N - Commuting Maps on Semiprime Rings

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Abstract

Let R be a ring with center $Z(R)$, and n, m are arbitrary positive integers. We show that a semiprime ring R with suitable - restriction must contain a nonzero central ideal, if it admits a derivation d which is nonzero on a non trivial left ideal U of R and the map $x \rightarrow [x^m, d(x)]$ satisfies one of the following:

i- n - commuting on U.

ii- n - skew - commuting on U.

الخلاصة

لتكن R حلقة مركزها Z(R), m,n اعداد صحيحة موجبة, بينا في هذا البحث ان الحلقة الاولية R تحت شروط مناسبة يجب ان تحوي على مثالي مركزي غير صفري, اذا سمحت R بوجود اشتقاق غير صفري d على مثالي يساري U من R غير تافه والدالة X← [x^m,d(x)] تحقق احد الشروط الاتية:

1. ابدالية – n على U.
2. ابدالية ملتوية – n على U.

Introduction

Let R be a ring with center $Z(R)$, S be a nonempty subset of R and n be a positive integer, A mapping F of R into itself is called n - centralizing on S (resp n-commuting), if $[x^n, F(x)] \in Z(R)$ for all $x \in S$ (resp $[x^n, F(x)] = 0$ for all $x \in S$). For $n = 1$, F is simply called centralizing on S (resp commuting on S) and F is n - skew - centralizing (resp n - skew - commuting) if $x^n F(x) + F(x) x^n \in Z(R)$ for all $x \in S$ (resp $x^n F(x) + F(x) x^n = 0$ for all $x \in S$). The classical result of Posner [6] states that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commuting. A lot of work has done during the last twenty-five years in this field (see [1, 2, 3, 4, 5, 7 and 8]). For $n > 1$, Majeed and Niufengwen studied these maps [5], and they proved the following theorem.

Theorem A

Let n be a positive integer, R be a semiprime ring, which is $(n+1)$ torsion - free if $n \geq 2$, and 6-torsion - free if $n = 1$, and let U be a nonzero left ideal of R. if R admits a derivation d which is nonzero on U, and the map $x \rightarrow [x, d(x)]$ is n-centralizing on U, then R contains a nonzero central ideal.

In this paper we generalize theorem A and we give an analogous result when the map $x \rightarrow [x^m, d(x)]$ is n - commuting or n - skew - commuting.

§ 1 Preliminaries

We begin with some definitions, remarks and lemmas, that we use in the proof of the main theorem.

Let R be a semiprime ring, U is a nonzero left ideal of R, d is a derivation on R, and $\Omega = \{P_\alpha / \alpha \in \Lambda\}$ be family of prime ideals of R, such that $\bigcap_{\alpha\in\Lambda} P_\alpha = \{0\}$. If P_α has any one of the following properties,

(a) $U \subseteq P_\alpha$.

(b) $d(U) \subseteq P_\alpha$,

(c) $0 < \text{char}\ (R / P_\alpha) \le m$ where m denotes an arbitrary positive integer, then we call P_α an extraordinary prime ideal. The ideals will be denoted by Ω_1.

Remark 1:

If $P_\alpha \in \Omega_1$ then $m![x^n,[x,d(x)]] \in P_\alpha$ for all $x \in U$.

Remark 2:

If for each $P_\alpha \in \Omega$, $m![x^n,[x,d(x)]] \in P_\alpha$ for all $x \in U$. Then

$m![x^n,[x,d(x)]] = 0$ for all $x \in U$.

Now we give several lemmas, that we need in the proof of the main result.

Lemma 1: [3]

Let n be a positive integer, R be an n! - torsion - free ring, and f be an additive map on R. For $i = 1,2,..., n$ let $F_i(X, Y)$ be a generalized polynomial which is homogenous of degree i in the nonzero commuting indeterminates X and Y.

Let $a \in R$, and $< a >$ the additive subgroup generated by a, if $F_n(x,f(x)) + F_{n-1}(x,f(x)) + ... + F_1(x,f(x)) \in Z(R)$ for all $x \in < a >$ then $F_i(a, f(a)) \in Z(R)$ for $i = 1,2,..., n$.

