Pseudo – Normal Operators

Adil G. Naoum and Sadiq N. Nassir
Department of Mathematics, College of Science, University of Baghdad. Baghdad-Iraq.

Abstract

In this paper we introduce a class of operators on a Hilbert space that contains properly the classes of normal operator, hyponormal operators, M – hyponormal operators, *- paranormal operators and dominat operators. We also give some properties of these operators. We call these operators Pseudonormal.

الخلاصة

في هذا البحث ندرس صنفاً من المؤثرات على فضاء هلبرت يضم كلا من صنف المؤثرات السوية المؤثرات فوق السوية ، المؤثرات فوق السوية من النمط M ، المؤثرات الموازية للسوية من النمط* والمؤثرات المهيمنة ويطلق على هذا الصنف اسم صنف المؤثرات السوية الكاذبة. كما ندرس بعض خواص هذه المؤثرات.

Introduction

Let H be a separable a complex Hilbert space and B(H) be the Banach algebra of all bounded linear operators on H as is customary, we refer toT*T as the self commutator of T, and is denoted by [T*,T] the operator is normal operator if [T*,T]=0.and T is hyponormal if $T^*T - TT^* \geq 0$. Or equivalently $\| T^*x \| \leq \| Tx \| \quad \forall x \in H$. The operator is called a dominant operator if for each $\lambda \in ¢$ there exists a number $M_\lambda > 0$ such that

$\| (T^* - \overline{\lambda})x \| \leq M_\lambda \| (T - \lambda)x \|$ for all $x \in H$. Furthermore if the constants M_λ are bounded by a positive number M, then T is called M-hyponormal

Operator [4], finally the operator Tis called a*- paranormal operator if $\| T^* x \|^2 \leq \| T^2 x \|$ for every unit vector x in H,[5].

In this paper we introduce a class of operators that contains the class of* - paranormal operators, dominant operators properly. In particular, it contains the class of normal operators, hyponormal operators and M- hyponormal operators. We call the elements of this class pseudo normal operators We study the basic properties of these operators of these operator and we give some sufficient conditions for a pseudo normal operator to be normal.

1 : Difinition 1.1

Let $T \in B(H)$. We call T a pseudo normal operator if $Tx = \lambda x$ for some $x \in H$, $\lambda \in ¢$, then $T^*x = \overline{\lambda}x$. i.e., if x is an eigen vector for T with eigen value λ then x is an eigenvector for T* with eigen- value $\overline{\lambda}$.

Examples 1.2

1. It is clear that if $\sigma_p(T) = \Phi$ then T is psendonormal operator where $\sigma_p(T)$ is the point spectrum of T.
2. Let $H = \ell_2(¢) = \{x : x = (x_1, x_2, x_3 \ldots\ldots xn, ..) : \sum_{i=l}^{\infty} |x_i|^2 < \infty\}$ the unilateral shift is the operator on H defined by $U(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, ..)$. It is known that U has no eigen-values hence is pseudo normal operator.
3. Let $H = L^2[0,1] = \{f; f:[0,1] \to R$ such that $\int_0^1 |f(x)|^2 < \infty\}$ the Volterra integration operator v is defined on H, as folloews:

$$(vf)_x = \int_0^x f(y)\,dy \quad [3.p.98],$$

It is known that every Volterra operator isquasinilpotent operator i.e $\sigma(T) = \{0\}$ and $\sigma_p(v) = \Phi$. hence v is pseudo normal operator .

Now we give an example of an operator that is not pseudo normal operator.

4. Let H = $\ell_2(¢)$, define T: H→ H as follows: $T(X_1, X_2, X_3, \ldots) = (0, X_1, 0,0,0,\ldots)$. It is easy to check that

$T^*(X_1, X_2, X_3, \ldots) = (X_2, 0,0,0,\ldots)$ and T is nilpotenet operator, σ(T) ={0}.

If we take $X=(0, X_2, X_3, \ldots)$ such that $X_2 \neq 0$ then TX = 0 but $T^*X = T^*(0,X_2, X_3\ldots) \neq 0$

Therefore, T is not pseudo normal operator .

Remark 1.3

1. It is easy to see that every dominant, in particular, every M- hyponormal normal operator is pseudo normal operator.
2. Every * paranormal operator is pseudo normal, to see this let Tx = λx,we my assume $\|x\| = 1$, then $\|T^*x\|^2 \leq \|T^2x\| = |\lambda|^2$. Now $\|(T^* - \bar{\lambda}I)x\|^2 \leq \|T^*x\|^2 - \bar{\lambda} < Tx, x > - \lambda < x, Tx > + |\lambda|^2 \leq 0$. Therefore $\|T^* - \bar{\lambda}Ix\|^2 = 0$ which implies $T^*x = \bar{\lambda}x$.

