matrix} 1^{(7+k)} & 2^{(m)} \\ 2^{(4-k-m)} & \end{matrix}\right)$$
$$= \binom{k+1+m}{m}\mathcal{Z}_{21}^{(k+1+m)}x\left(\begin{matrix} w \\ w' \end{matrix}\middle|\begin{matrix} 1^{(7+k+m)} \\ 2^{(4-k-m)} \end{matrix}\right),$$
and
$$\partial_x\mathcal{S}_1\left(\mathcal{Z}_{21}^{(k+1)}x\left(\begin{matrix} w \\ w' \end{matrix}\middle|\begin{matrix} 1^{(7+k)} & 2^{(m)} \\ 2^{(4-k-m)} & \end{matrix}\right)\right) = \partial_x\left(\mathcal{Z}_{21}^{(k+1)}x\mathcal{Z}_{21}^{(m)}x\left(\begin{matrix} w \\ w' \end{matrix}\middle|\begin{matrix} 1^{(7+k+m)} \\ 2^{(4-k-m)} \end{matrix}\right)\right)$$
$$= -\binom{k+1+m}{m}\mathcal{Z}_{21}^{(k+1+m)}x\left(\begin{matrix} w \\ w' \end{matrix}\middle|\begin{matrix} 1^{(7+k+m)} \\ 2^{(4-k-m)} \end{matrix}\right) + \mathcal{Z}_{21}^{(k+1)}x\left(\begin{matrix} w \\ w' \end{matrix}\middle|\begin{matrix} 1^{(7+k)} & 2^{(m)} \\ 2^{(4-k-m)} & \end{matrix}\right)$$

It is clear that $\mathcal{S}_0\partial_x + \partial_x\mathcal{S}_1 = \mathrm{id}_{\mathcal{M}_1}$.

$$\mathcal{S}_1\partial_x\left(\mathcal{Z}_{21}^{(k_1+1)}x\mathcal{Z}_{21}^{(k_2)}x\left(\begin{matrix} w \\ w' \end{matrix}\middle|\begin{matrix} 1^{(7+|k|)} & 2^{(m)} \\ 2^{(4-|k|-m)} & \end{matrix}\right)\right)$$
$$= \mathcal{S}_1\left[-\binom{|k|+1}{k_2}\mathcal{Z}_{21}^{(|k|+1)}x\left(\begin{matrix} w \\ w' \end{matrix}\middle|\begin{matrix} 1^{(7+|k|)} & 2^{(m)} \\ 2^{(4-|k|-m)} & \end{matrix}\right) + \mathcal{Z}_{21}^{(k_1+1)}x\partial_{21}^{(k_2)}\left(\begin{matrix} w \\ w' \end{matrix}\middle|\begin{matrix} 1^{(7+|k|)} & 2^{(m)} \\ 2^{(4-|k|-m)} & \end{matrix}\right)\right]$$
$$=$$
$$-\binom{|k|+1}{k_2}\mathcal{Z}_{21}^{(|k|+2)}x\mathcal{Z}_{21}^{(m)}x\left(\begin{matrix} w \\ w' \end{matrix}\middle|\begin{matrix} 1^{(7+|k|+m)} \\ 2^{(4-|k|-m)} \end{matrix}\right) + \binom{k_2+m}{m}\mathcal{Z}_{21}^{(k_1+1)}x\mathcal{Z}_{21}^{(k_2+m)}x\left(\begin{matrix} w \\ w' \end{matrix}\middle|\begin{matrix} 1^{(7+|k|+m)} \\ 2^{(4-|k|-m)} \end{matrix}\right),$$
and
$$\partial_x\mathcal{S}_2\left(\mathcal{Z}_{21}^{(k_1+1)}x\mathcal{Z}_{21}^{(k_2)}x\left(\begin{matrix} w \\ w' \end{matrix}\middle|\begin{matrix} 1^{(7+|k|)} & 2^{(m)} \\ 2^{(4-|k|-m)} & \end{matrix}\right)\right) = \partial_x\left(\mathcal{Z}_{21}^{(k_1+1)}x\mathcal{Z}_{21}^{(k_2)}x\mathcal{Z}_{21}^{(m)}x\left(\begin{matrix} w \\ w' \end{matrix}\middle|\begin{matrix} 1^{(7+|k|+m)} \\ 2^{(4-|k|-m)} \end{matrix}\right)\right)$$
$$= \binom{|k|+1}{k_2}\mathcal{Z}_{21}^{(|k|+1)}x\mathcal{Z}_{21}^{(m)}x\left(\begin{matrix} w \\ w' \end{matrix}\middle|\begin{matrix} 1^{(7+|k|+m)} \\ 2^{(4-|k|-m)} \end{matrix}\right) - \binom{k_2+m}{m}\mathcal{Z}_{21}^{(k_1+1)}x\mathcal{Z}_{21}^{(k_2+m)}x\left(\begin{matrix} w \\ w' \end{matrix}\middle|\begin{matrix} 1^{(7+|k|+m)} \\ 2^{(4-|k|-m)} \end{matrix}\right)$$
$$+$$
$$\mathcal{Z}_{21}^{(k_1+1)}x\mathcal{Z}_{21}^{(k_2)}x\left(\begin{matrix} w \\ w' \end{matrix}\middle|\begin{matrix} 1^{(7+|k|)} & 2^{(m)} \\ 2^{(4-|k|-m)} & \end{matrix}\right);\text{ where } |k| = k_1+k_2.$$

