Parameters Estimation in a Mixture of Two Modified Agarwal and Kalla's Distributions

Shawki Shaker Hussain
Department of Mathematics, College of Science, University of Baghdad. Baghdad-Iraq.

Abstract

In this paper we introduced a new distribution model defined by a mixture of two modified Agarwal and Kalla generalized gamma distributions. Its possible application in reliability is to study simultaneously both problems; the two types of failures and the displacement phenomenon. Moment and maximum likelihood methods were used to estimate the parameters of this distribution model, the non-linear systems obtained by these two methods are solved numerically using Newton-Raphson iterative technique.

الخلاصة

في هذا البحث تم تقديم أنموذج توزع جديد معرف من خليط توزعي اكروال وكالا المحور ذي الخمسة المعالم . إن هذا الأنموذج ملائم لدراسة معولية نظام ينطوي على نوعين من الفشل إضافة إلى الإزاحة . استخدم أسلوب العزوم وكذلك أسلوب الأرجحية العظمى لتخمين معالم هذا الأنموذج , وان كل من النظام اللاخطي المنبثق عن هذين الأسلوبين قد تم دراسة حلهما عددياً باستخدام أسلوب نيوتن – رافسون التكراري .

Keywords: Kobayashi's generalized gamma function; Agarwal and Kalla's generalized gamma distribution; Mixture distributions; Moments and Maximum likelihood methods.

Introduction

Generalized gamma distribution has proved to be of considerable interest in the field of reliability, it is a reasonable model for life-time distribution of a component (or a system of components). It was suggested by Stacy (1962) as the random variable X whose probability density function (pdf) with 3-parameters is of the form:

$$f\left(x;a,b,c\right)=\frac{cb^{a}}{\Gamma(a)}x^{ac-1}e^{-bx^{c}};\ \text{for } x,a,b,c>0 \quad \ldots (1)$$

Where $\Gamma(a)$ is the gamma function defined by:

$$\Gamma(a)=\int_{0}^{\infty}x^{a-1}e^{-x}dx;\ \text{for } a>0$$

and both a and c represent shape parameters, while b is a scale parameter. By varying the parameters of this distribution one can obtained many known probability distributions includes Weibull and gamma distributions as special cases. Kao (1959) stated (see also Falls (1970)); one has to use mixture of distributions to analyze the data from life-time experiments whenever there are two types of failures such as sudden catastrophic and wear-out failures. Therefore, Radhakrishna etal (1992) suggested to use the following pdf of a two component mixture generalized gamma distributions:

$f\left(x;a_1,b_1,c_1,a_2,b_2,c_2,p\right)=pf\left(x;a_1,b_1,c_1\right)+(1-p)f\left(x;a_2,b_2,c_2\right)$; for $0\le p\le 1$.

They derived both moment and maximum likelihood estimators of the parameters p,a_i,b_i,c_i; $i=1,2$.

Recently Agarwal and Kalla (1996) mentioned that there are many situations in industrial components were the effect of all parameters do not start in the beginning, some of them start playing their role after sometimes, i.e. displaced parameter. For example in a new machine system the corrosion problem will start after certain interval of time and similar is the case of metal fatigue. In order to study displacement effect a power is introduced to displaced parameter to observe the intensity of the effect of the corresponding parameter. Therefore, they defined a new type of generalization of gamma distribution whose pdf with 4-parameters is given by:

$$f\left(x;m,n,\lambda,\alpha\right)=\frac{1}{\Gamma\left(m,n,\lambda,\alpha\right)}x^{m-1}\left(x+n\right)^{-\lambda}e^{-\alpha x};$$

for $x,m,n,\lambda,\alpha>0$,

where m is a shape parameter, n is the displacement parameters, λ is the intensity parameter, α is a scale parameter, and the function;

$$\Gamma\left(m,n,\lambda,\alpha\right)=\int_0^{\infty}x^{m-1}\left(x+n\right)^{-\lambda}e^{-\alpha x}dx .$$

it is easy to verify that:

$\Gamma\left(m,n,\lambda,\alpha\right)=\alpha^{\lambda-m}\Gamma\left(m,\alpha n,\lambda,1\right)$

therefore;

$$f\left(x;m,n,\lambda,\alpha\right)=\frac{\alpha^{m-\lambda}}{\Gamma\left(m,\alpha n,\lambda,1\right)}x^{m-1}\left(x+n\right)^{-\lambda}e^{-\alpha x};$$

for $x,m,n,\lambda,\alpha>0$,

where the function $\Gamma\left(m,\alpha n,\lambda,1\right)$ is a modified form of the new type generalized gamma function introduced by Kobayashi (1991) defined by:

$$\Gamma\left(m,\alpha n,\lambda,1\right)=\int_0^{\infty}x^{m-1}\left(x+n\right)^{-\lambda}e^{-x}dx;\ \text{for } m,n,\lambda>0$$

This function occurs in many problems of diffraction theory, and its particular case is the ordinary gamma function when $\lambda=0$, i.e., $\Gamma\left(m,n,0,1\right)=\Gamma\left(m\right)$.

In terms of the confluent hyper geometric function of the second kind, Agarwal and Kalla have shown that for small αn:

$\Gamma\left(m,\alpha n,\lambda,1\right)=\Gamma\left(m-\lambda\right)$; for $m>\lambda$.

Thus for small αn:

$$f\left(x;m,n,\lambda,\alpha\right)=\frac{\alpha^{m-\lambda}}{\Gamma\left(m-\lambda\right)}x^{m-1}\left(x+n\right)^{-\lambda}e^{-\alpha x};\quad \text{for}$$

$x,m,n,\lambda,\alpha>0$.

In this paper we shall consider a distribution, which is a modified Agarwal and Kalla's distribution, whose pdf is defined with 5-parameters as follows:

$$f\left(x;m,n,\lambda,\alpha,\beta\right)=\frac{1}{\Lambda\left(m,n,\lambda,\alpha,\beta\right)}x^{m-1}\left(x^{\beta}+n\right)^{-\lambda}e^{-\alpha x^{\beta}}$$

; for $x>0$, …(2)

Where $m,n,\lambda,\alpha,\beta>0$, both m,β are shape parameters, α is a scale parameter, n and λ are, respectively, represents the displacement and intensity parameters, and:

$$\Lambda\left(m,n,\lambda,\alpha,\beta\right)=\int_0^{\infty}x^{m-1}\left(x^{\beta}+n\right)^{-\lambda}e^{-\alpha x^{\beta}}dx .$$

Using the transformation $y=\alpha x^{\beta}$, we can show that:

$$\Lambda\left(m,n,\lambda,\alpha,\beta\right)=\frac{\alpha^{\lambda-\frac{m}{\beta}}}{\beta}\Lambda\left(\frac{m}{\beta},\alpha n,\lambda,1,1\right)$$

$$=\frac{\alpha^{\lambda-\frac{m}{\beta}}}{\beta}\Gamma\left(\frac{m}{\beta},\alpha n,\lambda,1\right)$$

$$=\frac{\alpha^{\lambda-\frac{m}{\beta}}}{\beta}\Gamma\left(\frac{m}{\beta}-\lambda\right)$$

; for small αn. …(3)

Thus the pdf $f\left(x;m,n,\lambda,\alpha,\beta\right)$ becomes:

$$f\left(x;m,n,\lambda,\alpha,\beta\right)=\frac{\beta\alpha^{\frac{m}{\beta}-\lambda}}{\Gamma\left(\frac{m}{\beta},\alpha n,\lambda,1\right)}x^{m-1}\left(x^{\beta}+n\right)^{-\lambda}e^{-\alpha x^{\beta}}$$

; for $x>0$

$$=\frac{\beta\alpha^{\frac{m}{\beta}-\lambda}}{\Gamma\left(\frac{m}{\beta}-\lambda\right)}x^{m-1}\left(x^{\beta}+n\right)^{-\lambda}e^{-\alpha x^{\beta}};\ \text{for } x>0 \text{ and}$$

small αn.