3. Note that if λ is an eigen value for T and $\bar{\lambda}$ is eigenvalue for T* then it is not necessary that T is pseudo normal operator, see example(1.2)(4).

For any operator T∈ B(H). we set Re(T) = ½(T+T*) and Im(T) = ½i(T-T*)

Propositin 1.4

Let T be a pseudo normal operator, then

1- if $\Phi \neq \sigma_p(T) \subseteq R$ then $0 \in \sigma_p(ImT)$ and $\sigma_p(T) \subseteq \sigma_p(ReT)$.

2- $Re\,\sigma_p(T) \subseteq \sigma_p(ReT)$ and $Im\sigma_p(T) \subseteq \sigma_p(ImT)$

3- If $\sigma_p(T) \neq \varphi$, then $0 \in \sigma_p([T^*, T])$

Proof

1- Since $\sigma_p(T) \neq \Phi$, then there exists λ ∈ ¢ such that Tx= λx for x ≠ 0

Now

$$(ImT)x = \frac{1}{2i}(T - T^*)x = \frac{1}{2i}(Tx - T^*x)$$

$$= \frac{1}{2i}(\lambda x - \lambda x) = 0 \text{ hence}$$

$0 \in \sigma_p(ImT)$, let $\lambda \in \sigma_p(T)$, then TX= λx for x ≠ Now(ReT) x= ½ [T+T*] x = ½[Tx+T*x] = ½(λ + $\bar{\lambda}$)Ix = λx , hence $\lambda \in \sigma_p(ReT)$.

2- If $\sigma_p(T) = \phi$ there is nothing to prove.

Let $\lambda \in \sigma_p(T)$, then Tx = λx for some x ≠ 0.

$$\text{Now}[Re(T) - Re\lambda]\,x = \frac{1}{2}[(T+T^*) - (\lambda + \bar{\lambda})Ix]$$

$$= \frac{1}{2}[(T+T^*)\,x - (\lambda + \bar{\lambda})Ix]$$

$$= \frac{1}{2}[(\lambda x + \overline{\lambda x}) - (\lambda + \bar{\lambda})Ix] = 0$$

Therefore, $Re\lambda \in \sigma_p(ReT)$

$$[ImT - (Im\lambda)I]x = \frac{1}{2i}[(T-T^*) - (\lambda - \bar{\lambda})I]x$$

$$= \frac{1}{2i}[(T-T^*)x - (\lambda - \bar{\lambda})Ix] = \frac{1}{2i}[(\lambda x - \overline{\lambda x}) - \lambda x + \bar{\lambda}x]$$

=0, hence $Im(\lambda) \in \sigma_p(ImT)$.

3. Since $\sigma_p(T) \neq \phi$, then there exists λ∈¢ such that Tx = λx for some x ≠ 0∈H Therefore [T*,T]x = (T*T – TT*)x

$$= T^*(\lambda x) - T(\bar{\lambda}x) = |\lambda|^2 x - |\lambda|^2 x = 0$$

Hence , $0 \in \sigma_p([T^*, T])$.

Recall that if T is an operator and λ is a scalar, the null space of the operator T-λI is called the λ-th eigenspace of T and is denoted by $E_T(\lambda)$, $E_T(\lambda) = \{x \in H : Tx = \lambda x\}$, it is easy to check that $E_T(\lambda)$ is closed linear subspace of H, moreover,$E_T(\lambda) \neq \{0\}$ if and only if λ is an eigenvalue for T . A non zero vector x is an eigenvector for T if x belongs to some λ- eigenvector of T.

Proposition 1.5.

Let T be a pseudo normal operator , then :

1- $E_T(\lambda) \perp E_T(M)$ whenever λ ≠ M.

2- For a fixed scalar λ, $E_T(\lambda)$ redues T and $T \mid E_{T(\lambda)}$ is normal operator, morever $\overline{\sum_{\lambda \in \sigma_p(T)} \oplus E_T(\lambda)}$ reduces T.