Lemma 2: [3]

Let n be a positive integer, R be a ring, and P be a prime ideal of R such that $\text{char}\ (R / P) > 0$ or $\text{char}\ (R / P) = 0$. Let f, F be as in lemma 1, if $F_n(x,f(x)) + F_{n-1}(x,f(x)) + ... + F_1(x,f(x)) \in P$ for all $x \in < a >$, then $F_i(a, f(a)) \in P$ for $i = 1,2,..., n$.

Lemma 3: [3]

Let R be a ring and P be a prime ideal of R such that $\text{char}\ (R / P) \ge n$. if $a_1, a_2, ..., a_{n+1}$ are elements of R such that $a_1 x a_2 x a_3 ... a_n x_{n+1} \in P$ for all $x \in R$, then $a_i \in P$ for some $i = 1,2,..., n+1$.

Lemma 4: [3]

Let R be a ring and U is a nonzero left ideal of R containing no nonzero nilpotint elements, then U contains nonzero elements which are left zero divisors in R.

§ 2 A Theorem on Commuting Maps

Lemma 5:

Let n be a positive integer and R is a ring. If R admits a nonzero derivation d then $[x^n, d(x)] = [x, d(x^n)]$ for all $x \in U$.

Proof:

The proof is by induction, if $n = 1$, the relation is clear; suppose that the relation is true for $n - 1$ that is $[x^{n-1}, d(x)] = [x, d(x^{n-1})]$.

We have to prove the relation is true for n.

$$[x^n, d(x)] = x^{n-1}[x, d(x)] + [x^{n-1}, d(x^2)]x$$
$$= [x, x^{n-1}d(x)] + [x, d(x, d(x^{n-1})x]$$
$$= [x, x^{n-1}d(x)] + d(x^{n-1})x = [x, d(x^n)]$$

One can easily prove the following remark.

Remark 3:

Let n be a positive integer and R is a ring that admits a nonzero map d. if d is n - centralizing on R then $[x, d(x)]$ is n-commuting.

The main results.

Theorem 1:

Let n, m be a positive integers and R is semiprime ring. Which is $(n + m)!$ - torsion - free, let U be a left ideal of R. Suppose R admits a derivation d which is nonzero on U. if the map $x \to [x^m, d(x)]$ satisfies one of the following conditions:

i) n - commuting on U.
ii) skew - n - commuting on U.

Then R contains a nonzero central ideal.

In order to prove the theorem, we need the following lemma.

Lemma 6:

Let R satisfy the hypothesis of the above theorem and let $\Omega = \{P_\alpha / \alpha \in \Lambda\}$ be family of prime ideals such that $\bigcap_{\alpha\in\Lambda} P_\alpha = \{0\}$.

Let $\Omega_1 = \{P_\alpha \in \Omega / d(U) \subseteq P_\alpha\}$ and $P \in \Omega / \Omega_1$. If $a \in U$ and $a^2 \in P$ then $a \in P$.

Proof:
If d satisfies part (I), then

$[x^n,[x^m,d(x)]] = 0$ ……… (1) for all $x \in U$

$x^{n+m}d(x) - x^n d(x)x^m - x^m d(x)x^n + d(x)x^{m+n} = 0$ ……..(2)
for all $x \in U$. Replacing x by ra in (2), we get

$(ra)^{n+m}d(ra) - (ra)^n d(ra)(ra)^m - (ra)^m d(ra)(ra)^n + d(ra)(ra)^{m+n} = 0$ … (3)
for all $r \in R$. Right multiplying (3) by a, and using the hypothesis $a^2 \in P$, we get $(ra)^{2n} rd(a)a \in P$.
Then by lemma (3).

