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The Dynamics of A Square Root Prey-Predator Model with Fear

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Abstract

An ecological model consisting of prey-predator system involving the prey's fear is proposed and studied. It is assumed that the predator species consumed the prey according to prey square root type of functional response. The existence, uniqueness and boundedness of the solution are examined. All the possible equilibrium points are determined. The stability analysis of these points is investigated along with the persistence of the system. The local bifurcation analysis is carried out. Finally, this paper is ended with a numerical simulation to understand the global dynamics of the system.

Keywords: Prey-predator; Fear; Stability; Bifurcation.

ديناميكية نموذج الفريسة والمفترس ذو الجذر التربيعي مع الخوف

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قسم الرياضيات ، كلية العلوم ، جامعة بغداد ، بغداد ، العراق

الخلاصة

تم اقتراح ودراسة نظام بيئي يتكون من نظام الفريسة والمفترس والمتضمن الخوف. ان المفترس يفترس الفريسة بموجب دالة الاستجابة ذات الجذر التربيعي. وجود ووحداية وحدود الحل درست. جميع نقاط التوازن الممكنة حسبت. تحليل الاستقرارية لتلك النقاط بحثت مع اصرار النظام. كما تم اجراء تحليل التفرع المحلي. واخيرا اختتم البحث بحل النظام عدديا لفهم الديناميكية الشاملة للنظام

1. Introduction

It is well known that the dynamic relationship between prey and predator is very essential in both ecology and mathematical ecology due to its universal existence and importance. Mathematical modeling is one of the tools to study the effects of biological factors on the ecological systems, including the prey-predator system. This is the main reason in developing and studying different types of ecological models [1-3].

Many prey-predator models have been proposed and studied extensively in which either the predator kills the prey (direct effect) or that the presence of the predator affects the behavior of the prey population due to the fear of predation process (indirect effect). Most of these studies have been conducted with prey-dependent type of functional responses [4-6], general predator-dependent functional responses [7-8], and ratio-dependent functional responses [9-12]. In all the previously mentioned references, the functional responses only reflect the direct effect.

Later on, Zanette *et al.* [13] reached the conclusion that "fear of predation risk is powerful enough to affect wildlife populations even when predators are prevented from directly killing any prey".

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Therefore, the indirect effect is also important for understanding the behavior of the prey-predator system and needs to be investigated in more details. Accordingly, a number of researchers have recently taken into account studying the prey-predator system with the effects of fear to understand the effect of this factor on the dynamical behavior of the system [13-16]. A predator-prey model incorporating the cost of fear into prey reproduction was proposed and studied by Wang *et al.* [14]. They showed that high levels of fear can stabilize the predator-prey system by excluding the existence of periodic solutions. However, Panday *et al.* [15] investigated the impact of fear in a tri-trophic food chain model with Holling type II of functional responses. They concluded that chaotic dynamics can be controlled by fear factors. Later on, Pal *et al.* [16] investigated the impact of fear in a predator-prey model, where predator-prey interaction follows Beddington-DeAngelis functional response. They concluded that fear of predation risk can have both stabilizing and destabilizing effects.

Keeping the above in view, in this paper we modified the prey-predator model proposed by Pal *et al.* [17] so that it involves the effects of fear and then discussed their dynamical behavior in details.

2. The Model and Analysis

In this section, the prey-predator real-world system is formulated mathematically using square root functional response for describing the predation process. It is assumed that the prey grows logistically in the absence of predator while the predator's population decays exponentially in the absence of their prey. Moreover, the growth rate of prey is reduced due to fear of the predator. Hence it is reduced in the presence of predator so that the modified intrinsic growth rate of prey becomes $\frac{r}{1+ky}$ a monotonic decreasing function of both k and y [14]. Here k is the fear parameter of prey. Accordingly, the dynamics of such a prey-predator system can be described using the following set of differential equations:

$$\begin{aligned}\frac{dx}{dt} &= rx(1-x)\left(\frac{1}{1+ky}\right) - y\sqrt{x} \\ \frac{dy}{dt} &= -\alpha y + \beta\sqrt{x}y\end{aligned}\quad (1)$$

with initial condition $x(t) \geq 0$ and $y(t) \geq 0$; where $x(t)$ and $y(t)$ are the densities of prey and predator populations, respectively, at anytime t . Here the parameters r, α, β are the growth rate of the prey, death rate of the predator in the absence of prey, and conversion rate of prey to predator, respectively.

