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Some Chaotic Results of Product on Zero Dimension Spaces

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Abstract
In this work , we study different chaotic properties of the product space on a one-step shift of a finite type, as well as other spaces. We prove that the product $\mathcal{H} \times \mathcal{L}$ is Lyapunove ε –unstable if and only if at least one $\mathcal{H}$ or $\mathcal{L}$ is Lyapunove ε – unstable. Also, we show that $\mathcal{H}$ and $\mathcal{L}$ are locally everywhere onto *(l.e.o)* if and only if $\mathcal{H} \times \mathcal{L}$ is locally everywhere onto *(l.e.o)* .

Keywords: Devaney Chaos; Lypanoveε –unstable ; Topological Mixing; Shift of finite type ; weakly blending; strongly blending .

بعض النتائج الفوضوية في دوال فضاءات البعد الصفري

فرح وطن كامل* ، افتخار مضر طالب الشرع
قسم الرياضيات ، جامعة بابل ، كلية التربية للعلوم الصرفة ، بابل ، العراق

الخلاصة
في هذا العمل , نحن ندرس الخصائص الفوضوية المختلفة لفضاء الضرب على دالة التزحيف من النوع المحدد وفضاءات اخرى . نثبت ان ضرب $\mathcal{H} \times \mathcal{L}$ هو ليبانوف $\varepsilon-$ غير مستقر اذا وفقط على الاقل $\mathcal{H}$ او $\mathcal{L}$ هو ليبانوف $\varepsilon-$ غير مستقر أيضا , نظهر اذا كان $\mathcal{H}$ و $\mathcal{L}$ محليا في كل مكان وشاملة اذا وفقط اذا كان $\mathcal{H} \times \mathcal{L}$ محليا في كل مكان وشاملة .

Introduction

A discrete dynamical system is a way of describing the passage in discrete times of all points in a given space X. A continuous map $\mathcal{H}$ describes the rule of changes of each point. Therefore we can define a discrete dynamical system by a map $\mathcal{H}: X \rightarrow X$.
We analyze its dynamical behavior to determine if the system settles steady to equipoise, goes on in repeating cycles, or does something more complex.

A chaotic dynamical system is an unpredictable system which can be found within complicated dynamical systems, as well as the almost trivial systems. Several efforts have been made to give the notion of chaos a mathematically precise meaning. However, chaos is not simple to define and has no universally concordant definition [1].

Until the end of the 1980s the subject of chaotic dynamics was limited mainly to research-oriented publications by Devaney, where his famous definition of chaos was (A map $\mathcal{H}: X \rightarrow X$ is said to be Devaney chaotic if it satisfies; $\mathcal{H}$ is topologically transitive, the periodic points of $\mathcal{H}$ is dense, and $\mathcal{H}$ posses sensitive dependence on initial conditions or simply (*SDIC*)) [1,2].

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In another work [3], Dzul-kifli and Good showed that the set of points with prime period at least n is dense for each n if $\mathcal{H}$ is Devaney chaotic on a compact metric space with no isolated points . In their article [4], Baloush and Dzul-kifli introduced six various one-step shifts of finite types, with totally different dynamic demeanor, and clearified the dynamics of each space Other authors [5] showed that the expression "Locally Everywhere Onto" implies many other chaos properties such as mixing, totally transitive ,and blending. Another study investigated how chaos conditions on maps hold over to their products [6].

Finally, Iftichar Talb [7] proved that some properties on a map were carried over to their product, and he also contracted the conditions of two maps to achieve that the product is chaotic.