Proof

1. Let $x \in E_T(M)$ and $y \in E_T(\lambda)$ then

   $M(x,y) = (Mx,y) = (Tx,y) = (x,T^*y) = (x, \bar{\lambda}y) = \lambda(x,y)$

   Thus(M- λ)(x,y)= 0 but M ≠ λ then (x,y)=0

2- First we prove $E_T(\lambda)$ reduces T . We show that $E_T(\lambda)$ is invariant under T and T* .

T(T*x) = T($\bar{\lambda}$ x) = $\bar{\lambda}$ (Tx)= $\bar{\lambda}\lambda$ = λ(T*x).

Hence E(λ) = λ(Tx) and E(λ) is invariant under T.

T(T*x)=T($\bar{\lambda}$x)= $\bar{\lambda}$(Tx)= $\bar{\lambda}\lambda$x= λ(T*x). Hence $E_T(\lambda)$ is invariant under T* and $E_T(\lambda)$ reduces T.

To prove $T|E_{T(\lambda)}$ is normal , Let $x\in E_T(\lambda)$

$T^*Tx=T^*(Tx) = T^*(\lambda x)= \lambda(T^*x)=|\lambda|^2x$.

On the other hand

$TT^*x = T(T^*x) = T(\bar{\lambda} x)= \bar{\lambda} (Tx) =|\lambda|^2x$.

Therefore.

$TT^*x= T^*Tx$ $\forall x \in E_T(\lambda)$ and $T|E_{T(\lambda)}$ is normal, we leave the proof of the last assertion to the reader.

Before we give our next result we need a lemma.

Lemma 1.6

Every *-paranormal operator T has an approximate eigenualue λ such that $|\lambda| =\|T\|$,

Proof see [5].

Now by the above lemma. It is clear that example(1.2)(2) is not *- paranormal operator but it is pseudo normal operator.

Proposition 1.7

Let $T \in B(H)$ such that for all $\lambda\in \sigma_p(T)$ $|\lambda|=\|T\|$ then T is pseudo normal operator.

Proof

Let $x \in H$ such that Tx = λx

Now $\|(T^* - \bar{\lambda} I) x\|^2$ = <(T* - $\bar{\lambda}$ I) x, (T*- $\bar{\lambda}$ I)x>

= <T*x –T*x> - <T*x, $\bar{\lambda}$ x> - <$\bar{\lambda}$ x,T*x > + <$\bar{\lambda}$ x, $\bar{\lambda}$ x>

$= \|T^*x\|^2 - \lambda <x,Tx> - \bar{\lambda} <Tx,x> + |\lambda|^2\|x\|^2$

$\leq \|T^*\|^2\|x\|^2 - |\lambda|^2\|x\|^2 - |\lambda|^2\|x\|^2 + |\lambda|^2\|x\|^2 = 0$

Therefor T*x = $\bar{\lambda}$x hence T is pseudo normal operator.

Theorem 1.8

If T is pseudo normal operator then T can be expressed uniquely as the direct sum $T = T_1 \oplus T_2$ defined on the space $H=H_1\oplus H_2$ with the following properties:

1. H_1 is the closure of the space spanned by the eigen vectors of T.
2. T_1 is normal.
3. $\sigma_p(T_2) = \phi$
4. T is normal if and only if T_2 is normal.

Proof

1. Let $H_1 = \sum_{\lambda\in\sigma_p(T)} \oplus E_T(\lambda)$ Since $H_1^{\perp}$ is a closed linear subspace,

Then $H = H_1\oplus H_2$ where $H_2 = H_1^{\perp}$ Let $T_1 = T|_{H1}$

And $T_2 = T|_{H2}$

Then $T = T_1 \oplus T_2$ uniquely.

2. Let $H_1 = E_T(\lambda_1)\oplus E_T(\lambda_2)\oplus E_T(\lambda_3)\oplus....$ and $x \in H_1$. Then $x = x_{\lambda_1}+x_{\lambda_2}+x_{\lambda_3}+.....$ for any $x_{\lambda i} \in E_T(\lambda i)$, i. e., $x = \sum_{\lambda\in\sigma_p(T_1)} x_{\lambda i}$ for each I and $\sum |x|^2_{\lambda i} < \infty$.

$$T_1{}^*T_1\left[\sum_{\lambda i\in\sigma_p(T)_1} x_{\lambda i}\right]=T_1{}^*\left[\sum_{\lambda i\in\sigma_p(T)_1} T_1x_{\lambda i}\right]$$

$$=\lambda_1T_1{}^*x_{\lambda_1}+\lambda_2T_2{}^*x_{\lambda_2}+\lambda_3T_3{}^*x_{\lambda_3}+.....$$

$$=|\lambda_1|^2x_{\lambda_1}+|\lambda_2|^2x_{\lambda_2}+|\lambda_3|^2x_{\lambda_3}+....$$