When $\lambda = 0$, we have $\Lambda(m,n,\lambda,\alpha,\beta) = \Gamma\left(\frac{m}{\beta}\right)$ and the pdf $f(x;m,n,\lambda,\alpha,\beta)$ becomes:

$$f(x;m,n,\lambda,\alpha,\beta) = \frac{\beta\alpha^{\frac{m}{\beta}}}{\Gamma\left(\frac{m}{\beta}\right)} x^{m-1} e^{-\alpha x^{\beta}}; \text{ for } x > 0 .$$

Several well known distributions are particular cases of this distribution, for example by setting $a = \frac{m}{\beta}$, $b = \alpha$ and $c = \beta$ we obtained the pdf of (Stacy) generalized gamma distribution given in (1). While, by setting $m = \beta$ and $\lambda = 0$, we have:

$$f(x;m,n,\lambda,\alpha,\beta) = \beta\alpha x^{\beta-1} e^{-\alpha x^{\beta}}; \text{ for } x > 0 .$$

which is the Weibull pdf with 2-parameters. Also, we can have Lee and Gross (1989) distribution and many others.

Based on the modified Agarwal and Kalla's distribution whose pdf is defined in (2), we introduced a distribution model defined by a mixture of two such distributions whose pdf with 11-parameters is given by:

$$f(x;\theta) = pf(x;\theta_1) + (1-p)(x;\theta_2); \text{ for } x > 0 \quad \ldots(4)$$

where $0 \le p \le 1$, $\theta_i = (m_i, n_i, \lambda_i, \alpha_i, \beta_i)$; $i = 1,2$, and $\theta = (p, \theta_1, \theta_2)$. This distribution can be used to analyze life-time data whenever there are two kinds of failures and a phenomenon of displacement.

The following two sections gives the derivation of moment and maximum likelihood estimates of the 11-parameters of the proposed mixture distributions.

Moment Estimates

Let $X_1, X_2, ..., X_N$ be a random sample from a random variable X with a pdf as given in (4) which can be rewritten an the following from:

$$f(x;\theta) = \sum_{i=1}^{2} p^{2-i}(1-p)^{i-1} f(x;\theta_i); \text{ for } x > 0 .$$

The expression of the k-th moment about the origin of X can be derived as follows:

$$\mu_k(\theta) = EX^k = \sum_{i=1}^{2} p^{2-i}(1-p)^{i-1} \int_0^{\infty} x^k f(x;\theta_i) dx$$

$$= \sum_{i=1}^{2} p^{2-i}(1-p)^{i-1} \frac{\Lambda(m_i + k, n_i, \lambda_i, \alpha_i, \beta_i)}{\Lambda(m_i, n_i, \lambda_i, \alpha_i, \beta_i)}$$

By (3) it follows that:

$$\mu_k(\theta) = \sum_{i=1}^{2} p^{2-i}(1-p)^{i-1} \alpha_i^{-\frac{k}{\beta_i}} \frac{\Gamma\left(\frac{m_i + k}{\beta_i}, \alpha_i n_i, \lambda_i, 1\right)}{\Gamma\left(\frac{m_i}{\beta_i}, \alpha_i n_i, \lambda_i, 1\right)}$$

And for small $\alpha_i n_i$; $i = 1,2$, we have:

$$\mu_k(\theta) = \sum_{i=1}^{2} p^{2-i}(1-p)^{i-1} \alpha_i^{-\frac{k}{\beta_i}} \frac{\Gamma\left(\frac{m_i + k}{\beta_i} - \lambda_i\right)}{\Gamma\left(\frac{m_i}{\beta_i} - \lambda_i\right)}$$

By using the Stirling's formula (see Lindgren (1976)):

$$\frac{\Gamma(g+h)}{\Gamma(g)} \approx g^h, \text{ for large } h .$$

implies for large $\left(\frac{m_i}{\beta_i} - \lambda_i\right)$ that:

$$\mu_k(\theta) = \sum_{i=1}^{2} p^{2-i}(1-p)^{i-1} \alpha_i^{-\frac{k}{\beta_i}} \left(\frac{m_i}{\beta_i} - \lambda_i\right)^{\frac{k}{\beta_i}}$$

