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Large Small Submodules with Chain Conditions

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Abstract.
We consider R is any ring with identity and M is a non-zero unitary left R-module. In this article we study the notions of large small submodules with ascending (descending) chain conditions on large small submodules. Also, we will discuss and study many basic properties about this concept that fulfills the chain condition with its generalization to the ring. Where a submodule X of M is said to be large small of M denoted by $X \ll_L M$ if for $H \leq M$ such that $X + H = M$ then H is essential in M.

Keywords: Chained R -module, Noetherian module, Large small Submodules with ascending (respectively, descending) Chain condition.

المقاسات الجزئية الصغيرة الاساسية مع شرط السلسلة

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الخلاصة
في هذا البحث اعتبرنا R هي حلقة ذات عنصر محايد و M هو مقاس احادي ايسر ليس صفريا معرفا على R. قمنا بدراسة مفهوم المقاسات الجزئية الصغيرة الاساسية مع شرط السلسلة التصاعدية (المتناقصة, على التوالي)على هذا النوع من المقاسات الجزئية الصغيرة الاساسية. وسنقوم بمناقشة ودراسة العديد من الخصائص الاساسية حول هذا المفهوم الذي يحقق شرط السلسلة مع تعميمها على الحلقات.على اعتبار ان Xهو مقاس جزئي من M يسمى جزئي اساسي صغير ويرمز له بالرمز $X \ll_L M$ اذا كان كل مقاس جزئي H من M بحيث $X + H = M$ فان H هو جوهري في M .

1.Introduction

Suppose M is a unitary left R-module and R is an associative ring with identity. A proper submodule X of M is named small in M $(X \ll M)$, if for any submodule S of M such that $X + S = M$ implies that $M = S$, see [1] and [2]. A non-zero submodule X of M considered as essential in $M(X \leq_e M)$ if for every $0 \neq S \leq M$ then $X \cap S \neq 0$ [3], [4]. A non-zero module M called uniform if all its non-zero submodule are essential in M see [5], [6] and [7]. The annihilator of a module M is the set $ann(M) = \{r \in R : rM = 0\}$, as well as $M$is said to be faithful if $ann(M) = 0$, see these [8], [9], and [10]. A module M is called multiplication module if for all submodule B of M, $B = JM$ for some ideal J in R. Equivalently if for all $L \leq M, L = [L: M]. M$, where $[L: M] = \{r \in R: rM \subseteq L\}$ see [11], [12] and [13]. Many authors

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present generalizations of a small submodule, to see more of these generalizations, note the following sources [14-20].

A. Abduljaleel in [21] and [22] introduced the definition of large small submodule (LS-submodule) as a submodule X of M called large small (LS) of M denoted by $(X \ll_L M)$ if for $H \leq M$ such that $X + H = M$, then H is essential in M. And the ideal A is called LS-ideal if for $A \subseteq R$ such that $A + S = R$, then S is essential ideal in R. Let us recall the most important definitions which are the essential concepts for our study such as Noetherian module is a module that achieves Ascending Chain Condition on its submodules. And, Artinian module is a module that achives the Descending Chain Condition on its submodules [23], [24], and [25]. Remember that a module M is said to be chained module if for each submodules A, B of M either A ≤ B or, B ≤ A [26], [27].

A.G Naoum in [28] and [29] gives the notion of module that achieves ascending (descending) chain conditions on small submodules, as a module M is called M with ascending chain condition (ACC) on small submodules, respectively descending chain condition (DCC) on small submodules if each ascending (descending) chain of small submodules $S_1 \subseteq S_2 \subseteq S_3 \subseteq \dots . S_n \subseteq \dots$, respectively $S_1 \supseteq S_2 \supseteq S_3 \dots \supseteq S_n \supseteq \cdots$ is finite. That is there exist $t \in Z^+$, such that $S_n = S_t \; \forall \; n \geq t$. Many authors introduced a lot of generalizations of this kind of modules see [30] and [31].