$d(a)a \in P$ ……..(4)
Linearizing (1) and using lemma (1) we get
$[x^{n-1}y + x^{n-2}yx + ... + yx^{n-1},[x^m,d(x)]] + [x^n,[x^{m-1}y + x^{m-2}xy + ... + yx^{m-1},d(x)]] + [x^n,[x^m,d(y)]] = 0$ …(5)
for all $x, y \in U$.
Replacing x by ra and y by a in (5), then by (4) and $a^2 \in P$, we obtain
$[a(ra)^{n-1},(ra)^m,d(ra)]] + [(ra)^n,[a(ra)^{m-1},d(ra)]] + [(ra)^m,d(a)]]$
$= a(ra)^{n+m-1}d(ra) - a(ra)^{n-1}d(ra)(ra)^m - (ra)^m d(ra)a(ra)^{n-1} + d(ra)(ra)^m d(ra)^{n-1} + (ra)^n a(ra)^{m-1}d(ra) - (ra)^n d(ra)a(ra)^{m-1}a(ra)^{m-1}d(ra)(ra)^n + d(ra)a(ra)^{m-1}(ra)^n + (ra)^{n+m}d(a) - (ra)^n d(a)a(ra)^m (ra)^m d(a)(ra)^n + d(a)(ra)^{m+n} = a(ra)^{n+m-1} d(ra) - a(ra)^{n-1}d(ra)(ra)^m - a(ra)^{m-1}d(ra)(ra)^n (ra)^{n+m}d(a) - (ra)^n d(a)(ra)^m - (ra)^n d(a)(ra)^m - (ra)^m d(a)(ra)^n + d(ra)(ra)^{m+n} = 0$.
Thus
$a(ra)^{n+m-1}d(ra) - a(ra)^{n-1} d(ra)(ra)^m - a(ra)^{m-1}d(ra)(ra)^n + [(ra)^n,[(ra)^m,d(a)]] \in P$ ….. (6)

Replacing x by ra and y by ara in (5), then by (3), (4) and using $a^2 \in P$, we get $[(ara)(ra)^{n-1},(ra)^m,d(ra)]] + [(ra)^n,[(ara)(ra)^{m-1},d(ra)]] + [(ra)^n,[(ra)^m,d(ara)]] = a(ra)^{n+m}d(ra) - a(ra)^n d(ra)(ra)^m - (ra)^m d(ra)a(ra)^n + d(ra)(ra)^m (ara)(ra)^{n-1} + (ra)^n a(ra)^m d(ra) - (ra)^n d(ra)a(ra)^m - a(ra)^m d(ra)(ra)^n + d(ra)(ra)^{m+n} + (ra)^{n+m}d(ara) - (ra)^n d(ara)(ra)^m - (ra)^m d(ara)(ra)^n + d(ara)(ra)^{m+n} = a(ra)^{n+m}d(ra) - a(ra)^n d(ra)(ra)^{m+1} - a(ra)^m d(a)(ra)^n + (ra)^{n+m}d(a)(ra) - (ra)^n d(a)(ra)^{m-1} - (ra)^m d(a)(ra)^{n+1} + ad(ra)(ra)^{n+m} + d(a)(ra)^{m+n+1} = a[(ra)^n,[(ra)^m,d(a)] + [(ra)^m,d(a)]]ra = 0$.
From (1), we get

$[(ra)^n,[(ra)^m,d(a)]]ra \in P$. …….. (7)
For all $r \in R$.
Right multiply (6) by ra and comparing the result with (7), we obtain
$a(ra)^{n+m+1}d(ra)(ra) - a(ra)^n - 1d(ra)(ra)^{m+1} - a(ra)^{m-1}d(ra)(ra)^{n+1} \in P$, left multiplying the result by r and using (2), we get

$d(ra)(ra)^{n+m+1} \in P$ …….. (8) for all $r \in R$
Linearizing (8) and using lemma (2), we obtain $d(xa + ya)(xa + ya)^{n+m+1} \in P$, implies that

$d(xa)\{(xa)^{n+m}(ya) + ... + (ya)(xa)^{n+m}\} + d(ya)(xa)^{n+m+1} \in P$ …… (9)
for all $x, y \in R$.
Taking $y = a$ in (9), we conclude that $d(a^2)(xa)^{n+m+1} \in P$ for all $x \in R$, by lemma (3), we get $d(a^2) \in P$ then $d(a)a + ad(a) \in P$, and by (4), yield

$ad(a) \in P$. ……… (10)
From (8), (7), we obtain

$d(a)(ra)^{n+m+1} \in P$ for all $r \in R$, by lemma (3), either $a \in P$ or $d(a) \in P$. We may assume that $d(a) \in P$ otherwise we are finished.