Let M_k be the k-th sample moment, i.e.:

$$M_k = \frac{1}{N}\sum_{i=1}^{N} X_i^k ; \; k = 1,2,...$$

Then the moment estimates are obtained by solving simultaneously for each component of θ the following system of 11-nonlinear equations:

$$f_k(\theta) = \mu_k(\theta) - M_k = 0 ; \; k = 1,2,...,11, .$$

where an iterative numerical method is required to obtain a solution:

when $\alpha_i n_i$; $i = 1,2$ are small we have to solve simultaneously for $\theta = (p, m_1, \lambda_1, \alpha_1, \beta_1, m_2, \lambda_2, \alpha_2, \beta_2)$ the following system of 9-nonlinear equations:

$$f_k(\theta) = \left\{\sum_{i=1}^{2} p^{2-i}(1-p)^{(i-1)} \alpha_i^{\frac{k}{\beta_i}} \left(\frac{m_i}{\beta_i} - \lambda_i\right)^{\frac{k}{\beta_i}}\right\} - M_k = 0$$

; $k = 1,2,...,9$,

whenever, $\left(\frac{m_i}{\beta_i} - \lambda_i\right) > 0$; $i = 1,2$, are large. This system can be solved using Newton-Raphson iterative method. The solution is given by the following procedure: given a small value for $\alpha_i n_i$; $i = 1,2$, and an initial approximate value $\hat{\theta}^{(0)}$ of θ (any given set of positive values that satisfied the distribution model conditions) . Then this iterative method is given by:

$$\hat{\theta}^{(i+1)} = \hat{\theta}^{(i)} - J^{-1}\left(\hat{\theta}^{(i)}\right) f\left(\hat{\theta}^{(i)}\right);\ i = 1,2,\ldots .$$

where $f(\theta) = \left(f_1(\theta) f_2(\theta) \ldots f_9(\theta)\right)^T$, and $J(\theta) = \left[J_{kj}\right]_{(\theta)};\ 1 \le k,\ i \le 9$, is the Jacobean matrix.

By continuing iterative on i, this procedure is terminated as soon as the norm $\left\|\hat{\theta}^{(i+1)} - \hat{\theta}^{(i)}\right\| \le \in$, where $\in$ is a pre-assigned small positive value. Whence, $\hat{\theta}^{(i+1)}$ is the moment estimate of θ, and if $\alpha_i n_i = d$; $i = 1,2$, where d is a small given value, then the estimate of n_i is given by $\hat{n}_i = \frac{d}{\hat{\alpha}_i};\ i = 1,2$.

As for the elements of the Jacobean matrix they can be obtained as follows: for $k = 1,2,\ldots,9$, the k-th column elements are given by:

$$J_{k1} = \frac{\partial f_k(\theta)}{\partial p} = \sum_{i=1}^{2} (-1)^{i+1} \alpha_i^{-\frac{k}{\beta_i}} A_i^{\frac{k}{\beta_i}},$$

$$J_{k2} = \frac{\partial f_k(\theta)}{\partial m_1} = \frac{pk}{\beta_1^2} \alpha_1^{-\frac{k}{\beta_1}} A_1^{\frac{k}{\beta_1}},$$

$$J_{k3} = \frac{\partial f_k(\theta)}{\partial \lambda_1} = -\frac{pk}{\beta_1} \alpha_1^{-\frac{k}{\beta_1}} A_1^{\frac{k}{\beta_1}-1},$$

$$J_{k4} = \frac{\partial f_k(\theta)}{\partial \alpha_1} = -\frac{pk}{\beta_1} \alpha_1^{-\frac{k}{\beta_1}-1} A_1^{\frac{k}{\beta_1}},$$