2. Preliminaries

*Let $\mathcal{H}: X \to X$ be a map , let $p \in X$ then $\mathcal{H}(p) =$ the first iterate of p for $\mathcal{H}$. More generally ,if n is any an integer , and a_n is the n-th iterate of p for $\mathcal{H}$, then $\mathcal{H}(a_n)$ is the $(n+1)$ st iterate of p for $\mathcal{H}$. Let X be any metric space and $\mathcal{H}: X \to X$ be a map , **the orbit** of p is the set of points $p, \mathcal{H}(p), \mathcal{H}^2(p), \ldots$, and is symbolized by $orb(p) = \{\mathcal{H}^n(p) | n \in N_\circ\}$ such that $N_\circ = N \cup \{0\}$. A point $p \in X$ is said **a fixed point** of $\mathcal{H}$* if $\mathcal{H}(\mathrm{p}) = \mathrm{p}$.

Definition 2.1. [5]

Let X be any metric space with a metric d and $\mathcal{H}: \mathrm{X} \to \mathrm{X}$ be a map , $\mathcal{H}$ is said to be topologically transitive if $\exists\, \mathrm{n} > 0$ such that $\mathcal{H}^{\mathrm{n}}(\mathrm{U}) \cap \mathrm{V} \neq \emptyset$, where U, V are any two non-empty open subsets of X.

Definition 2.2. [5]

Let $\mathcal{H}: \mathrm{X} \to \mathrm{X}$ be a map with metric a d , if $\exists\, \delta > 0$ for this any $\mathrm{x} \in \mathrm{X}$ and neighborhood N of x , $\exists\, \mathrm{y} \in \mathrm{N}$ and $\mathrm{n} > 0$ where $\mathrm{d}(\mathcal{H}^{\mathrm{n}}(\mathrm{x}), \mathcal{H}^{\mathrm{n}}(\mathrm{y})) > \delta$, then $\mathcal{H}$ has **a** Sensitive Dependence on Initial Conditions (SDIC).

Definition 2.3. [5]

Let X be any metric space and $\mathcal{H}: \mathrm{X} \to \mathrm{X}$ be a continuous a map , if for every pair non-empty open subsets U and V in X , there is a positive integer n such that $\mathcal{H}^{\mathrm{k}}(\mathrm{U}) \cap \mathrm{V} \neq \emptyset$ for every $\mathrm{k} > n$, then we call that $\mathcal{H}$ as topological mixing .

Definition 2.4. [3]

Let $\mathcal{H}: \mathrm{X} \to \mathrm{X}$ be a map , if for any pair of non-empty open sets U and V in X , there exists some $\mathrm{n} > 0$ such that $\mathcal{H}^{\mathrm{n}}(\mathrm{U}) \cap \mathcal{H}^{\mathrm{n}}(\mathrm{V}) \neq \emptyset$, then we call that $\mathcal{H}$ as weakly blending, and call it as strongly blending if for any pair of non-empty open sets U and V in X , there exists some $\mathrm{n} > 0$ where $\mathcal{H}^{\mathrm{n}}(\mathrm{U}) \cap \mathcal{H}^{\mathrm{n}}(\mathrm{v})$ contains a non-empty open subset.

Definition 2.5. [8]

Let $\mathcal{H}: \mathrm{X} \to \mathrm{X}$ be a map with a metric . Given $\varepsilon > 0$, if for every neighborhood N(x) of x , there exist $\mathrm{y} \in \mathrm{N(x)}$ and $\mathrm{n} \geq 0$ with $\mathrm{d}(\mathcal{H}^{\mathrm{n}}(\mathrm{x}), \mathcal{H}^{\mathrm{n}}(\mathrm{y})) > \varepsilon$, then the map $\mathcal{H}$ is said Lyapunov ε-unstable at a point $\mathrm{x} \in \mathrm{X}$.

Definition 2.6. [5]

Let $\mathcal{H}: X \to X$ be a map that is said to be *locally everywhere onto* if for every open set U of X , there exists a positive integer n such that $\mathcal{H}^n(U) = X$.

