R-annihilator-Coessential and R-annihilator-Coclosed Submodules

Omar K. Ibrahim*, Alaa A. Elewi
Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq

Received: 8/7/2019 Accepted: 21/9/2019

Abstract:
Let $W$ be a unitary left $R$-module on associative ring $R$ with identity. A submodule $F$ of $W$ is called $R$-annihilator small if $F + T = W$, where $T$ is a submodule of $W$, implies that $\text{ann}(T) = 0$, where $\text{ann}(T)$ indicates annihilator of $T$ in $R$. In this paper, we introduce the concepts of $R$-annihilator-coessential and $R$-annihilator-coclosed submodules. We give many properties related with these types of submodules.

Keywords: Essential submodules, coclosed submodule, coessential submodule, coclosed submodule, $R$-annihilator-coessential and $R$-annihilator-coclosed.

1. Introduction
Let $W$ be a unitary left of $R$-module on associative ring $R$ with identity. The concept of $R$-annihilator-small ($R$-ann-small) submodule was introduced in an earlier study [1]. A submodule $W$ of an $R$-module $W$ is called $R$-ann-small if $F + T = W$, $T$ is a submodule of $W$, implies that $\text{ann}_R(T) = 0$, where $\text{ann}_R(T) = \{r \in R: r \cdot T = 0\}$ and denoted by $F \ll_W T$. A submodule $F$ of $W$ is said to be essential submodule in $W$ (denoted by $F \subseteq W$) if for any $X \subseteq W$, $X \cap F = 0$, implies that $X = 0$ [2].

*Email: omarkhaelelibrahim@gmail.com

ISSN: 0067-2904
coessential), if \( K \leq \frac{W}{F} \) (denoted by \( F \leq ace K \)). Also, a submodule \( W \) of an \( R \)-module \( W \) is called \( R \)-annihilator coclosed (\( R \)-a-coclosed) submodule in \( W \) (denoted by \( F \leq ace K \)), if \( F \) has no proper coessential submodule in \( W \). In other words, if whenever \( K \subseteq F \) with \( \frac{K}{F} \leq \frac{W}{F} \), implies that \( F = K \).

We give the same properties of these kinds of submodules.

2. \( R \)-annihilator-coessential submodules

In this section, we introduce the concept of \( R \)-annihilator-coessential submodules which is a generalization of coessential submodules [4] [5]. We also give some basic properties of this class of submodules.

**Definition 2.1:** Let \( W \) be an \( R \)-module, for \( A \subseteq B \subseteq W \), \( A \) is said to be \( R \)-annihilator coessential submodule of \( B \) in \( W \), briefly \( R \)-a-coessential (denoted by \( A \leq ace B \) in \( W \)), if \( \frac{B}{A} \leq \frac{W}{A} \).

**Examples and Remarks 2.2:**

1- Consider that \( Z_6 \) as \( Z \)-module. It is clear that \( \{0\} \) is not \( Z \)-a-coessential submodule of \( \{0, 3\} \) in \( Z_6 \), since \( \frac{0,3}{0} = \{0, 3\} \) and \( \frac{3}{0} = Z_6 \). But \( \{0, 3\} \) is not \( Z \)-annihilator-small in \( Z_6 \).

2- Consider that \( Z \) as \( Z \)-module, then \( 0 \) is \( Z \)-a-coessential of \( 2Z \) in \( Z \), since \( \frac{2Z}{0} \approx 2Z \), \( \frac{Z}{0} \approx Z \) and \( 2Z \leq \frac{a}{a} Z \), where \( 2Z + 3Z = Z \) and \( n(ax, x) = \{n \in Z; n(3Z) = 0\} = 0 \).

3- Let \( W \) be an \( R \)-module and let \( F \) be a submodule of \( W \), then \( F \leq \frac{a}{a} W \) iff \( O \leq ace F \).

**Proof:** Suppose that \( F \leq \frac{a}{a} W \) then \( F(0) = W(0) = W \). Thus, \( \frac{F}{W} \leq \frac{a}{a} \). This means that \( (0) \leq ace F \) in \( W \).

\( \Leftarrow \) Now, suppose that \( (0) \leq ace F \) in \( W \). To prove that \( F \leq \frac{a}{a} W \), suppose that \( W = F + T \), where \( T \leq W \). Thus, \( \frac{W}{T} = \frac{F}{T} \), but \( (0) \leq ace F \) in \( W \). Therefore, \( F \leq \frac{a}{a} W \).

