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# R-annihilator-Coessential and R-annihilator-Coclosed Submodules

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**Abstract:**
Let $W$ be a unitary left R-module on associative ring $R$ with identity. A submodule $F$ of $W$ is called $R$-annihilator small if $F+T=W$, where $T$ is a submodule of $W$, implies that ann($T$)=0, where ann($T$) indicates annihilator of $T$ in $R$. In this paper, we introduce the concepts of $R$-annihilator-coessential and $R$-annihilator - coclosed submodules. We give many properties related with these types of submodules.

**Keywords**: Essential submodules, coclosed submodule, coessential submodule, coclosed submodule, $R$-annihilator - coessential and $R$- annihilator -coclosed.

## المقاسات الجزئيه ضد الجوهريه من النمط–R والمقاسات الجزئيه الضد المغلقه من النمط–R

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**الخلاصه**
ليكن Wمقاس احادي ايسر معرف على حلقة تجميعيه R ذات عنصر محايد. المقاس الجزئي F من المقاس W يدعى تالف صغير من النمط–R اذا كان W=F+T حيث Tمقاس جزئي من Wيقود الى ann(T)=0حيث ann(T)هو تالف ل Tفي R.
في هذا البحث عرفنا مفهومين الاول هو المقاسات الجزئيه ضد الجوهرية من النمطR والثاني هو والمقاسات الجزئيه الضد مغلقه التالفة من النمط–R. اعطينا العديد من الخصائص ذات العلاقة لهذه الانواع من المقاسات الجزئيه.

## 1. Introduction

Let $W$ be a unitary left of $R$-module on associative ring $R$ with identity. The concept of $R$-annihilator-small ($R$-ann-small) submodule was introduced in an earlier study [1]. A submodule $W$ of an $R$-module $W$ is called $R$-ann-small if $F+T=W$, $T$ is a submodule of $W$, implies that $ann_R(T)=0$, where $ann_R(T)=\{r\in R: r.T=0\}$ and denoted by $F \ll_a W$. A submodule $F$ of $W$ is said to be essential submodule in $W$ (denoted by $F \ll_e W$) if for any $X\subseteq W$, $X\cap F=0$, implies that $X=0$ [2]. For $F\subseteq K\subseteq W$, if $F\leq K$, then $K$ is called an essential extension of $F$ in $W$.

A submodule $F$ is said to be closed in $W$, if $F$ has no proper essential extension in $W$ [3].

Dually, for $F\subseteq K\subseteq W$, $F$ is said to be a coessential submosule of $K$ in $W$ (denoted by $F\subseteq_{ce} K$ in $W$) if $\frac{K}{F}\ll\frac{W}{F}$ [4] [5].

$F$ is said to be coclosed in $W$ (denoted by $F\subseteq_{cc} W$), if $F$ has no proper coessential submodule in $W$ [6]. In this paper, we give the concept of $R$-annihilator-coessential and $R$-annihilator coclosed submodules. For $F\subseteq K\subseteq W$, $F$ is said to be a $R$-annihilator -coessential submodule of $K$ in $W$ ($R$-a-

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coessentiale), if $\frac{K}{F} \ll_a \frac{W}{F}$ (denoted by $F \leq_{ace} K$ ). Also, a submodule $W$ of an $R$-module $W$ is called $R$-annihilator coclosed ($R$-a-coclosed) submodule in $W$ (denoted by $F \leq_{acc} K$), if $F$ has no proper coessential submodule in $W$. In other words, if whenever $K \subseteq F$ with $\frac{K}{F} \ll_a \frac{W}{F}$, implies that $F = K$. We give the same properties of these kinds of submodules.

## 2. R-annihilator-coessential submodules

In this section, we introduce the concept of $R$-annihilator-coessential submodules which is a generalization of coessential submodules [4] [5]. We also give some basic properties of this class of submodules.

**Definition 2.1:** Let $W$ be an $R$-module, for $A \subseteq B \subseteq W$, $A$ is said to be $R$-annihilator coessential submodule of $B$ in $W$, briefly $R$-a-coessential (denoted by $A \subseteq_{ace} B$ in $W$), if $\frac{B}{A} \ll_a \frac{W}{A}$.