3. On The Shift of Finite Type

The product space $\{0,1\}^{\mathbb{N}} = \{s = \{s_i\}_{i \in \mathbb{N}} : s_i \in \{0,1\}\ for\ all\ i \in \mathbb{N}\}$ is a topological space. A convenient basis for $\{0,1\}^{\mathbb{N}}$ is given by the cylinder sets

$$C[b_1 \ldots b_m] = \{x \in \{0,1\}^{\mathbb{N}} : b_i = x_i\ for\ \ i \leq m\}$$

Where $m \in \mathbb{N}$ and $b_i \in \{0,1\}$ for each $i \leq m$. when equipped with the metric d defined as

$$d(s,t) = \begin{cases} 0 & if\ \ s = t \\ 2^{-j} & if\ s \neq t \end{cases}$$

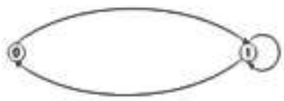
where $j \in \mathbb{N}$ is the minimal number such that $s_j \neq t_j$ since this gives the cylinder sets as open ball $B_m(x) = C[x_1 \ldots x_m]$, [8]. An *l-block* is a finite sequence of symbols of the length l, i.e. $s_0 s \ldots s_{l-1}$. We now define a continuous map σ on$\{0,1\}^{\mathbb{N}}$, called the *shift map* as $\sigma(s_0 s_1 s_2 \ldots) = s_1 s_2 s_3 \ldots$. Shift map σ deletes the first entry of the sequence in $\{0,1\}^{\mathbb{N}}$ to produce the image of the sequence under σ . *Shift space* is a closed subset of full-2-shift which is invariant under σ [9] . We write $\sigma\sigma$ instead of $\times\, \sigma$. The most widely studied shift spaces are called *shifts of finite type,* defined as follows

Definition 3.1 [9]

A shift space $X \subset \{0,1\}^{\mathbb{N}}$ is a shift of finite type if there is a finite number of blocks over symbols 0 and 1 where the blocks do not occur in any element of . The blocks are said forbidden blocks in .

Because we only have four possible distinct blocks of length two, i.e. 00, 01,10 and 11, then we have 16 sets of forbidden blocks. For each $i = \{1,2,\dots,16\}, X_i \subset \{0,1\}^{\mathbb{N}}$ is the one-step SFT with a set of forbidden blocks $\mathcal{F}_i$. Then , there exists that some of them are singletons, empty set or the whole $\{0,1\}^{\mathbb{N}}$, which have trivial dynamics and are not of our interest. There exists six distinct one-step shift of finite types over two symbols, X_2, X_3, X_6, X_7, X_8 and X_9, with sets of forbidden blocks $\mathcal{F}_2 = \{00\}$, $\mathcal{F}_3 = \{01\}$, $\mathcal{F}_6 = \{00,01\}$, $\mathcal{F}_7 = \{00,10\}$, $\mathcal{F}_8 = \{00,11\}$ and $\mathcal{F}_9 = \{01,10\}$, respectively [1]. They are shown below through matrices and their own graph

$$X_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

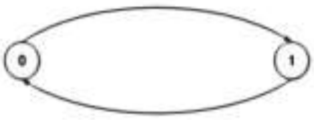


$$X_3 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$X_6 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$X_7 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$X_8 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



$$X_9 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

4. Main Results

In this section , some results on $\{0,1\}^{\mathbb{N}}$ are proved.

Theorem 4.1

Let the one-step SFT X_2 , X_8 and X_9 which has sets of forbidden blocks $\mathcal{F}_2 = \{00\}$, $\mathcal{F}_8 = \{00,11\}$ and $\mathcal{F}_9 = \{01,10\}$, respectively , $\sigma: X_i \to X_i \quad i = \{2,8,9\}$ be a map . The periodic points of $\sigma\sigma: X_i \times X_j \to X_i \times X_j \quad i,j = \{2,8,9\}$ are dense if and only if the periodic points of X_i are dense.