For a giving, $r \in R$, $ara \in U$ and $(ara)^2 \in P$, so by the same way, either $ara \in P$ or $d(ara) \in P$.

The sets of r which alternatives hold are additive subgroup of R, therefore, either $aRa \subseteq P$ or $d(ara) \in P$ for all $r \in R$.

The first of these forces $a \in P$, so we assume hence forth that $d(a) \in P$ and $d(ara) \in P$ for all $r \in P$.

Replacing r by xy, we get

$$d(axya) = d(ax)ya + axd(ya) \in P \cdots (11)$$

for all $x, y \in R$.

Noting that by (3), we have

$d(ya)(ya)^{n+m} = (ya)^m d(ya)(ya)^n + (ya)^n d(ya)(ya)^m - (ya)^{n+m} d(ya) = (ya)^{m-1} y(d(aya) - d(a)ya)(ya)^n + (ya)^{n-1} y(d(aya) - d(a)(ya)(ra)^m - (ya)^{n+m-1} y(d(aya) - d(a)ya) \in P$.

For all $y \in R$.

Right multiply (11) by $(ya)^{n+m}$ and using the last result, we have

$d(ax)(ya)^{n+m+1} \in P$ for all $x, y \in R$, hence, $d(ax) \in P$ for all $x \in R$. Therefore, $ad(x)d(a)x \in P$, thus, $ad(x) \in P$ for all $x \in R$. Therefore,

$ad(RR) = ad(R)R + aRd(R) \subseteq P$, hence $aRd(R) \subseteq P$, then $d(R) \subseteq P$, contradicting the fact that $P \in \Omega/\Omega_1$, therefore, $a \in P$.

Now if d satisfies (ii) then

$$x^n[x^m, d(x)] + [x^m, d(x)]x^n = 0 \ldots.. (12)$$

for all $x \in U$.

The proof can be completed in a way similar to the proof of part (i), except replacing equation (1) by (12).

<u>Proof of theorem 1:</u>

Since R is semiprime then R has a family $\Omega = \{P_\alpha / \alpha \in \Lambda\}$ of prime ideals. such that $\bigcap_{\alpha \in \Lambda} P_\alpha = \{0\}$ .and

$\Omega_1 = \{P_\alpha \in \Omega / d(U) \subseteq P_\alpha\}$ [2]

i) for each $P_\alpha \in \Omega$ we have to show that

$$(n+m)![x^m, [x^m, d(x)]] \in P_\alpha \cdots \quad (13)$$

for all $x \in U$.

By remark 1, the equation (13) holds for each $P_\alpha \in \Omega_1$, so we have to show that equation (13) holds for each $P \in \Omega/\Omega_1$. Substituting $y = x^{m+1}$ in (5) and using lemma (1), we have

$[nx^{n+m}, [x^m, d(x)]] + [x^n, [mx^{2m}, d(x)]] + [x^n, [x^m, d(x^{m+1})]] =$
$nx^n[x^m, [x^m, d(x)]] + n[x^n, [x^m, d(x)]]x^m + mx^m[x^n, [x^m, d(x)]] + m[x^n, [x^m d(x)]]x^m + [x^n, [x^m, d(x)^m]]x + x^m[x^n, [x^m, d(x)]] \in P$.

Using (1), we obtain

$nx^n[x^m, [x^m, d(x)]] + [x^n, [x^m, d(x)]]x \in P$; using lemma (5) on the last result, we get $nx^n[x^m, [x^n, d(x)]] + [x^n, [x^{2m}, d(x)]]x \in P$. Therefore, $nx^n[x^m, [x^m, d(x)]] \in P$, also

$$(n+m)x^n[x^m, [x^m, d(x)]] \in P \cdots (14)$$

for all $x \in U$.