**Examples and Remarks 2.2:**

1- Consider that $Z_6$ as $Z$-module. It is clear that $\{\overline{0}\}$ is not $Z$-a-coessential submodule of $\{\overline{0},\overline{3}\}$ in $Z_6$, since $\frac{\{\overline{0},\overline{3}\}}{\{\overline{0}\}} \simeq \{\overline{0},\overline{3}\}$ and $\frac{Z_6}{\{\overline{0}\}} \simeq Z_6$. But $\{\overline{0},\overline{3}\}$ is not $Z$-annihilator-small in $Z_6$.

2- Consider that $Z$ as $Z$-module, then (0) is $Z$-a-coessential of $2Z$ in $Z$, since $\frac{2Z}{(0)} \simeq 2Z$, $\frac{Z}{(0)} \simeq Z$ and $2Z \ll^a Z$, where $2Z + 3Z = Z$ and $ann_Z(3Z) = \{n \in \mathbb{Z}: n(3Z) = 0\} = 0$.

3- Let $W$ be an $R$-module and let $F$ be a submodule of $W$, then $F \ll^a W$ iff $O \subseteq_{ace} F$

**Proof :** $\Rightarrow$) Suppose that $F \ll_a W$ then $\frac{F}{(0)} = F$, $\frac{W}{(0)} = W$. Thus, $\frac{F}{(0)} \ll^a \frac{W}{(0)}$. This means that $(0) \subseteq_{ace} F$ in $W$.

$\Leftarrow$) Now, suppose that $(0) \leq_{ace} F$ in $W$. To prove that $F \ll_a W$, suppose that $W = F + T$, where $T \leq W$. Thus, $\frac{W}{(0)} = \frac{F}{(0)} + \frac{T}{(0)}$, but $(0) \leq_{ace} F$ in $W$. Then, $ann(T) = 0$. Therefore, $F \ll_a W$.

4- The concepts of coessential and $R$-a-coessential are independent since, in $Z$ as $Z$-module, $4Z \subset 2Z \subset Z$, $\frac{2Z}{4Z} \simeq \{\overline{0},\overline{2}\}$ in $Z_4$ and $\frac{Z}{4Z} \simeq Z_4$. But, $\{\overline{0},\overline{2}\}$ is not $Z$-a-small in $Z_4$, since $\{\overline{0},\overline{2}\}+Z_4 = Z_4$ and $ann_Z Z_4 = \{n \in \mathbb{Z}: nZ_4 = 0\} = 4Z \neq 0$. We know that $\{\overline{0},\overline{2}\} \ll Z_4$, thus $4Z \subseteq_{ce} 2Z$ in $Z$, but $4Z \nsubseteq_{ace} 2Z$ in $Z$.

In this module, $Z$ as $Z$-module $\{0\}$ is $R$-a-coessential of $2Z$ in $Z$, as we shows in (2), but $\{0\}$ is not coessential of $2Z$ in $Z$ since $2Z$is not small in $Z$.

**Proposition 2.3:** Let $W$ be an R-module. If $A \subseteq_{ace} C$, then $A \subseteq_{ace} B$, where $A \subseteq B \subseteq C$ and $A, B, C$ are submodules of $W$.

**Proof:** Let $A \subset X \subset W$ with $\frac{B}{A} + \frac{X}{A} = \frac{W}{A}$, thus $B + X = W$. But $B \subseteq C$, therefore $W = C + X$ and then $\frac{W}{A} = \frac{C}{A} + \frac{X}{A}$. $A \leq_{ace} C$, then $\frac{C}{A} \ll_a \frac{W}{A}$, thus $ann\left(\frac{X}{A}\right) = 0$ and hence $\frac{B}{A} \ll^a \frac{W}{A}$, i.e. $A \subseteq_{ace} B$ in $W$.

**Proposition 2.4:** Let $W$ be an $R$-module and $A$, $B$, $N$ are submodules of $W$. If $A \subseteq_{ace} B$ and $N \ll W$, then $A \subseteq_{ace} B + N$ in $W$.

**Proof :** Suppose that $A \subset X \subset W$ with $\frac{B+N}{A} + \frac{X}{A} = \frac{W}{A}$ then $B + N + X = W$, but $N \ll W$, therefore $B + X = W$ and hence $\frac{B}{A} + \frac{X}{A} = \frac{W}{A}$. But $\frac{B}{A} \ll_a \frac{W}{A}$, Thus $ann\left(\frac{X}{A}\right) = 0$. This means that $A \subseteq_{ace} B + N$ in $W$.