Proof:

Let us assume that the periodic points of X_i are dense. To prove the periodic points of $X_i \times X_j$ are dense, let $W \subset X_i \times X_j$ be any non- empty cylinder set so there are two non- empty cylinder sets $U = C[s_1 \dots s_m]$, $V = C[t_1 \dots t_m] \subset X_i$ with $U \times V \subset W$. According to the hypothesis , there is a point $s = \{s_0 s_1 s_2 \dots s_n \dots\} \in U$ where $\sigma^n(s) = s$ with $n \geq 1$. Similarly , there exists $t = \{t_0 t_1 t_2 \dots t_q \dots\} \in V$ such that $\sigma^q(t) = t$ with $q \geq 1$, for $p = (s,t) \in W$ and $k = qn$, we get

$(\sigma\sigma)^k(p) = (\sigma\sigma)^k(s,t) = \left(\sigma^k(s), \sigma^k(t)\right) = (s,t) = p$ then W has a periodic point and the periodic points of $X_i \times X_j$ are dense .
Conversely, let $U = C[s_1 \dots s_m]$ and $V = C[t_1 \dots t_m] \subset X_i$ be non-empty cylinder sets so $U \times V$ is a non-empty cylinder set in $X_i \times X_j$ $\forall\, i,j = \{2,8,9\}$, the set of periodic points of $\sigma\sigma$ is dense in $X_i \times X_j$, there is a point $p(s,t)$ in $U \times V$ such that $(\sigma\sigma)^n(s,t) = \left(\sigma^n(s), \sigma^n(t)\right) = (s,t)$ for some n , from the lest equality we obtain $\sigma^n(s) = s$ for $s = \{s_0 s_1 s_2 \dots s_n \dots\} \in U$ and $\sigma^n(t) = t$ for $t = \{t_0 t_1 t_2 \dots t_n \dots\} \in V$. ■

Theorem 4.2

Let the one-step SFT X_2 , X_8 and X_9 which has a set of forbidden blocks $\mathcal{F}_2 = \{00\}$, $\mathcal{F}_8 = \{00,11\}$ and $\mathcal{F}_9 = \{01,10\}$, respectively , the map $\sigma: X_i \to X_i$ $i = \{2,8,9\}$ is SDIC if and only if $\sigma\sigma: X_i \times X_j \to X_i \times X_j$, $i,j = \{2,8,9\}$ is SDIC .

Proof:

Let σ is SDIC, to see that the same is correct for , let $p = (s,t) \in X_i \times X_j$ be any point and $W = C[p_1 \dots p_m]$ any cylinder set of p . Then there exist two non- empty cylinder sets $U = C[s_1 \dots s_m]$ of $s = \{s_0 s_1 s_2 \dots s_n \dots\}$ in X_i and $V = C[t_1 \dots t_m]$ of $t = \{t_0 t_1 t_2 \dots t_n \dots\}$ in X_j $\forall\, i,j = \{2,8,9\}$ where $\times V \subset W$. As σ is SDIC, there is $\varepsilon > 0$ for this any $\acute{s} = \{\acute{s}_0 \acute{s}_1 \acute{s}_2 \dots \acute{s}_n \dots\} \in U$ and integer $n \geq 1$. The inequality $d_1\left(\sigma^n(s), \sigma^n(\acute{s})\right) > \varepsilon$ holds then for any $\acute{t} = \{\acute{t}_0 \acute{t}_1 \acute{t}_2 \dots \acute{t}_n \dots\} \in V$, $\acute{p} = (\acute{s}, \acute{t})$ belong to W and

$$\begin{aligned} d\left((\sigma\sigma)^n(p), (\sigma\sigma)^n(\acute{p})\right) &= max\left\{d_1\left(\sigma^n(s), \sigma^n(\acute{s})\right), d_2\left(\sigma^n(t), \sigma^n(\acute{t})\right)\right\} \\ &\geq d_1\left(\sigma^n(s), \sigma^n(\acute{s})\right) > \varepsilon \end{aligned}$$

This means that $\sigma\sigma$ is SDIC.