By lemma (4), the left ideal $U + P/P$ of R/P contains no elements which are nonzero left zero divisors in R/P, hence, (4) shows that $(n+m)![x^m, [x^m, d(x)]]P_\alpha \in \Omega$, for all $x \in U$. And $P \in \Omega/\Omega_1$. Therefor, $(n+m)![x^m, [x^m, d(x)]] \in P$, for each $P \in \Omega$. Hence,

$(n+m)![x^m, d(x)] \in \bigcap_{\alpha \in \Lambda} P_\alpha = \{0\}$

since R is $(n+m)!$ - torsion - free then $[x^m, [x^m, d(x)]] = 0$ for all $x \in U$.

By the same way we can get $[x^m, [x, d(x)]] = 0$ for the last equation. Then by theorem A the proof is finished.

ii) linearizing (12) and using lemma 1 give

$(x^{n-1}y + \ldots + yx^{n-1})[x^m, d(x)] + [x^m, d(x)](x^{n-1}y + \ldots + yx^{n-1}) + x^n[x^{m-1}y + \ldots + yx^{m-1}, d(x)] +$

$[x^{m-1}y + ... + yx^{m-1}, d(x)]x^n + x^n[x^m, d(y)]$
$[x^m, d(y)]x^n = 0$ ….. (15)
for all $x \in U$.
We have to show that for each $P \in \Omega / \Omega_1$ $(n+m)![x^m,[x^m,d(x)]] \in P$ for all $x \in U$.
Substituting $y = x^{m+1}$ in (15), we get
$nx^{n+m}[x^m, d(x)] + n[x^n, d(x)]x^{n+m}$
$+ mx^n[x^{2m}, d(x)] + m[x^{2m}, d(x)]x^n +$
$x^n[x^m, d(m+1)] + x^m, d(x^{m+1})]x^n$
$= n(x^{n+m}[x^m, d(x)] + x^n[x^m, d(x)]x^m$
$- x^n[x^m, d(x)]x^m + [x^m, d(x)]$
$x^{n+m}) + m(x^{n+m}[x^m, d(x)] +$
$x^n[x^m, d(x)]x^m + x^m[x^m, d(x)]x^n - x^n$
$[x^m d(x)]x^{n+m}) + x^n[x^m, d(x^m)]x +$
$x^{n+m}[x^m, d(x)] + x^m[x^m, d(x^m)]x^n$
$[x^m, d(x^m)]x^{n+1} = n(x^n(x^m[x^m, d(x)]$
$-[x^m, d(x)]x^m) + (x^n[x^m, d(x)] +$
$[x^m, d(x)]x^n)x^m) + m(x^m(x^n[x^m, d(x)]$
$+[x^m, d(x)]x^n) + (x^n[x^m, d(x)]$
$+[x^m, d(x)]x^n)x^m) + x^n[x^m, d(x^m)]x$
$+[s^m, d(x^m)]x^{n+1} + x^m(x^n$
$[x^m, d(x) + [x^m, d(x)]x^n) = 0$
using (1) and lemma 5, we get

$n(x^n(x^m[x^m,d(x)] - [x^m,d(x)]x^m) = 0$ ….. (16)
therefore,
$(n+m)!\, x^n[x^m,[x^m, d(x)]] \in P$.
By lemma (6) and lemma (4), the left ideal $U + P/P$ of R/P contains no nonzero divisors in R/P, hence, (15) shows that $(n+m)![x^m,[x^m,d(x)]] \in P$. So we have
(17) ….. $(n+m)![x^m,[x^m,d(x)]] \in P$ for all $x \in U$, for each $P \in \Omega/\Omega_1$. By remark 2, (17) holds for each $P \in \Omega_1$ then $(n+m)![x^m,[x^m,d(x)]] \in \bigcap_{\alpha \in \Lambda} P_\alpha$ for all $x \in U$.
Therefor $(n+m)![x^m,[x^m,d(x)]] = 0$. Since R is $(n+m)!$ - torsion - free we have $[x^m,[x^m,d(x)]] = 0$ for all $x \in U$.