According to the form of right-hand side functions of system 1, it is clear that they are continuous and have continuous partial derivatives. Hence they are Lipschitzian. Therefore, system 1 has a unique solution. Moreover, all the solutions of system 1 are positive and uniformly bounded as shown in the following theorems, and hence system 1 is a dissipative system.

Theorem 1: The domain of system 1, \mathbb{R}_+^2 is positively invariant.

Proof: Let $(x(t), y(t))$ be any solution of system 1. Since the solution $(x(t), y(t))$ of it with initial conditions in \mathbb{R}_+^2 exists and unique on $[0, \delta)$, where $0 < \delta \leq +\infty$, then from the equations of system 1 we have

$$x(t) = x(0)e^{\int_0^t \left[r(1-x(s))\left(\frac{1}{1+ky(s)}\right) - \frac{y(s)}{\sqrt{x(s)}} \right] ds} \geq 0,$$

and

$$y(t) = y(0)e^{\int_0^t [-\alpha + \beta\sqrt{x(s)}] ds} \geq 0$$

This completes the proof.

Theorem 2: All solutions of system 1 that are initiated in \mathbb{R}_+^2 are uniformly bounded.

Proof: From the first equation of system 1, it is observed that

$$\frac{dx}{dt} \leq rx(1-x)$$

Then it is easy to verify that for all values of t we have $x(t) \leq \frac{r}{4}$. Define the function $W = x + \frac{y}{\beta}$, then

$$\frac{dW}{dt} \leq rx(1-x) - \frac{\alpha}{\beta}y$$

Using the bound of x gives that

$$\frac{dW}{dt} + \alpha W \leq \frac{(r+\alpha)r}{4}$$

Therefore, direct computation shows that as $t \rightarrow \infty$ we obtain that $W(t) \leq \frac{(r+\alpha)r}{4\alpha}$.

The proof is completed.

3. Existence of equilibrium points and their local stability analysis

The equilibrium points of system 1 are given by the trivial equilibrium point $E_0 = (0,0)$ and the axial equilibrium point $E_1 = (1,0)$, which always exist. While the coexistence equilibrium point $E_2 = (\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{\alpha^2}{\beta^2}, \bar{y} = \frac{-1}{2k} + \frac{1}{2k} \sqrt{1 + 4k \frac{r\alpha}{\beta^3} (\beta^2 - \alpha^2)} \quad (2)$$

exists provided that

$$\beta^2 > \alpha^2 \quad (3)$$

The method of Jacobian matrix is used to study the local stability of the above equilibrium points. The equilibrium point is locally asymptotically stable, if all the eigenvalues of the Jacobian matrix at that point have negative real parts. Since the Jacobian matrix of system 1 about arbitrary point (x, y) is determined by

$$J(x, y) = \begin{pmatrix} \frac{r-2rx}{1+ky} - \frac{y}{2\sqrt{x}} & \frac{-rxk(1-x)}{(1+ky)^2} - \sqrt{x} \\ \frac{\beta y}{2\sqrt{x}} & \beta\sqrt{x} - \alpha \end{pmatrix} \quad (4)$$

then by substituting the trivial equilibrium point in Eq. 4 and then determining the eigenvalues, the following eigenvalues are obtained: $\lambda_{10} = r > 0$ and $\lambda_{20} = -\alpha < 0$ and hence E_0 is an unstable saddle point.

The Jacobian matrix of system 1 at the axial equilibrium point $E_1 = (1,0)$ can be written as

$$J(1,0) = \begin{pmatrix} -r & -1 \\ 0 & \beta - \alpha \end{pmatrix} \quad (5)$$