**Proposition 2.5:** Let $A \subseteq X \subseteq B \subseteq W$, $X \subseteq_{ace} B$ if and only if $\frac{X}{A} \subseteq_{ace} \frac{B}{A}$ in $\frac{W}{A}$.

**Proof** $\Rightarrow$) Assume that $X \subseteq_{ace} B$ in $M$. Since $\frac{B}{X} \simeq \frac{B/A}{X/A}$ and $\frac{W}{X} \simeq \frac{W/A}{x/A}$ (by the 3rd isomorphism theorem). Thus, $\frac{B/A}{X/A} \ll_a \frac{W/A}{X/A}$ and hence $\frac{X}{A} \subseteq_{ace} \frac{B}{A}$ in $\frac{W}{A}$.

$\Leftarrow$) Now, suppose that $\frac{X}{A} \subseteq_{ace} \frac{B}{A}$ in $\frac{W}{A}$. Since $\frac{B/A}{x/A} \simeq \frac{B}{X}$ and $\frac{W/A}{x/A} \simeq \frac{W}{X}$ (by the 3rd isomorphism theorem). Then $\frac{B}{X} \ll_a \frac{W}{X}$ and this means that $X \subseteq_{ace} B$ in $W$.

**Lemma 2.6:** Let $W$ be an $R$-module and $A \subseteq B \subseteq C \subseteq W$. If $B \subseteq_{ace} C$ in $W$, then $A \subseteq_{ace} C$ in $W$.

**Proof:** Suppose that $B \subseteq_{ace} C$ in $M$. To prove that $A \subseteq_{ace} C$ in $W$, suppose that $\frac{W}{A} = \frac{C}{A} + \frac{T}{A}$, where $A \subseteq T$, thus $W = C + T$ and then $\frac{W}{B} = \frac{C}{B} + \frac{T+B}{B}$. But $B \subseteq_{ace} C$, and this means that $\frac{C}{B} \ll_a \frac{W}{B}$.

Therefore, $ann\left(\frac{T+B}{B}\right) = 0$. To prove that $ann\left(\frac{T}{A}\right) = 0$, let $r \in ann\left(\frac{T}{A}\right)$, thus $rT \subseteq A$ and hence $rT \subseteq B$ since $A \subseteq B$, therefore $rT + B = B$. Thus, $r \in ann\left(\frac{T+B}{B}\right) = 0$ which means that $ann\left(\frac{T}{A}\right) = 0$. Therefore, $A \subseteq_{ace} C$ in $W$.

**Proposition 2.7:** If $W$ is an $R$-module and $A$, $B$ and $C$ are submodules of $W$, such that $A + C \subseteq_{ace} B + C$ in $W$, then $A \subseteq_{ace} B$ in $W$.

**Proof:** Let $T$ be a submodule of $W$, such that $A \subseteq T$, and suppose that $\frac{M}{A} = \frac{B}{A} + \frac{T}{A}$. So $W = B + T$ and then $\frac{W}{A+C} = \frac{B+C}{A+C} + \frac{T+C}{A+C}$, where $A + C\ ann\left(\frac{T}{A}\right) = 0$. Let $r \in ann(\frac{T}{A})$, thus $rT \subseteq A \subseteq A + C$ and hence $r(T + A + C) = A + C$. Then, $r \in ann\left(\frac{T+C}{A+C}\right) = 0$, thus $ann\left(\frac{T}{A}\right) = 0$. So $A \subseteq_{ace} B$ in $W$.

**Proposition 2.8:** Let $A$, $B$, $C$ and $X$ be submodules of an $R$-module $W$. The following statements are the same:

1) If $A \subseteq_{ace} A + B$ in $W$, then $A \cap B \subseteq_{ace} B$ in $W$.
2) If $A \subseteq_{ace} B$ in $W$ and $Y \leq W$, then $A \cap Y \subseteq_{ace} B \cap Y$ in $W$.
3) If $A \subseteq_{ace} B$ in $W$ and $X \subseteq_{ace} C$ in $W$, then $A \cap X \subseteq_{ace} B \cap C$ in $W$.