4- The concepts of coessential and \( R \)-a-coessential are independent since, in \( Z \) as \( Z \)-module, \( 4Z \subset 2Z \subset 3Z \approx \{0, 2\} \) in \( Z_4 \), and \( \frac{Z}{4Z} \approx \{0, 2\} \). But, \( \{0, 2\} \) is not \( Z \)-a-small in \( Z_4 \), since \( \{0, 2\} + Z_4 = Z_4 \) and \( \frac{Z}{4Z} = \{n \in Z; n(4Z) = 0\} = 4Z \neq 0 \). We know that \( \{0, 2\} \leq ace \), thus \( 4Z \leq ace 2Z \) in \( Z_4 \), but \( 4Z \leq ace 2Z \) in \( Z_2 \).

In this module, \( Z \) as \( Z \)-module \( \{0\} \) is \( R \)-a-coessential of \( 2Z \) in \( Z \), as we shows in (2), but \( \{0\} \) is not coessential of \( 2Z \) in \( Z \) since \( 2Z \) is not small in \( Z \).

**Proposition 2.3:** Let \( W \) be an \( R \)-module. If \( A \leq ace C \), then \( A \leq ace B \), where \( A \subseteq B \subseteq C \) and \( A, B, C \) are submodules of \( W \).

**Proof:** Suppose that \( A \subseteq X \subseteq W \) with \( \frac{B}{A} + \frac{X}{A} = \frac{W}{A} \), thus \( B + X = W \). But \( B \subseteq C \), therefore \( W = C + X \) and then \( \frac{W}{C + X} = \frac{C}{A} + \frac{X}{A} \leq ace C \), thus \( \frac{B}{A} \leq ace B \), and \( \frac{X}{A} \leq ace \), thus \( ann \frac{X}{A} = 0 \) and hence \( \frac{B}{A} \leq ace \frac{W}{A} \), i.e. \( A \leq ace B \) in \( W \).

**Proposition 2.4:** Let \( W \) be an \( R \)-module and \( A, B, N \) are submodules of \( W \). If \( A \leq ace B \) and \( N \leq ace \), then \( A \leq ace B + N \) in \( W \).

**Proof:** Suppose that \( A \subseteq X \subseteq W \) with \( \frac{B + N}{A} + \frac{X}{A} = \frac{W}{A} \) then \( B + N + X = W \), but \( N \leq ace \), therefore \( B + X = W \) and hence \( \frac{B}{A} + \frac{X}{A} = \frac{W}{A} \). But \( \frac{B}{A} \leq ace \), thus \( ann \frac{X}{A} = 0 \). This means that \( A \leq ace B + N \) in \( W \).

**Proposition 2.5:** Let \( A \subseteq X \subseteq B \subseteq W \), \( X \leq ace B \) if and only if \( \frac{X}{A} \leq ace \frac{W}{A} \).

**Proof:** Suppose that \( X \leq ace B \) in \( M \) and \( \frac{B}{A} \leq ace \frac{W}{A} \) in \( M \), where \( A \subseteq T \), thus \( W = C + T \) and then \( \frac{W}{B} = \frac{C}{B} + \frac{T}{B} + \frac{B}{B} \). But \( \frac{B}{A} \leq ace \frac{C}{A} \), and this means that \( \frac{1}{B} \leq ace \frac{W}{B} \).
Therefore, \( \text{ann}(\frac{T+B}{B}) = 0 \). To prove that \( \text{ann}(\frac{T}{A}) = 0 \), let \( r \in \text{ann}(\frac{T}{A}) \), thus \( rT \subseteq A \) and hence \( rT \subseteq B \) since \( A \subseteq B \), therefore \( rT + B = B \). Thus, \( r \in \text{ann}(\frac{T+B}{B}) = 0 \) which means that \( \text{ann}(\frac{T}{A}) = 0 \). Therefore, \( A \subseteq ace \) C in W.

**Proposition 2.7:** If \( W \) is an \( R \)-module and \( A, B \) and \( C \) are submodules of \( W \), such that \( A + C \subseteq \text{ace} B + C \) in \( W \), then \( A \subseteq ace B \) in \( W \).

**Proof:** Let \( T \) be a submodule of \( W \), such that \( A \subseteq T \), and suppose that \( M = \frac{B \cap T}{A} \). So \( W = B + T \) and then \( \frac{W}{A + C} = \frac{B + C}{A + C} \), where \( A + C \subseteq \text{ann}(\frac{T}{A}) = 0 \). Let \( r \in \text{ann}(\frac{T}{A}) \), thus \( rT \subseteq A \subseteq A + C \) and hence \( r(T + A + C) = A + C \). Then, \( r \in \text{ann}(\frac{T+C}{A+C}) = 0 \), thus \( \text{ann}(\frac{T}{A}) = 0 \). So \( A \subseteq ace B \) in \( W \).