Hence the eigenvalues of $J(1,0)$ are given by $\lambda_{11} = -r < 0$ and $\lambda_{21} = \beta - \alpha$. Hence, $E_1 = (1,0)$ is locally asymptotically stable provided that

$$\beta < \alpha \quad (6)$$

Now the Jacobian matrix of system 1 at the coexistence equilibrium point $E_2 = (\bar{x}, \bar{y})$ can be determined as follows

$$J(\bar{x}, \bar{y}) = [a_{ij}]_{2 \times 2} \quad (7)$$

where

$$a_{11} = \frac{(r-2\bar{x}r)}{1+k\bar{y}} - \frac{\bar{y}}{2\sqrt{\bar{x}}}; \quad a_{12} = \frac{-rk\bar{x}(1-\bar{x})}{(1+k\bar{y})^2} - \sqrt{\bar{x}} < 0$$

$$a_{21} = \frac{\beta\bar{y}}{2\sqrt{\bar{x}}} > 0; \quad a_{22} = 0$$

The characteristic equation of $J(\bar{x}, \bar{y})$ can be written as

$$\lambda^2 + A_1\lambda + A_2 = 0 \quad (8)$$

where $A_1 = -\frac{(r-2\bar{x}r)}{1+k\bar{y}} + \frac{\bar{y}}{2\sqrt{\bar{x}}}$ and $A_2 = \left[\frac{rk\bar{x}(1-\bar{x})}{(1+k\bar{y})^2} + \sqrt{\bar{x}} \right] \frac{\beta\bar{y}}{2\sqrt{\bar{x}}} > 0$. According to the Routh-Hurwitz criterion, Eq. 8 has two eigenvalues with negative real parts provided that $A_1 > 0$ and $A_2 > 0$. Hence the coexistence equilibrium point $E_2 = (\bar{x}, \bar{y})$ is locally asymptotically stable provided that

$$\frac{\bar{y}}{2\sqrt{\bar{x}}} > \frac{(r-2\bar{x}r)}{1+k\bar{y}} \quad (9)$$

It is well known that persistence ensures the long term survival of all populations, starting from any initial population. In the next theorem, the condition that guarantees the uniform persistence of system 1 is established.

Theorem 3: System 1 is uniformly persistent provided that

$$\beta > \alpha \quad (10)$$

Proof. Assume that p is a point in the positive quadrant and $O(p)$ is the orbit through p and Ω is the omega limit set of the orbit through p . Note that $\Omega(p)$ is bounded due to the uniform boundedness of system 1.

The claim is that $E_0 \notin \Omega(p)$. Assume that $E_0 \in \Omega(p)$, then by the Butler–McGehee lemma [6] there exists a point q in $\Omega(p) \cap W^s(E_0)$, where $W^s(E_0)$ denotes the stable manifold of E_0 . Since $O(q)$ lies in $\Omega(p)$ and $W^s(E_0)$ is the y -axis, it is concluded that $O(q)$ is unbounded, which is a contradiction.

Now, the claim is that $E_1 \notin \Omega(p)$, otherwise $E_1 \in \Omega(p)$. Since E_1 is an unstable saddle point under condition 10 then by the Butler–McGehee lemma there exists a point q_1 in $\Omega(p) \cap W^s(E_1)$. Since $O(q_1)$ lies in $\Omega(p)$ and $W^s(E_1)$ is the x -axis, it implies that an unbounded orbit lies in $\Omega(p)$, a contradiction. Therefore system 1 is uniformly persistent. ■

4. Global Stability Analysis

In this section, the global stability of the locally asymptotically stable equilibrium points is investigated using suitable Lyapunov functions as shown in the following theorems.

Theorem 4: Assume that the axial equilibrium point $E_1 = (1, 0)$ is locally asymptotically stable, then it is a globally asymptotically stable provided that

$$\sqrt{x} > \frac{\beta}{\alpha} \quad (11)$$

Proof: Let $v_1 = c_1[x - 1 - \ln x] + c_2 y$ be a positive definite real valued function. Then we have

$$\begin{aligned} \frac{dv_1}{dt} &= \frac{\partial v_1}{\partial x} \frac{dx}{dt} + \frac{\partial v_1}{\partial y} \frac{dy}{dt} \\ \frac{dv_1}{dt} &= -\frac{c_1 r}{1+ky} (x-1)^2 - (c_1 - c_2 \beta) \sqrt{x} y - \left(c_2 \alpha - \frac{c_1}{\sqrt{x}}\right) y \end{aligned}$$

Then by choosing $c_1 = 1$ and $c_2 = \frac{1}{\beta}$, we obtain that

$$\frac{dv_1}{dt} = -\frac{r}{1+ky} (x-1)^2 - \left(\frac{\alpha}{\beta} - \frac{1}{\sqrt{x}}\right) y$$