Let σ is not SDIC , this means that given any $\varepsilon > 0$ there exists $s = \{s_0 s_1 s_2 \dots s_n \dots\} \in X_i$ $\forall\, i = \{2,8,9\}$ such that for a certain cylinder set $U = C[s_1 \dots s_m] \subset X_i$ containing s , the inequality $d_1\left(\sigma^n(s), \sigma^n(\acute{s})\right) < \varepsilon/2$ holds for every $\acute{s} = \{\acute{s}_0 \acute{s}_1 \acute{s}_2 \dots \acute{s}_n \dots\} \in U$ and positive integer n .

Identically, there is $t \in X_j$ $j = \{2,8,9\}$ such that for a certain cylinder set $V = C[t_1 \dots t_m] \subset X_j$ containing t, the inequality $d_2\left(\sigma^n(t), \sigma^n(\acute{t})\right) < \varepsilon/2$ holds for every $\acute{t} = \{\acute{t}_0 \acute{t}_1 \acute{t}_2 \dots \acute{t}_n \dots\} \in V$ and positive integer n .
Then we get

$$d\left((\sigma\sigma)^n(p), (\sigma\sigma)^n(\acute{p})\right) = \max\left\{d_1\left(\sigma^n(s), \sigma^n(\acute{s})\right), d_2\left(\sigma^n(t), \sigma^n(\acute{t})\right)\right\} < \varepsilon$$

For $(\acute{s}, \acute{t}) \in U \times V$ so that it is not SDIC, which disagrees with the hypothesis.

Theorem 4.3

Let the one-step SFT X_2 and X_8 which has the sets of forbidden blocks $\mathcal{F}_2 = \{00\}$, $\mathcal{F}_8 = \{00,11\}$, let $\sigma: X_i \to X_i$, $i = \{2,8\}$ be a map and let the product $\sigma\sigma$ is topologically transitive on $X_i \times X_j$, $\forall\, i,j = \{2,8\}$, then the map σ is topologically transitive on X_i.
Proof:

Let $U_1 = C[\acute{x}_1, \acute{x}_2, \dots, \acute{x}_m], V_1 = C[x_1, x_2, \dots, x_n]$ be non-empty cylinder sets in X_i , so the sets $U = U_1 \times X_j$ and $V = V_1 \times X_j$ are cylinder sets in $X_i \times X_j$. As $\sigma\sigma$ is transitive , there is a positive integer k for this $(\sigma\sigma)^k(U) \cap V \neq \emptyset$. then

$$\begin{aligned} (\sigma\sigma)^k(U) \cap V &= \left(\sigma^k(U_1) \times \sigma^k(X_j)\right) \cap \left(V_1 \times X_j\right) \\ &= \left(\sigma^k(U_1) \cap V_1\right) \times \left(\sigma^k(X_j) \cap X_j\right) \end{aligned}$$

It follows that $\left(\sigma^k(U_1) \cap V_1\right) \times \left(\sigma^k(X_j) \cap X_j\right) \neq \emptyset$, so $\sigma^k(U_1) \cap V_1 \neq \emptyset$, thus σ is topologically transitive.

Theorem 4.4

Let the one-step SFT $X_2 \subset \sum_2$ which has a set of forbidden blocks $\mathcal{F}_2 = \{00\}$, the map $\sigma: X_2 \to X_2$ is topologically mixing then $\sigma\sigma : X_2 \times X_2 \to X_2 \times X_2$ is topologically mixing .
Proof:

Let $\sigma : X_2 \to X_2$ be a topologically mixing map . Given $W_1, W_2 \subset X_2 \times X_2$, there exists cylinder sets $p_1, p_2, q_1, q_2 \subset X_2$ such that $p_1 \times q_1 \subset W_1$ and $p_2 \times q_2 \subset W_2$.