Our work is to study the large small submodules with ascending chain condition (ACC) and descending chain condition (DCC). Also, we generalize it on the rings. And discuss some of these results on finitely generated faithful multiplication modules (FGFM).

2. Large small submodules with chain conditions

This section provides an overview of Large Small submodules that is achieves ascending chain condition (ACC) and descending chain condition (DCC) on LS-submodules as generalization on small submodules and study its properties. which also provides remarks to help understand the topic and the relationships with some other modules. Also, we study the relation between the ring that satisfies (ACC) and descending chain condition (DCC) on LS-ideals.

In the following lemma we give the properties the most important characteristics of LS-submodules that we needed it in our work. For more characteristics, we recommend referring to the source referred to below.

Lemma 2.1 [22]**:** Let X, B be submodules of a module M then:

1. If $X \leq B \leq M$ and $B \ll_L M$, then $X \ll_L M$.
2. If $X \leq C \leq M$, $X$is closed in M if $C \ll_L M$, then $C/X \ll_L M/X$.
3. If $X_1 \ll_L M$ and $X_2 \ll_L M$, then $X_1 \cap X_2 \ll_L M$.
4. Let $X \leq B \leq M$, if $B/X \ll_L M/X$, then $B \ll_L M$.
5. If $M$is uniform module then each submodule of M is an LS- submodule in M.

Definition 2.2: An R-module M is named M with ascending chain condition (ACC) on LS-submodules, respectively descending chain condition (DCC) on LS- submodules if for each ascending (descending) chain of LS- submodules $H_1 \subseteq H_2 \subseteq \dots \subseteq \dots H_n \subseteq \dots$, and $H_1 \supseteq H_2 \dots \supseteq H_n \supseteq \cdots$, respectively is finite. That is there exist $t \in Z^+$, such that $H_n = H_t$ for all $n \geq t$.

Remarks 2.3:
1. Each small submodule is LS- submodule [4]. So, if M with ACC (DCC, respectively) on LS-submodules then, M with ACC (DCC, respectively) on small submodules.
2. Each Noetherian (Artinian) module M, is M with ACC (DCC, respectively) on LS-submodules.
3. Every uniform module M, with ACC (DCC, respectively) on submodules of M, then M with ACC (DCC, respectively,) on LS- submodules. By Lemma 2.1(5).
4. ACC (DCC) on LS - submodules closed under isomorphism.