**Proposition 2.8:** Let \( A, B, C \) and \( X \) be submodules of an \( R \)-module \( W \). The following statements are the same:

1. If \( A \subseteq ace A + B \) in \( W \), then \( A \cap B \subseteq ace B \) in \( W \).
2. If \( A \subseteq ace B \) in \( W \) and \( Y \subseteq W \), then \( A \cap Y \subseteq ace B \cap Y \) in \( W \).
3. If \( A \subseteq ace B \) in \( W \), then \( A \cap X \subseteq ace B \cap C \in W \).

**Proof:** (1) \( \Rightarrow \) (2) Let \( A \subseteq_{ace} B \) in \( W \) and \( Y \subseteq W \). Since \( A + (B \cap Y) \subseteq B \) and \( A \subseteq ace B + (B \cap Y) \) (by proposition (1.3)). By (1), \( A \subseteq ace B \cap Y \), this implies that \( A \cap Y \subseteq ace B \cap Y \) in \( W \).

(2) \( \Rightarrow \) (3) Let \( A \subseteq ace B \) in \( W \) and \( X \subseteq ace C \) in \( W \). By (2) \( A \cap X \subseteq ace B \cap X \) in \( W \). Also, \( X \subseteq ace C \) and \( B \subseteq W \), \( B \cap X \subseteq ace B \cap C \) in \( W \). Thus, \( A \cap X \subseteq ace B \cap C \) in \( W \).

(3) \( \Rightarrow \) (1) Suppose that \( A \subseteq ace A + B \) in \( W \). Since \( B \subseteq ace B \) in \( W \), then by (3), \( A \cap B \subseteq ace (A + B) \) \( \cap B \). Thus, \( A \cap B \subseteq ace B \) in \( W \).

3. R-annihilator-closed submodules

In this section, we introduce the concept of \( (R \)-annihilator) \( R \)-a-closed as a generalization of coclosed submodules, where a submodule \( N \) of an \( R \)-module \( W \) is called coclosed submodule in \( W \) (denoted by \( F \subseteq ace W \)) if whenever \( K \subseteq F \) with \( \frac{F}{K} \ll \frac{W}{K} \) implies \( F = K \).

**Definitions 3.1:**

Let \( W \) be an \( R \)-module, then a submodule \( F \) of \( M \) is called R-annihilator-closed in \( W \) (briefly \( R \)-a-closed) if whenever \( K \subseteq F \) with \( \frac{F}{K} \ll \frac{W}{K} \) implies \( F = K \). (denoted by \( F \subseteq ace W \)).

**Examples and Remarks 3.2**

1. The submodule \( \{0, 2\} \) of a \( Z \)-module \( Z_4 \) is \( Z \)-a-closed in \( Z_4 \) since \( \{0, 2\} \subseteq \{0, 2\} \) and \( \{0, 2\} \approx \{0, 2\} \), also \( \{0, 2\} \approx Z_4 \). But \( \{0, 2\} \) is not \( Z \)-a-small in \( Z_4 \) because \( \{0, 2\} + Z_4 = Z_4 \) and \( \text{ann}_Z Z_4 = 4Z \neq 0 \).
2. The submodule \( 2Z \) of a \( Z \)-module \( Z \) is not \( Z \)-a-closed of \( Z \). To recognize that, let \( \{0\} \subseteq 2Z \) and notice that \( \frac{2Z}{\{0\}} = 2Z \), \( \frac{Z}{\{0\}} \approx Z \). We also know that \( 2Z \ll_a Z \) [1, Rem. & Ex (1.2.13)].
3. The concepts of coclosed and \( R \)-a-closed submodules are independent, since \( \{0, 2\} \) is \( Z \)-a-closed in the \( Z \)-module \( Z_4 \) but it is not coclosed in \( Z_4 \) as \( Z \)-module [7, Rem. & Ex. (1.2.3), p15].
4. Consider the \( Z_8 \)-module \( Z_8 \), the submodule \( \{0, 2, 4, 6\} \) is \( Z_8 \)-a-closed but not coclosed since \( \{0\} \subseteq \{0, 2, 4, 6\} \subseteq Z_8 \) \( \frac{\{0, 2, 4, 6\}}{\{0\}} \approx \{0, 2, 4, 6\} \). Also \( \frac{Z_8}{\{0\}} \approx Z_8 \), but \( \{0, 2, 4, 6\} \) is not \( Z_8 \)-a-closed since \( \{0, 2, 4, 6\} \) is not \( Z_8 \)-a-small. But \( \{0, 2, 4, 6\} \) \( \ll a Z_8 \)-module, thus it is coclosed in \( Z_8 \).