$$J_{k5} = \frac{\partial f_k(\theta)}{\partial \beta_1} = \frac{pk}{\beta_1^2} \alpha_1^{-\frac{k}{\beta_1}} A_1^{\frac{k}{\beta_1}} \left(\ln\frac{\alpha_1}{A_1} - \frac{m_1}{\beta_1 A_1}\right),$$

$$J_{k6} = \frac{\partial f_k(\theta)}{\partial m_2} = \frac{(1-p)k}{\beta_2^2} \alpha_2^{-\frac{k}{\beta_2}} A_2^{\frac{k}{\beta_2}-1},$$

$$J_{k7} = \frac{\partial f_k(\theta)}{\partial \lambda_2} = -\frac{(1-p)k}{\beta_2} \alpha_2^{-\frac{k}{\beta_2}} A_2^{\frac{k}{\beta_2}-1},$$

$$J_{k8} = \frac{\partial f_k(\theta)}{\partial \alpha_2} = -\frac{(1-p)k}{\beta_2} \alpha_2^{-\frac{k}{\beta_2}-1} A_2^{\frac{k}{\beta_2}},$$

$$J_{k9} = \frac{\partial f_k(\theta)}{\partial \beta_2} = \frac{(1-p)k}{\beta_2^2} \alpha_2^{-\frac{k}{\beta_2}} A_2^{\frac{k}{\beta_2}} \left(\ln\frac{\alpha_2}{A_2} - \frac{m_2}{\beta_2 A_2}\right),$$

where:

$$A_i = \left(\frac{m_i}{\beta_i} - \lambda_i\right);\ i = 1,2 .$$

It is easy to see that J is a singular matrix when $\beta_i = 1;\ i = 1,2$, therefore these two cases showed be excluded from this procedure. Also, we can show that when $\beta_i = 1$ and $\alpha_i n_i$; $i = 1,2$ are small the system of non-linear equations can be written in the following form:

$$f_k(\theta) = \left\{\sum_{i=1}^{2} p^{2-i} (1-p)^{i-1} \alpha_i^{-k} \mathop{\pi}_{S=1}^{k} (m_i - \lambda_i + k - S)\right\} - M_k = 0$$

; for $k = 1,2,\ldots,7$.

To obtain a solution for this system of equations, again, we can show that the Jacobean matrix is singular, thus we have to look for other numerical methods to solve this new system.

In all cases, it is interesting to notice that when $p = 1$ the above procedure reduced to the moment method of estimation of the parameters of the modified Agarwal and Kalla's distribution as well as the Agarwal and Kalla's distribution by setting $\beta = 1$.

Maximum Likelihood Estimates

Let $X_1, X_2, \ldots, X_N$ be a random sample from a random variable X with a pdf as given in (4). Then the likelihood function is defined by:

$$L(\theta) = \mathop{\pi}_{i=1}^{N} f(x_i;\theta) = \mathop{\pi}_{i=1}^{N} \left[p f(x_i;\theta_1) + (1-p) f(x_i;\theta_2)\right].$$

Hence, the natural logarithm of $L(\theta)$ can be written as:

$$LnL(\theta) = \sum_{i=1}^{N} \ln f(x_i,\theta)$$

By taking the partial derivative of $LnL(\theta)$ with respect to each k-th component of θ; $k = 1,2,\ldots,11$, and set them equal zero, we obtain the following system of non-linear equations:

$$f_k(\theta) = \sum_{i=1}^{N} \frac{A_k}{p f(x_i;\theta_1) + (1-p) f(x_i;\theta_2)} = 0;\ k = 1,2,\ldots,11,$$

where A_k represents the partial derivative of $f(x_i,\theta)$ with respect to the k-th component of θ, each is given by:

$$A_1 = \frac{\partial f(x_i;\theta)}{\partial p} = f(x_i;\theta_1) - f(x_i;\theta_2),$$

and,

$$A_k = \begin{cases} \left[p / \Lambda^2(\theta_1)\right] B_{k-1}(x_i,\theta_1) & ; k = 2,3,4,5,6, \\ \left[(1-p) / \Lambda^2(\theta_2)\right] B_{k-6}(x_i,\theta_2) & ; k = 7,8,9,10,11 \end{cases}$$