Proposition 2.4: If M is a chained module, and $X \leq M$ such that M satisfied ACC (DCC, respectively) on LS- submodules, then X with ACC (DCC) on LS- submodules.
Proof: Let M with ACC on LS- submodules and $X_1 \subseteq X_2 \subseteq \cdots$ is ascending chain of LS-submodules in X. Since M satisfies chained condition, so X is an LS- submodule of M then by Lemma 2.1(1) X_i is LS- submodule of M for all $i = 1, 2, \ldots$. Hence, $X_1 \subseteq X_2 \ldots$ be ascending chain of LS- submodules of M. Since M with ACC on LS- submodules, thus $\exists t \in Z^+$, such that $X_n = X_t$ for all $n \geq t$.Thus, X with ACC on LS- submodules. The similar proof for (DCC).
Proposition 2.5: Let M_1 and M_2 be two R-modules and $R = ann\ M_1 + ann\ M_2$. Then $M_1 \oplus M_2$ with ACC (DCC) on LS-submodules iff M_1, and M_2, with ACC (DCC) on LS-submodules.
Proof: Since $R = ann\ M_1 + ann M_2$, let $D_1 \oplus X_1 \subseteq D_2 \oplus X_2 \subseteq D_3 \oplus X_3 \subseteq \cdots \subseteq D_n \oplus X_n$ be ascending chain on LS- submodules of $M_1 \oplus M_2$ hence, $D_1 \subseteq D_2 \subseteq D_3 \subseteq \ldots \subseteq D_n \subseteq \ldots$ is ascending chain on LS- submodules of M_1 and $X_1 \subseteq X_2 \subseteq X_3 \subseteq \ldots X_n \ldots$ be ascending chain on LS- submodules of M_2. Since M_1 and M_2 with ACC on LS-submodules that is $\exists r, t \in Z^+$ such that, $D_r = D_{r+i} = \ldots \forall i = 1, 2, 3, \ldots$ and $X_t = X_{t+i}$ $\forall i = 1, 2, 3, \ldots$. Put $s = max\{r, t\}$, therefore, $D_s + X_s = D_{s+i} + X_{s+i}$ $\forall i = 1, 2, 3, \ldots$.
On other hand, let $D_1 \subseteq D_2 \subseteq D_3 \subseteq \ldots \subseteq D_n \subseteq \cdots$ be ascending chain of LS- submodules of M_1, thus $D_1 \oplus \{0\} \subseteq D_2 \oplus \{0\} \subseteq D_3 \oplus \{0\} \subseteq \ldots \subseteq D_n \oplus \{0\} \subseteq \cdots$ is an ascending chain of LS- submodules of $M_1 \oplus M_2$, then $\exists u \in Z^+$ such that $D_u \oplus \{0\} = D_{u+i} \oplus \{0\}$, $\forall i = 1, 2, 3, \ldots$. So, $D_u = D_{u+i}\ \forall i$.
Similarly, for (DCC).
We can generalize the proposition above for finite index of direct sum of LS-submodules.
Remark 2.6: Let $M_1, \ldots, M_t$, be modules and $R = ann(M_1) + \ldots + ann\ (M_t)$, if M_i, with ACC(DCC) on LS- submodules, ($\forall i = 1, \ldots, t$), then $M_1 \oplus \ldots \oplus M_t$ is with ACC (DCC) on LS- submodules.
Proof: Clear by induction**.**
Theorem 2.7: Let $X_1, X_2, \ldots, X_t$, be R-submodules of a module M such that $R = ann(M_1) + ann\ (M_2) + \ldots + ann\ (M_t)$, if M/X_i with ACC (DCC) on LS- submodules, for each $i = 1, 2, \ldots, t$, then $\frac{M}{X_1 \cap X_2 \cap \ldots \cap X_t}$ is with ACC (DCC) on LS-submodules.
Proof: Let $\frac{M}{X_t}$ with (DCC) on LS- submodules, for each $i = 1, 2, \ldots, t$, then by Remark 2.6 $\frac{M}{X_1} \oplus \frac{M}{X_2} \oplus \ldots \oplus \frac{M}{X_t}$ satisfies (DCC) on LS- submodules. Let $\varphi_i: M \longrightarrow \frac{M}{X_t}$ be the natural epimorphism, for all $i = 1, 2, \ldots, t$. Define $\varphi : M \longrightarrow \frac{M}{X_1} \oplus \ldots \oplus \frac{M}{X_t}$ by $\varphi(m) = (\varphi_1(m), \ldots, \varphi_t(m))$. By the First Isomorphism Theorem [32], we have $M/Ker\varphi \cong Im\varphi$. We can easily note that $Ker\varphi = \{m \in M: \varphi(m) = 0\} = X_1 \cap X_2 \cap \ldots \cap X_t$, but $Im\varphi$ satisfies (DCC) on LS- submodule, therefore $\frac{M}{X_1 \cap X_2 \cap \ldots \cap X_t}$ with (DCC) on LS- submodules. Similarly, for (ACC).

Theorem 2.8 [22]**:** Let M be FGFM R- module and let I be any ideal in R then $I \ll_L R$ if and only if, $IM \ll_L M$.

Theorem 2.9: If M is finitely generated chained module with ACC (DCC) on LS-submodules, then $\frac{R}{ann(M)}$ with ACC(DCC) on LS-submodules.