The proof can be completed in a way similar to the proof of (I), we get
$[x^m, d(x)] = 0$ for all $x \in U$.
Then by theorem A, R contains a nonzero central ideal.
Before we state next theorem we need the following lemma

Lemma 8:
Let R be semiprime ring, R is (2n)! - torsion - free if $n \geq 2$ and 6 - torsion - free if $n = 1$, where n denotes an arbitrary positive integer. Let U be an additive subgroup closed under squaring and d a derivation on R. if the map $x \to [x^n, d(x)]$ is n - centralizing, then this map is n - commuting on U.

Proof:
We have
$[x^n,[x^n,d(x)]] \in Z(R)$ ………. (18)
for all $x \in U$.
Linearizing (18) and applying lemma 1 give
$[x^{n-1}y + x^{n-2}yx + ... + yx^{n-1}[x^n, d(x)]]$
$+[x^n,[x^{n-1}y + x^{n-2}yx + ... + yx^{n-1},$
$d(x)]] + [x^n[x^n, d(y)]] \in Z(R)$ for all $x \in U$.
Replacing y by x^{2n+1}
$[nx^{3n},[x^n, d(x)]] + n[x^n,[x^{3n}, d(x)]]$
$+[x^n, d(x^{2n+1})]] = nx^n[x^{2n}[x^n, d$
$(x)]] + nx^n[x^n,[x^{2n}, d(y)]] +$
$2n[x^n,[x^n,d(x)]]x^{2n} + n[x^n,[x^n,d(x)]x^{2n}]$
$+[x^n, x[x^n, d(x^{2n})]] =$
$(6n+1)x^{2n}[x^n,[x^n,d(x)]] + [x^n,d(x)^n]]x^{n-1} +$
$x^n[x^n,[x^n,d(x^n)]x \in Z(R)$.
By the lemma 4 yields
$(6n+1)x^{2n}[x^n,[x^n,d(x)]] +$
$nx^{n-1}[x^n,[x^n,d(x)]]x^{n+1} + nx^{2n-1}[x^n,[x^n,$
$d(x)]x = (8n+1)x^{2n}[x^n,[x^n,d(x)]] \in Z(R)$
.Since R is $(8n+1)$ - torsion - free, we get
$x^{2n}[x^n,[x^n,d(x)]] \in Z(R)$.
Commuting (19) with $[x^n, d(x)]$ and commuting the result with $[x^n, d(x)]$, we get
$[x^n,[x^n,d(x)]]^3 = 0$ for all $x \in U$.
Since the center al semiprime ring contains no nonzero nilpotent elements, we have $[x^n,[x^n,d(x)]]^3 = 0$ for all $x \in U$.

The following theorem is a new generalization of theorem 1 in [8]; the proof is easy and hence, is omitted.

Theorem 2:
Let n be a positive integer, R is a semiprime ring which is $2n!$ - torsion - free when $n \geq 2$, and 6 - torsion - free when $n = 1$, and let U be a nonzero left ideal of R. suppose that R admits a derivation d which is nonzero on U. if the map $x \to [x^n, d(x)]$ satisfies one of the following conditions

i) n - centralizing on U.
ii) Skew - n - commuting on U.

Then R contains a nonzero central ideal.

The following corollaries show that under certain conditions, R is commutative. The proofs are simple.

Corollary 1:
Let n be a positive integer, R is a prime ring such that $\text{char}R > 2n$ or $\text{char}R = 0$, and U is a nonzero left ideal of R. Suppose R admits a nonzero derivation d. if the map $x \to [x^n, d(x)]$ satisfies one of the following conditions:

i) n - commuting on U.
ii) Skew - n - commuting on U.

Then R is commutative.

Corollary 2:
Let n, m be a positive integers, R is a prime ring such that $\text{char}R > n + m$ or $\text{char}R = 0$, and d is derivation. If the map $x \to [x^m, d(x)]$ satisfies one of the conditions:

iii) n - commuting on U.
iv) Skew - n - commuting on U.

Then R is commutative.

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