And

$$T_1T_1{}^*x = T_1T_1{}^*(x_{\lambda_1}+x_{\lambda_2}+x_{\lambda_3}+..)$$

$$=\bar{\lambda_1}\, Tx_{\lambda_1}+\bar{\lambda_2}\, Tx_{\lambda_2}+\bar{\lambda_3}\, Tx_{\lambda_3}+....$$

$$=|\lambda_1|^2x_{\lambda_1}+|\lambda_2|^2x_{\lambda_2}+|\lambda_3|^2x_{\lambda_3}+....$$

hence T_1 is normal

3. Suppose that, $\sigma_p(T_2)\neq\varnothing$ and let $M\in\sigma_p(T_2)$. Then there exists

$x\neq 0 \in H_2$ such that $(T_2-M)x=0$

Now $T(0+x)=T_2x=Mx=M(0+x)$.

$H_2=H_1$ and $x\neq o$ therefor , $\sigma_p(T_2)=\varnothing$

and T_2 is pseudo normal operator .

4- Since

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \in B(H_1 \oplus H_1^{\perp}) \text{ and}$$

$$T = \begin{bmatrix} T_1^* & 0 \\ 0 & T_2^* \end{bmatrix}$$

$$\text{Then} T^*T = \begin{bmatrix} T_1^*T_1 & 0 \\ 0 & T_2^*T_2 \end{bmatrix} \text{ and}$$

$$TT^* = \begin{bmatrix} T_1T_1^* & 0 \\ 0 & T_2T_2^* \end{bmatrix}$$

Thus $T^*T=TT^*$ if and only if $T_2{}^*T_2=T_2T_2{}^*$

2 .Recall that a family of closed linear subspaces in H is said to be a total family in case the only vector x in H , which is orthogonal to every subspace of H belonging to the family is X=0. [l,p.l67].

The following theorem gives a condition under which a pseudo normal operator is normal operator.

Theorem 2.1

Let $T \in B(H)$ be a pseudo normal operator, if the eigen spaces , $E_T(\lambda)$ of T form a total family, then T is a normal operator

Proof

Let H_0 be the null space of TT* -T*T, the problem is to show that H_0=H or equivalently $H_0^{\perp} = \{0\}$ claim $E_T(M) \subseteq H_0$ for all M.

Let $x \in E_T(M)$, thus $T^* x \in E_T(M)$. therefore TT*=M(T*x)=T*(MX)=T*Tx.

Thus (TT*-T*T)x=0 and $x \in H_o$

It follows that if $x \perp H_0$, then $x \perp E_T(M)$ for all M

But the eigenspaces form a total family. Hence x=o then

(TT*-T*T)x=0 $\forall$ $x \in H$, thus T*T=TT* and T is normal operator.

Corollary 2.2

Let T be a *-paranormal or dominant operator. If the eigen spaces, $E_T(\lambda)$ of T from a total family then T is normal operator.

Lemma 2.3 [3,P 96]

Let $T \in B(H)$ be a compact operator and

$\lambda \neq 0 \in \sigma(T)$ Then $\lambda \in \sigma_p(T)$

Theorem 2.4

Let T be a pseudo normal operator which satisfies the following conditions, then T is normal operator

1 - T is compact.

2- If $T_1=T/_{M_0}$ where M_0 is an invariant subspace of T then

$\left\|T_1^n\right\| = \left\|T_1\right\|^n$ for every positive integer n

Proof

Since T is pseudo normal operator , then by theorem 1.8

$T = T_1 \oplus T_2$, where T_1 is normal operator and $\sigma_p(T_2)=\varnothing$

It is enough to show that T_2 is normal operator since T is compact then T_2 is compact. By lemma 2.3 and theorem 1.8.

$\sigma(T_2)=\{0\}$

Hence

$$0 = r(T_2) = \lim_{n\to\infty} \left\|T_2^n\right\|^{\frac{1}{n}} =$$

$$\lim_{n\to\infty} \left\|T_2\right\| = \left\|T_2\right\|, [2, P.5]$$

Thus T_2 =0 and T is normal operator .

Remark 2.5

If T is a pseudo normal compact operator on H it is not necessary T is normal operator, in fact, in example 1.2 (3) we see that V is a pseudo normal compact operator but V is not normal operator.

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