**Proof: (1)⇒(2)** Let $A \subseteq_{ace} B$ in $W$ and $Y \subseteq W$. Since $A + (B \cap Y) \subseteq B$, then $A \subseteq_{ace} A + (B \cap Y)$ (by proposition (1.3)). By (1), $A \cap (B \cap Y) \subseteq_{ace} B \cap Y$, this implies that $A \cap Y \subseteq_{ace} B \cap Y$ in $W$.

**(2)⇒(3)** Let $A \subseteq_{ace} B$ in $W$and $X \subseteq_{ace} C$ in $W$. By (2) $A \cap X \subseteq_{ace} B \cap X$ in $W$. Also, $X \subseteq_{ace} C$ and $B \subseteq W$, $B \cap X \subseteq B \cap C$ in $W$. Thus, $A \cap X \cap C$ in $W$ by proposition (1.3).

**(3)⇒(1)** Suppose that $A \subseteq_{ace} A + B$ in $W$. Since $B \subseteq_{ace} B$ in $W$, then by (3), $A \cap B \subseteq_{ace} (A + B) \cap B$. Thus, $A \cap B \subseteq_{ace} B$ in $W$.

**3. R-annihilator-coclosed submodules**

In this section, we introduce the concept of ($R$-annihilator) $R$-a-coclosed as a generalization of coclosed submodules, where a submodule $N$ of an $R$-module $W$ is called coclosed submodule in $W$ (denoted by $F \leq_{cc} W$) if whenever $K \leq F$ with $\frac{F}{K} \ll \frac{W}{K}$ implies $F = K$.

**Definitions 3.1:**

Let $W$ be an $R$-module, then a submodule $F$ of $M$ is called $R$-annihilator coclosed in $W$ (briefly $R$-a-coclosed) if whenever $K \leq F$ with $\frac{F}{K} \ll^{a} \frac{W}{K}$ implies that $F = K$. (denoted by $F \leq_{acc} W$) .

**Examples and Remarks 3.2**

1- The submodule $\{\bar{0}, \bar{2}\}$ of a $Z$-module $Z_4$ is $Z$-a-coclosed in $Z_4$ since $\{0\} \subsetneqq \{\bar{0}, \bar{2}\}$ and $\frac{\{\bar{0},\bar{2}\}}{\{\bar{0}\}} \simeq \{\bar{0}, \bar{2}\}$, also $\frac{Z_4}{\{\bar{0}\}} \simeq Z_4$. But $\{\bar{0}, \bar{2}\}$ is not $Z$-a-small in $Z_4$ because $\{\bar{0}, \bar{2}\} + Z_4 = Z_4$ and $ann_Z Z_4 = 4Z \neq 0$.

2- The submodule $2Z$ of a $Z$-module $Z$ is not $Z$-a-coclosed of $Z$. To recognize that, let $\{0\} \subsetneq 2Z$ and notice that $\frac{2Z}{\{0\}} \simeq 2Z$, $\frac{Z}{\{0\}} \simeq Z$. We also know that $2Z \ll_a Z$ [1, Rem. & Ex (1.2.p13)].

3- The concepts of coclosed and $R$-a-coclosed submodules are independent, since $\{\bar{0}, \bar{2}\}$ is $Z$-a-coclosed in the $Z$-module $Z_4$ but it is not coclosed in $Z_4$ as $Z$-module [7, Rem. & Ex. (1.2.3), p15].

4- Consider the $Z_8$-module $Z_8$, the submodule $\{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$ is $Z_8$-a-coclosed but not coclosed since $\{0\} \subsetneq \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\} \subseteq Z_8$ $\frac{\{\bar{0},\bar{2},\bar{4},\bar{6}\}}{\{\bar{0}\}} \simeq \{\bar{0}, \bar{2}, \bar{4}, \bar{6},\}$. Also $\frac{Z_8}{\{\bar{0}\}} \simeq Z_8$, but $\{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$ is not $Z_8$-a-coclosed since $\{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$ is not $Z_8$-a-small. But $\{\bar{0}, \bar{2}, \bar{4}, \bar{6}\} \ll Z_8$ is $Z_8$-module, thus it is coclosed in $Z_8$.