Clearly under the condition 11, the derivative $\frac{dv_1}{dt}$ is a negative definite. Moreover, since v_1 is a radially unbounded function, then the axial equilibrium point $E_1 = (1, 0)$ is globally asymptotically stable. ■

Theorem (5): Assume that the coexistence equilibrium point $E_2 = (\bar{x}, \bar{y})$ is locally asymptotically stable, then it is globally asymptotically stable provided that

$$x < \frac{1}{3} \text{ or } \frac{1}{3} < x \quad (12)$$

Proof: Consider the Dulac function given by $\frac{1}{\sqrt{x}y}$. Then by using the Bendixson–Dulac theorem on system 1, if the expression $\frac{\partial}{\partial x}(Bf) + \frac{\partial}{\partial y}(Bg)$, where f and g are the right-hand side functions of system 1, has the same sign and not equal to zero almost everywhere in a simply connected region of the plane, then the plane autonomous system 1 has no non-constant periodic solutions lying entirely in the positive quadrant.

Now since

$$\frac{\partial}{\partial x} \left[\frac{1}{\sqrt{x}y} \left(\frac{rx(1-x)}{1+ky} - y\sqrt{x} \right) \right] + \frac{\partial}{\partial y} \left[\frac{1}{\sqrt{x}y} (-\alpha y + \beta y\sqrt{x}) \right] = \frac{1}{2} \frac{r}{y(1+ky)} \left[\frac{1}{\sqrt{x}} - 3\sqrt{x} \right]$$

then under condition 12, system 1 has no non-constant periodic solutions in the positive quadrant. Since the system is bounded and has a unique coexistence equilibrium point, then according to the Poincaré–Bendixson theorem, the coexistence equilibrium point $E_2 = (\bar{x}, \bar{y})$ is globally asymptotically stable under condition 12 and hence the proof is complete.

Bifurcation analysis

The effect of varying the parameter values on the dynamics of system 1 is studied in this section using the local bifurcation analysis with the help of Sotomayor theorem [18].

Now for simplifying the notations, system 1 in the vector form is written as follows

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X}), \text{ with } \mathbf{X} = (x, y)^T \text{ and } \mathbf{F} = (f, g)^T$$

Then the second derivative of \mathbf{F} with respect to \mathbf{X} can be written as

$$D^2F(V, V) = \begin{bmatrix} -\frac{2r}{1+ky} v_1^2 + \frac{y}{4x^{\frac{3}{2}}} v_1^2 - \frac{2rk(1-2x)}{(1+ky)^2} v_1 v_2 - \frac{1}{\sqrt{x}} v_1 v_2 + \frac{2rk^2 x(1-x)}{(1+ky)^3} v_2^2 \\ \frac{\beta y}{4x^{\frac{3}{2}}} v_1^2 + \frac{\beta}{\sqrt{x}} v_1 v_2 \end{bmatrix} \quad (13)$$

here $V = (v_1, v_2)^T$ be a general vector.

Theorem (6): Assume that $\beta = \alpha$ ($\equiv \beta^*$), then system 1 at the axial equilibrium point E_1 undergoes a transcritical bifurcation but neither saddle node nor pitchfork bifurcation can occur.

Proof. For $\beta = \alpha$ ($\equiv \beta^*$) we have the following Jacobian matrix

$$J(E_1, \beta^*) = J_1 = \begin{pmatrix} -r & -1 \\ 0 & 0 \end{pmatrix}$$

So J_1 has the following eigenvalues: $\lambda_{11}^* = -r < 0$ and $\lambda_{12}^* = 0$, hence the necessary but not sufficient condition for bifurcation is satisfied and E_1 is a nonhyperbolic point.

Let $\mathbf{V}_1 = (v_1, v_2)^T$ be the eigenvector of J_1 corresponding to the eigenvalue $\lambda_{12}^* = 0$. Then straightforward computation gives that $\mathbf{V}_1 = (-\frac{1}{r}v_2, v_2)^T$, where v_2 represents any nonzero real number.

Also, let $\Psi_1 = (\psi_1, \psi_2)^T$ be the eigenvector of J_1^T that corresponding to the eigenvalue $\lambda_{12}^* = 0$. Then direct calculation shows that $\Psi_1 = (0, \psi_2)^T$, where ψ_2 is any nonzero real number.