It is clear that $\mathcal{S}_1\partial_x + \partial_x\mathcal{S}_2 = \mathrm{id}_{\mathcal{M}_2}$.

$$\mathcal{S}_2\partial_x\left(\mathcal{Z}_{21}^{(k_1+1)}x\mathcal{Z}_{21}^{(k_2)}x\mathcal{Z}_{21}^{(k_3)}x\left(\begin{matrix} w \\ w' \end{matrix}\middle|\begin{matrix} 1^{(7+|k|)} & 2^{(m)} \\ 2^{(4-|k|-m)} & \end{matrix}\right)\right)$$
$$= \mathcal{S}_2\left[\binom{k_1+k_2+1}{k_2}\mathcal{Z}_{21}^{(k_1+k_2+1)}x\mathcal{Z}_{21}^{(k_3)}x\left(\begin{matrix} w \\ w' \end{matrix}\middle|\begin{matrix} 1^{(7+|k|)} & 2^{(m)} \\ 2^{(4-|k|-m)} & \end{matrix}\right) - \right.$$
$$\binom{k_2+k_3}{k_3}\mathcal{Z}_{21}^{(k_1+1)}x\mathcal{Z}_{21}^{(k_2+k_3)}x\left(\begin{matrix} w \\ w' \end{matrix}\middle|\begin{matrix} 1^{(7+|k|)} & 2^{(m)} \\ 2^{(4-|k|-m)} & \end{matrix}\right)$$
$$\left. + \mathcal{Z}_{21}^{(k_1+1)}x\mathcal{Z}_{21}^{(k_2)}x\partial_{21}^{(k_3)}\left(\begin{matrix} w \\ w' \end{matrix}\middle|\begin{matrix} 1^{(7+|k|)} & 2^{(m)} \\ 2^{(4-|k|-m)} & \end{matrix}\right)\right]$$
$$= \binom{k_1+k_2+1}{k_2}\mathcal{Z}_{21}^{(k_1+k_2+1)}x\mathcal{Z}_{21}^{(k_3)}x\mathcal{Z}_{21}^{(m)}x\left(\begin{matrix} w \\ w' \end{matrix}\middle|\begin{matrix} 1^{(7+|k|+m)} \\ 2^{(4-|k|-m)} \end{matrix}\right) -$$
$$\binom{k_2+k_3}{k_3}\mathcal{Z}_{21}^{(k_1+1)}x\mathcal{Z}_{21}^{(k_2+k_3)}x\mathcal{Z}_{21}^{(m)}x\left(\begin{matrix} w \\ w' \end{matrix}\middle|\begin{matrix} 1^{(7+|k|+m)} \\ 2^{(4-|k|-m)} \end{matrix}\right) +$$
$$\binom{k_3+m}{m}\mathcal{Z}_{21}^{(k_1+1)}x\mathcal{Z}_{21}^{(k_2)}x\mathcal{Z}_{21}^{(k_3+m)}x\left(\begin{matrix} w \\ w' \end{matrix}\middle|\begin{matrix} 1^{(7+|k|+m)} \\ 2^{(4-|k|-m)} \end{matrix}\right),$$
and