Proof: Let $M = Rm_1 + Rm_2 + \ldots + Rm_t$, where $m_1 \ldots . m_t \in M$. For each $i = 1,2, \ldots t$, define: $\varphi_i: R \longrightarrow Rm_i$, by $\varphi(r) = rm$, for all $r \in R$. Notice that φ_i is an epimorphism. By The First Isomorphism Theorem $\frac{R}{Ker\varphi} \cong Rm_i \ \forall i = 1,2, \ldots, t$. But $Ker\varphi_i = \{r \in R;\ \varphi_i(r) = 0\} = ann(m_i)$, so $\frac{R}{ann(m_i)} \cong Rm_i$. Since M is chained then every submodule of M is small, hence LS- small and M satisfies DCC on LS- submodules, then by Proposition 2.4, Rm_i satisfies DCC on LS-submodules, for all $(i = 1,2, \ldots, t)$. Since by [33, Proposition 2.3(4)], $ann(M) = ann(m^1) \cap ann(m) \cap \ldots \cap ann(m_t)$, so by Theorem 2.7, $\frac{R}{ann(M)}$ with DCC on LS- submodules. Since DCC on LS - submodules is closed under isomorphism. Similarly, for (ACC).

Proposition 2.10: If M is FGFM R-module, then R with ACC (DCC) on LS- ideal if and only if M with ACC (DCC) of LS- submodules.

Proof: Let $N_1 \subseteq N_2 \subseteq N_3 \subseteq \ldots \subseteq N_k \ldots$ is an ascending chain of LS- submodule of M and because M is multiplication module thus, $N_i = I_i M$, for some ideal I_i of R for each i. So, $MI_1 \subseteq MI_2 \subseteq MI_3 \subseteq \ldots MI_k \subseteq \cdots$, but M is finitely generated then by [25, Theorem 1.7] and $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots \subseteq I_k \subseteq \ldots$ is ascending chain of LS- ideals in R by Theorem 2.8. And because R with ACC on LS- ideal, then $\exists s \in Z^+$, such that $I_s = I_{s+1} = \ldots$, hence $MI_s = MI_{s+1} = \ldots$ that is $N_s = N_{s+1} = \cdots$ which is M with ACC on LS- submodules. On other hand, let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots \subseteq I_s \subseteq \ldots$ is an ascending chain LS- ideals of R, so through Theorem 2.8 $MI_1 \subseteq MI_2 \subseteq MI_3 \subseteq \ldots MI_s \subseteq \cdots$ is an ascending chain of LS- submodule of M. But M is originally with ACC on LS submodules then $\exists s \in Z^+$, such that $MI_s = MI_{s+1} = \ldots$. And because M is FGFM module thus, $I_s = I_{s+1} = \cdots$, [34]. So, R with ACC on LS- ideals of R. Similarly, for (DCC).

Proposition 2.11: Let M be a module, with ACC on LS- submodules and X is a submodule of M, then M/X with ACC on LS- submodules of M.

Proof: Let $X_1/X \subseteq X_2/X \subseteq \cdots$ be ACC on LS- submodules of M/X then $X_1 \subseteq X_2 \subseteq \cdots$. But $X_i/X \ll_L M/X$ then $X_i \ll_L M$ for all i by Lemma 2.1(4), thus $X_1 \subseteq X_2 \subseteq \cdots$ is ascending chain of LS- submodule of M. Hence, there exits $s \in Z_+$, such that $X_s = X_{s+1} = \cdots$, therefore M/X with ACC on LS- submodules. Similarly, for (DCC). Therefore, we get the following outcome:

Theorem 2.12: These statements are equivalent if M is FGFM R-module.
1. M an R-module with ACC (DCC) on LS-submodules;
2. R with ACC (DCC) on LS- ideals;
3. $W = End_R(M)$ with ACC (DCC) on LS- ideals;
4. M with ACC (ACC) on LS- submodules as W- module.