An $R$-module $W$ is called $R$-a-hllow if every proper submodule of $W$ is $R$-a-small [8], where a submodule $F$ of $W$ is $R$-a-small if whenever $W = F + T$, where $T \leq W$, then $ann(T) = 0$. [1]

**Proposition 3.3**

Let $W$ be an $R$-module and $L$ be a nonzero submodule of $W$ which is $R$-a-hollow, then either $L$ is $R$-a-small submodule of $W$ or $L$ is $R$-a-coclosed submodule of $W$, but not both.

**Proof:** Let $(0) \neq L \leq W$ and suppose that $L$ is not $R$-a- coclosed of $W$. So, there exists $K < L$ such that $\frac{L}{K} \ll_a \frac{W}{K}$, but [by 1,cor.(2.1.6), p34] $L \ll_a W$.

Now, if $L$ is $R$-a-coclosed submodule of $W$, and suppose that $L \ll_a W$, then $\frac{L}{(0)} \ll_a \frac{W}{(0)} \simeq W$ and hence $L = (0)$ which is contradiction.

**Proposition 3.4:** Let $W$ be an $R$-module and $K \leq L$ be submodules of $W$. If $L \leq_{acc} W$, then $\frac{L}{K} \leq_{acc} \frac{W}{K}$.

**Proof :**Let $\frac{N}{K} \leq_{ace} \frac{L}{K}$ in $\frac{W}{K}$. We must prove that $\frac{N}{K} = \frac{L}{K}$ . We have $\frac{L/K}{N/K} \ll^{a} \frac{W/K}{N/K}$. This means that $\frac{L}{N} \ll^{a} \frac{W}{N}$, but $L \leq_{acc} W$. Therefore, $L = N$ and hence $\frac{N}{K} = \frac{L}{K}$. Thus $\frac{L}{K} \leq_{acc} \frac{W}{K}$.

**Lemma 3.5**:

Let be an $R$-module and let $X$, $K$ and $L$ be submodules of $W$ such that $X \subseteq K \subseteq L \subseteq W$. If $\frac{K}{X} \ll_{a} \frac{L}{X}$ and $\frac{L}{X} \ll_{a} \frac{W}{X}$, then $\frac{K}{X} \ll_{a} \frac{W}{X}$.

**Proof:** Suppose that $\frac{W}{X} = \frac{K}{X} + \frac{T}{X}$, where $T$ is a submodule of $W$ such that $X \subseteq T$. Thus, $W = K + T$. But $K \subseteq L$, therefore $W = L + T$. So, $\frac{W}{X} = \frac{L}{X} + \frac{T}{X}$. But $\frac{L}{X} \ll_{a} \frac{W}{X}$, thus $ann\left(\frac{T}{X}\right) = 0$, which means that $\frac{K}{X} \ll_{a} \frac{W}{X}$.

**Proposition 3.6**

Let $W$ be an $R$-module and $K \subseteq L$ be submodules of $W$. If $K \leq_{acc} W$ and $\frac{L}{K} \ll_{a} \frac{W}{K}$, then $K \leq_{acc} L$.

**Proof:** Let $X \leq K$ such that $\frac{K}{X} \ll_{a} \frac{L}{X}$. Since $\frac{L}{K} \ll_{a} \frac{W}{X}$ then $\frac{K}{X} \ll_{a} \frac{W}{X}$ by lemma 3.5. But $K \leq_{acc} W$. Therefore, $K = X$ and hence $K \leq_{acc} L$.

**Proposition 3.7:** Let $W$ be an $R$-module and $L$, $N$ are submodule. If $L \leq_{acc} W$ then $\frac{L+N}{N} \leq_{acc} \frac{W}{N}$.

**Proof** Suppose that $\frac{X}{N} \leq_{ace} \frac{L+N}{N}$ in $\frac{W}{N}$ where $N \subseteq X$, then $X = N + (L \cap X) \leq_{ace} N + L$ in $W$ [by prop. (2.5)].

But $N \leq W$, which implies that $(L \cap X) \leq_{ace} L$ in $W$, by [prop. 2.7] As $L \leq_{acc} W$, we get $X = L + N$. Then $\frac{L+N}{N} \leq_{acc} \frac{W}{N}$ .

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