An \( R \)-module \( W \) is called \( R \)-a-hollow if every proper submodule of \( W \) is \( R \)-a-small [8], where a submodule \( F \) of \( W \) is \( R \)-a-small if whenever \( W = F + T \), where \( T \subseteq W \), then \( \text{ann}(T) = 0 \). [1]

**Proposition 3.3**

Let \( W \) be an \( R \)-module and \( L \) be a nonzero submodule of \( W \) which is \( R \)-a-hollow, then either \( L \) is \( R \)-a-small submodule of \( W \) or \( L \) is \( R \)-a-closed submodule of \( W \), but not both.

**Proof:** Let \( 0 \neq L \subseteq W \) and suppose that \( L \) is not \( R \)-a-closed of \( W \). So, there exists \( K < L \) such that \( \frac{L}{K} \ll_a \frac{W}{K} \), but [by 1, cor.(2.1.6), p34] \( L \ll_a W \).

Now, if \( L \) is \( R \)-a-closed submodule of \( W \), and suppose that \( L \ll_a W \), then \( \frac{L}{0} \ll_a \frac{W}{0} \approx W \) and hence \( L = 0 \) which is contradiction.

822
Proposition 3.4: Let $W$ be an $R$-module and $K \leq L$ be submodules of $W$. If $L \leq_{acc} W$, then $\frac{L}{K} \leq_{acc} \frac{W}{K}$.

**Proof:** Let $\frac{N}{K} \leq_{acc} \frac{L}{K}$ in $\frac{W}{K}$. We must prove that $\frac{N}{K} = \frac{L}{K}$. We have $\frac{L}{K} \ll_{a} \frac{W/K}{N/K}$. This means that $\frac{L}{N} \ll_{a} \frac{W}{N}$, but $L \leq_{acc} W$. Therefore, $L = N$ and hence $\frac{N}{K} = \frac{L}{K}$. Thus $\frac{L}{K} \leq_{acc} \frac{W}{K}$.

Lemma 3.5:

Let $W$ be an $R$-module and let $X$, $K$ and $L$ be submodules of $W$ such that $X \subseteq K \subseteq L \subseteq W$. If $\frac{K}{X} \ll_{a} \frac{L}{X}$ and $\frac{L}{X} \ll_{a} \frac{W}{X}$, then $\frac{K}{X} \ll_{a} \frac{W}{X}$.

**Proof:** Suppose that $\frac{W}{X} = \frac{L}{X} + \frac{T}{X}$, where $T$ is a submodule of $W$ such that $X \subseteq T$. Thus, $W = K + T$. But $K \subseteq L$, therefore $W = L + T$. So, $\frac{W}{X} = \frac{L}{X} + \frac{T}{X}$. But $\frac{L}{X} \ll_{a} \frac{W}{X}$, thus $ann\left(\frac{T}{X}\right) = 0$, which means that $\frac{K}{X} \ll_{a} \frac{W}{X}$.

Proposition 3.6

Let $W$ be an $R$-module and $K \leq L$ be submodules of $W$. If $K \leq_{acc} W$ and $\frac{L}{K} \ll_{a} \frac{W}{K}$, then $K \leq_{acc} L$.

**Proof:** Let $X \leq K$ such that $\frac{K}{X} \ll_{a} \frac{L}{X}$. Since $\frac{L}{K} \ll_{a} \frac{W}{X}$ then $\frac{K}{X} \ll_{a} \frac{W}{X}$ by lemma 3.5. But $K \leq_{acc} W$. Therefore, $K = X$ and hence $K \leq_{acc} L$.

Proposition 3.7: Let $W$ be an $R$-module and $L$, $N$ are submodule. If $L \leq_{acc} W$ then $\frac{L+N}{N} \leq_{acc} \frac{W}{N}$.

**Proof** Suppose that $\frac{X}{N} \leq_{acc} \frac{L+N}{N}$ in $\frac{W}{N}$ where $N \subseteq X$, then $X = N + (L \cap X) \leq_{acc} N + L$ in $W$ [by prop. (2.5)]. But $N \leq W$, which implies that $(L \cap X) \leq_{acc} L$ in $W$, by [prop. 2.7] As $L \leq_{acc} W$, we get $X = L + N$. Then $\frac{L+N}{N} \leq_{acc} \frac{W}{N}$.

References