Because $\frac{\partial \mathbf{F}}{\partial \beta} = \mathbf{F}_\beta = (0, \sqrt{xy})^T$, hence we obtain that $\mathbf{F}_\beta(E_1, \beta^*) = (0, 0)^T$, which yields

$$\Psi_1^T [\mathbf{F}_\beta(E_1, \beta^*)] = 0$$

Thus system 1 at E_1 with $\beta = \beta^*$ does not experience saddle-node bifurcation in view of Sotomayor theorem. Moreover, we have

$$\Psi_1^T [D\mathbf{F}_\beta(E_1, \beta^*)\mathbf{V}_1] = v_2\psi_2 \neq 0$$

where $D\mathbf{F}_\beta$ represents the derivative of \mathbf{F}_β with respect to \mathbf{X} . Also by using eq. 13 at (E_1, β^*) with the eigenvector \mathbf{V}_1 we obtain that

$$\Psi_1^T [D^2\mathbf{F}(E_1, \beta^*)(\mathbf{V}_1, \mathbf{V}_1)] = -\frac{\beta^*}{r}\psi_2v_2^2 \neq 0$$

Accordingly by Sotomayor theorem, system 1 near the equilibrium point E_1 with $\beta = \beta^*$ undergoes a transcritical bifurcation but pitchfork cannot occur.

Theorem 7: Assume that $r = \frac{\bar{y}(1+k\bar{y})}{2\sqrt{\bar{x}}(1-2\bar{x})}$ ($\equiv r^*$), then system 1 at the coexistence equilibrium point E_2 undergoes a Hopf bifurcation provided that

$$\bar{x} < \frac{1}{2} \tag{14}$$

Proof. According to the Jacobian matrix and the characteristic equation given by Eq. 7 and Eq. 8, respectively, the eigenvalues can be written by

$$\lambda_{21}, \lambda_{22} = \frac{-A_1}{2} \pm \frac{1}{2}\sqrt{A_1^2 - 4A_2}$$

where $A_1 = -a_{11} = -\frac{r(1-2\bar{x})}{1+k\bar{y}} + \frac{\bar{y}}{2\sqrt{\bar{x}}}$; $A_2 = -a_{12}a_{21} = \frac{\beta\bar{y}}{2} \left(\frac{rk\sqrt{\bar{x}}(1-\bar{x})}{(1+k\bar{y})^2} + 1 \right) > 0$.

Clearly, for $r = r^*$ we obtain that $A_1 = 0$. Hence $\lambda_{21}, \lambda_{22} = \pm i\sqrt{A_2}$ which are pure imaginary. Thus the coexistence equilibrium point E_2 is a non-hyperbolic point. Moreover, since

$$\left. \frac{\partial A_1}{\partial r} \right|_{r=r^*} = -\frac{(1-2\bar{x})}{1+k\bar{y}} \neq 0 \text{ under condition 14.}$$

Hence the system undergoes a Hopf bifurcation.

5. Numerical simulation

In this section, the dynamics of the proposed system 1 is simulated numerically using the following hypothetical set of data. The objective is to verify our obtained analytical results and specify the set of parameters that control the global dynamics of the system.

$$r = 1, k = 1, \alpha = 0.4, \beta = 0.5 \tag{15}$$

It is observed that system 1 approaches asymptotically to the coexistence equilibrium point $E_2 = (0.64, 0.233)$ starting from different initial points as shown in Fig.1 below:

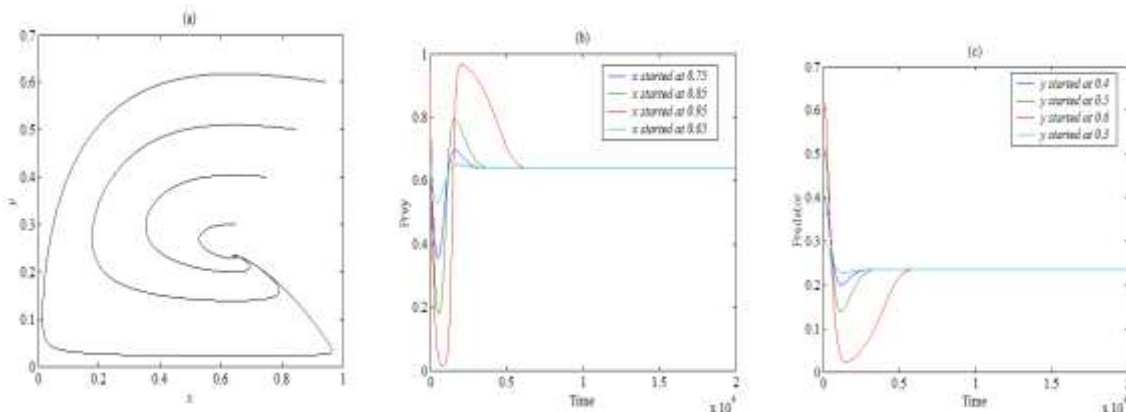


Figure 1-Phase portrait of system 1 using data given by Eq. 15. (a) The trajectories of system 1 approach asymptotically to the coexistence equilibrium point $E_2 = (0.64, 0.233)$ starting from different initial points. (b) Time series of trajectories of prey starting from different points. (c) Time series of trajectories of predator starting from different points.

Clearly, Figure-1 shows that the coexistence equilibrium point E_2 of system 1 is globally asymptotically stable.

Now the effect of varying the intrinsic growth rate of the prey species is numerically investigated and the obtained results are drawn in Figure-2 below. Obviously, as shown in Figure-2, varying the value of r leads to varying the position of the coexistence equilibrium point rather than extinction of prey or predator

Similarly, the effect of varying the carrying capacity of the prey species is numerically investigated and the obtained results are drawn in Figure-3 below. Obviously, Figure-3 shows the approach of the solution of system 1 to different coexistence equilibrium points depending on the value of carrying capacity.

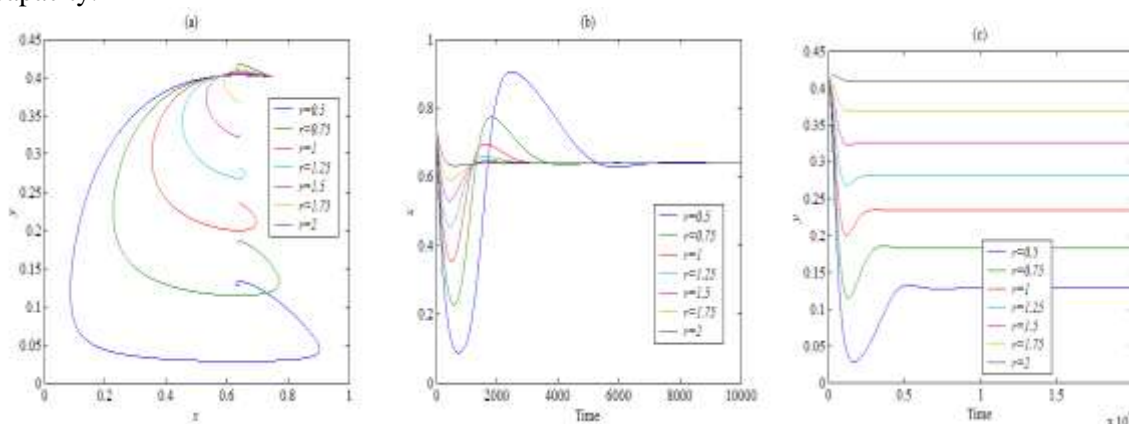


Figure 2-The trajectories of system 1 for the data (15) with different values of r . (a) The trajectories of system 1 approach asymptotically to the different coexistence equilibrium points depending on the value of r using the same initial point. (b) Time series of trajectories of the prey. (c) Time series of trajectories of the predator.