Proof: 1 ⇒) 2 Clear through Proposition 2.10.
2 ⇒)3 Acording to M is FGFM module, thus $R \cong W$ by [35, Theorem 3.2] so, R with ACC (DCC), $W = End\ (M)$ with ACC (DCC) on LS- ideals.
3 ⇒)4 Obtained from Proposition 2.10.
4 ⇒) 1 By Proposition 2.10, R with ACC (DCC) on LS-small ideals, $R \cong W$ [35]. So, R with ACC (DCC) on LS- ideals and by Proposition 2.10, M with ACC (DCC) on LS- submodules.

Proposition 2.13: The sum of any two LS-submodules of an R-module M is an LS-submodule. If X is LS and closed submodule in M such that X and $\frac{M}{X}$ with ACC (DCC) on LS- submodules, then M with ACC (DCC) on LS- submodules.

Proof: Suppose $B_1 \subseteq B_2 \subseteq \cdots$ be ascending chain of LS-submodules of M. Then by Lemma 2.1(3), $B_i \cap X$ is LS-submodules of M, for all $i = 1,2,..$ but $(B_i \cap X) \subseteq X$, so $B_i \cap X \ll_L X$, for each $i = 1,2,...$ by Lemma 2.1(1). Also, $B_i + X$ is an LS-submodule of M (by our assumption), hence $\frac{B_i + X}{X}$ is LS-submodule of $\frac{M}{X}$. $\forall i = 1,2,...$ by Lemma 2.1(2). Now, the two following ascending chain of LS- submodules are for X and $\frac{M}{X}$ respectively. $B_1 \cap X \subseteq B_2 \cap X \subseteq \cdots$, and $B_1 + X/X \subseteq B_2 + X/X \subseteq \cdots$, but X and $\frac{M}{X}$ are with ACC on LS-submodules. So there are $t_1, t_2 \in Z^+$, such that $B_n \cap X = B_{t1} \cap X$, for each $n \geq t_1$, and $B_n + X/X = B_{t2} + X/X$, for each $n \geq t_2$. By Second Isomorphism Theorem, $B + X/X \cong B/B \cap X$, so $B_n + X/X \cong B_n/B_n \cap X$. Hence, $B_n/B_n \cap X = B_{t2}/B_{t2} \cap X$ which means $B_n \cap X = B_{t2} \cap X$, for each $n \geq t_2$. Let $t = \max\{t_1, t_2\}$. Thus, $B_n \cap X = B_t \cap A$ for each $n \geq t$ and $B_n \cap X = B_t \cap B_n$, for each $n \geq t$. Now, for each $n \geq t$, $B_n = B_n \cap (B_n + X) = B_n \cap (B_t + X) = B_t \cap (B_t + X) = B_t$. Thus, M is with ACC on LS-submodules. By similarity can be prove that, M with DCC on LS- submodules.

Proposition 2.14: Let M be an R-module and $\bar{R} = \frac{R}{ann(M)}$, then an R-module M with (DCC) on LS-submodules if and only if an $\bar{R}$-module M with DCC on LS-submodules.

Proof: Assume that the R-module M with DCC on LS-submodules. We want to prove that M as $\bar{R}$-module with DCC on LS-submodules. Suppose that $X_1 \supseteq X_2 \supseteq ...$, is a descending chain of LS-submodules of an $\bar{R}$-module M. One can show easily that for all $n \in Z^+$, X_n is an LS-submodule of M as R-module. Since M as R-module with DCC on LS-submodules, then $\exists s \in Z^+$, such that $X_n = X_s$, for each $n \geq s$. Hence, M as $\bar{R}$-module with DCC on LS-submodules.

By the same argument of the proof, we can prove the converse

3. Conclusions

We defined large small submodules that is with ascending chain condition (ACC) and descending chain condition (DCC) on LS- submodules by generalization on small submodules and large small submodule, respectively. We also presented several key properties and illustrative examples, which will serve as a foundation for future research and establish a connection between our work and previous studies in our field of work.

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