$$\partial_x S_3\left(Z_{21}^{(k_1+1)}xZ_{21}^{(k_2)}xZ_{21}^{(k_3)}x\begin{pmatrix} w \\ w' \end{vmatrix}\begin{matrix} 1^{(7+|k|)} & 2^{(m)} \\ 2^{(4-|k|-m)} & \end{matrix}\right)\right) =$$

$$\partial_x\left(Z_{21}^{(k_1+1)}xZ_{21}^{(k_2)}xZ_{21}^{(k_3)}xZ_{21}^{(m)}x\begin{pmatrix} w \\ w' \end{vmatrix}\begin{matrix} 1^{(7+|k|+m)} \\ 2^{(4-|k|-m)} \end{matrix}\right)\right)$$

$$= -\binom{k_1+k_2+1}{k_2}Z_{21}^{(k_1+k_2+1)}xZ_{21}^{(k_3)}xZ_{21}^{(m)}x\begin{pmatrix} w \\ w' \end{vmatrix}\begin{matrix} 1^{(7+|k|+m)} \\ 2^{(4-|k|-m)} \end{matrix}\right) +$$

$$\binom{k_2+k_3}{k_3}Z_{21}^{(k_1+1)}xZ_{21}^{(k_2+k_3)}xZ_{21}^{(m)}x\begin{pmatrix} w \\ w' \end{vmatrix}\begin{matrix} 1^{(7+|k|+m)} \\ 2^{(4-|k|-m)} \end{matrix}\right) -$$

$$\binom{k_3+m}{m}Z_{21}^{(k_1+1)}xZ_{21}^{(k_2)}xZ_{21}^{(k_3+m)}x\begin{pmatrix} w \\ w' \end{vmatrix}\begin{matrix} 1^{(7+|k|+m)} \\ 2^{(4-|k|-m)} \end{matrix}\right) +$$

$$Z_{21}^{(k_1+1)}xZ_{21}^{(k_2)}xZ_{21}^{(k_3)}x\partial_{21}^{(m)}\begin{pmatrix} w \\ w' \end{vmatrix}\begin{matrix} 1^{(7+|k|+m)} \\ 2^{(4-|k|-m)} \end{matrix}\right)$$

$$= -\binom{k_1+k_2+1}{k_2}Z_{21}^{(k_1+k_2+1)}xZ_{21}^{(k_3)}xZ_{21}^{(m)}x\begin{pmatrix} w \\ w' \end{vmatrix}\begin{matrix} 1^{(7+|k|+m)} \\ 2^{(4-|k|-m)} \end{matrix}\right) +$$

$$\binom{k_2+k_3}{k_3}Z_{21}^{(k_1+1)}xZ_{21}^{(k_2+k_3)}xZ_{21}^{(m)}x\begin{pmatrix} w \\ w' \end{vmatrix}\begin{matrix} 1^{(7+|k|+m)} \\ 2^{(4-|k|-m)} \end{matrix}\right) -$$

$$\binom{k_3+m}{m}Z_{21}^{(k_1+1)}xZ_{21}^{(k_2)}xZ_{21}^{(k_3+m)}x\begin{pmatrix} w \\ w' \end{vmatrix}\begin{matrix} 1^{(7+|k|+m)} \\ 2^{(4-|k|-m)} \end{matrix}\right) +$$

$$Z_{21}^{(k_1+1)}xZ_{21}^{(k_2)}xZ_{21}^{(k_3)}x\begin{pmatrix} w \\ w' \end{vmatrix}\begin{matrix} 1^{(7+|k|)} & 2^{(m)} \\ 2^{(4-|k|-m)} & \end{matrix}\right); \text{ where } |k| = k_1+k_2+k_3.$$

.

It is clear that $S_2\partial_x + \partial_x S_3 = id_{M_3}$.

From the above, we get that $\{S_0, S_1, S_2, S_3\}$ is a contracting homotopy [7], which means that our complex is exact.

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