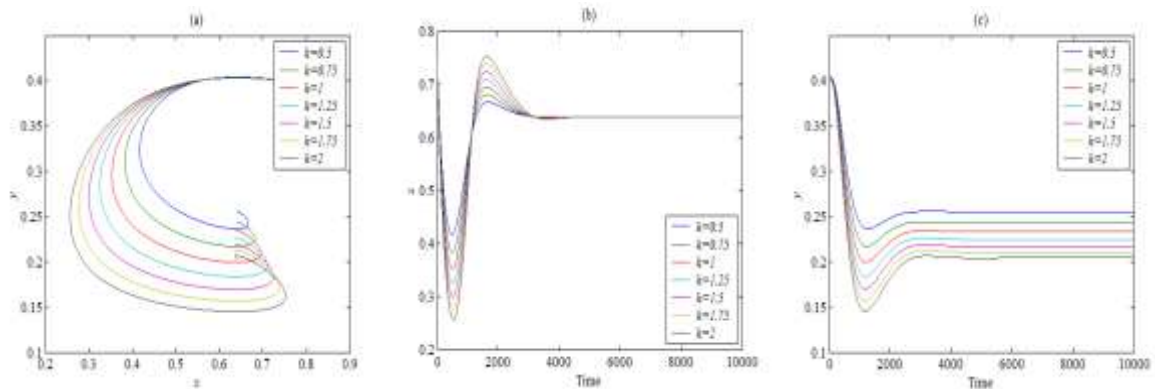


Figure 3-The trajectories of system 1 for the data (15) with different values of k . (a) The trajectories of system 1 approach asymptotically to the different coexistence equilibrium points depending on the value of k using the same initial point. (b) Time series of trajectories of the prey. (c) Time series of trajectories of the predator.

Finally, varying the value of α , so that condition 6 holds, leads to extinction in predator species and the solution of system 1 approaches asymptotically to the axial equilibrium point $E_1 = (1,0)$ as shown in Figure-4 for the typical value $\alpha = 0.75$, keeping the rest of values as given by Eq. 15.

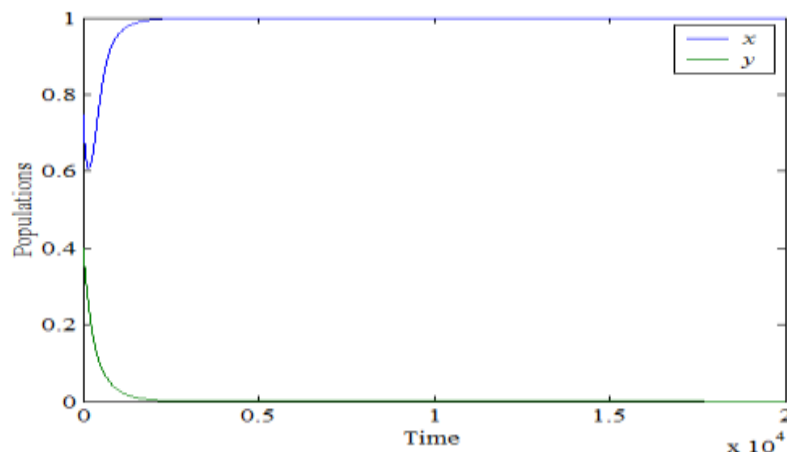


Figure 4- Time series of the trajectory of system 1 that approaches asymptotically to the axial equilibrium point $E_1 = (1,0)$ for data (15) with $\alpha = 0.75$.

6. Discussion and conclusions

In this paper, a mathematical model that describes the dynamics of a prey-predator system involving the effects of the prey's fear is proposed and investigated. It is observed that the model has at most three nonnegative equilibrium points. The local stability, persistence, global stability and local bifurcation for the proposed model are investigated analytically as well as numerically. In order to understand the global dynamics more precisely and confirm our analytical findings, a hypothetical set of parameters was selected so that the system is biologically realistic. Our obtained numerical simulation results that depend on the hypothetical parameters set 15 are summarized as follows:

1. The system has a globally asymptotically coexistence equilibrium point for the set of parameters 15, which satisfies the global stability condition and persistence condition.
2. Increasing the intrinsic growth rate of the prey leads to an increase in the predator population and the system still persists at the coexistence equilibrium point, the position of which changes depending on the value of intrinsic growth rate.
3. Increasing the prey's fear rate leads to a decrease in the predator population and the system still persists at the coexistence equilibrium point, the position of which changes depending on the value of prey's fear rate.
4. Once the death rate of predator exceeds the value of conversion rate of prey to predator, the system loses its persistence and undergoes a transcritical bifurcation by approaching to the axial equilibrium point rather than the coexistence equilibrium point.

5. Although the system undergoes a Hopf bifurcation as the intrinsic growth rate passes through a specific value analytically, we cannot show that with the set of parameters 15 due to the difficulty in the computing of the value of the intrinsic growth rate. However it is still possible to obtain the Hopf bifurcation using different sets of parameters.

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