where $B_k(x;\xi);\ \xi = (m,n,\lambda,\alpha,\beta)$ is the partial derivative of $f(x;\xi)$ with respect to the k-th component of ξ, given by:

$$B_1 = \frac{\partial f(x,\xi)}{\partial m} = \ln x g(\xi)\Lambda(\xi) - g(\xi)\underset{10}{\Lambda}(\xi)$$

$$B_2 = \frac{\partial f(x,\xi)}{\partial n} = -\lambda\left(x^{\beta}+n\right)^{-1} g(\xi)\Lambda(\xi) + \lambda g(\xi)\Lambda(m,n,\lambda+1,\alpha,\beta)$$

$$B_3 = \frac{\partial f(x,\xi)}{\partial \lambda} = -\ln\left(x^{\beta}+n\right) g(\xi)\Lambda(\xi) + g(\xi)\underset{\beta n}{\Lambda}(\xi)$$

$$B_4 = \frac{\partial f(x,\xi)}{\partial \alpha} = -x^{\beta} g(\xi)\Lambda(\xi) + g(\xi)\Lambda(m+\beta,n,\lambda,\alpha,\beta)$$

$$B_5 = \frac{\partial f(x,\xi)}{\partial \beta} = -x^{\beta}\ln x\left[\lambda\left(x^{\beta}+n\right)^{-1}+\alpha\right] g(\xi)\Lambda(\xi)$$
$$+g(\xi)\left[\lambda\underset{10}{\Lambda}(m+\beta,n,\lambda+1,\alpha,\beta)+\alpha\underset{10}{\Lambda}(m+\beta,n,\lambda,\alpha,\beta)\right]$$

where:

$$g(\xi) = x^{m-1}\left(x^{\beta}+n\right)^{-\lambda} e^{-\alpha x^{\beta}} ,$$

$$\underset{u\,v}{\Lambda}(\xi) = \int_0^{\infty} \ln\left(x^{u}+v\right) g(\xi) dx .$$

To obtain the maximum likelihood estimates of θ, the above (11) equations are have to be solved simultaneously for θ numerically using one of the iterative techniques such as Newton-Raphson method, where the elements of the Jacobean matrix can be obtained simply but lengthly to be written, therefore they are not be presented here. Finally this Jacobean matrix provides an approximate asymptotic variance-covariance matrix of the estimates.

Acknowledgment

The author would like to express his appreciation to Dr. Habeb Al-Doori for his valuable advise and assistance in the preparation of this paper.

References

1. Agarwal, S. K. and Kalla, S. L. (**1996**).: *A Generalized Gamma Distribution And Its Application In Reliability*. Commun. Statist. Theory Meth., 25 (1), 201-210.
2. Falls, L. W. (**1970**): *Estimation Of Parameters In Compound Weibull Distributions.* Technometrics, vol. 12, No. 2, 339-407.
3. Kao, J. H. (**1959**): *A Graphical Estimation Of Mixed Weibull Parameters In Life-Testing Of Electron Tubes*. Technometrics, vol. 1, 389-407.
4. Kobayashi, K. (**1991**): *On Generalized Gamma Functions Occurring In Different Theory*. J. physical society of Japan, 60, No. 5, 1501-1512.
5. Lee, M. and Gross, A. (**1989**): *Life-Time Distributions Under Unknown Environment*. J. Statist. Plan and inference, 29,137-143.
6. Lindgreen, B. W. (**1976**): *Statistical Theory*. Third edition, New York, Macmillan pub. Co.
7. Radhakrishna, C. et al (**1992**): *Estimation Of Parameters In A Two Component Mixture Generalized Gamma Distribution*. Commun. Stat. theory Meth., 21 (6), 1799-1805.
8. Stacy, E. W. (**1962**): *A Generalization Of The Gamma Distribution.* Annals of Math. Staist. 33, 1187-1192.