By the assumption that there exist n_1 and n_2 such that $\sigma^n(p_1) \cap p_2 \neq \emptyset$ for $n \geq n_1$ and $\sigma^n(q_1) \cap q_2 \neq \emptyset$ for $\geq n_2$. For $n \geq n_0 = \max\{n_1, n_2\}$ we get

$$\big((\sigma\sigma)^n(p_1 \times q_1)\big) \cap (p_2 \times q_2) = \big(\sigma^n(p_1) \times \sigma^n(q_1)\big) \cap (p_2 \times q_2)$$
$$= (\sigma^n(p_1) \cap p_2) \times (\sigma^n(q_1) \cap q_2) \neq \emptyset$$

Which means that $\sigma\sigma$ is topologically mixing. ∎

Theorem 4.5

Let the one-step SFT X_2 and X_7 which have the sets of forbidden blocks $\mathcal{F}_2 = \{00\}$,$\mathcal{F}_7 = \{00,10\}$, the map $\sigma: X_i \to X_i$, $i = \{2,7\}$ is weakly blending if and only if $\sigma\sigma: X_i \times X_j \to X_i \times X_j$, $i, j = \{2,7\}$ is weakly blending .

Proof:

If σ is weakly blending let U and V be non- empty cylinder sets in $\sigma\sigma$, then there are U_1 and V_1 non-empty cylinder of X_i and there are U_2 and V_2 cylinder in X_j such that $U = U_1 \times U_2$ and $V = V_1 \times V_2$. Since σ is weakly blending , then $\exists\, k_1 \in \mathbb{N}$ such that $\sigma^{k_1}(U_1) \cap \sigma^{k_1}(V_1) \neq \emptyset$ and $\exists\, k_2 \in \mathbb{N}$ such that $\sigma^{k_2}(U_2) \cap \sigma^{k_2}(V_2) \neq \emptyset$, choose $k = k_1 + k_2$, then $\sigma^k(U_1) \cap \sigma^k(V_1) \neq \emptyset$ and $\sigma^k(U_2) \cap \sigma^k(V_2) \neq \emptyset$, then

$$(\sigma\sigma)^k(U) \cap (\sigma\sigma)^k(V) = (\sigma\sigma)^k(U_1 \times U_2) \cap (\sigma\sigma)^K(V_1 \times V_2)$$
$$= \Big(\sigma^k(U_1) \times \sigma^k(U_2)\Big) \cap \Big(\sigma^k(V_1) \times \sigma^k(V_2)\Big)$$
$$= \Big(\sigma^k(U_1) \cap \sigma^k(V_1)\Big) \times \Big(\sigma^k(U_2) \cap \sigma^k(V_2)\Big) \neq \emptyset$$

Therefore, $\sigma\sigma$ is weakly blending.

Conversely , if $\sigma\sigma$ is weakly blending , to show that σ is weakly blending , let U_1 and V_2 are non-empty cylinder sets in X_i , thus $\exists\, U = U_1 \times X_j$ and $V = V_1 \times X_j$ are non-empty cylinder of $X_i \times X_j$. Since $\sigma\sigma$ is weakly blending , then $\exists\, k \in \mathbb{N} \ni$
$(\sigma\sigma)^k(U) \cap (\sigma\sigma)^K(V) \neq \emptyset$. We let

$$\emptyset \neq \Big(\sigma^k(U_1) \times \sigma^k(X_j)\Big) \cap \Big(\sigma^k(V_1) \times \sigma^k(X_j)\Big)$$
$$= \Big(\sigma^k(U_1) \cap \sigma^k(V_1)\Big) \times \Big(\sigma^k(X_j) \cap \sigma^k(X_j)\Big)$$

Thus $\sigma^k(U_1) \cap \sigma^k(V_1) \neq \emptyset$, then σ is weakly blending . ∎

Theorem 4.6

Let the one-step SFT X_2 and X_7 which have the sets of forbidden blocks $\mathcal{F}_2 = \{00\}$,$\mathcal{F}_7 = \{00,10\}$, the map $\sigma: X_i \to X_i$, $i = \{2,7\}$ is strongly blending if and only if $\sigma\sigma: X_i \times X_j \to X_i \times X_j$, $i, j = \{2,7\}$ is strongly blending .

Proof:

Let W_1 and W_2 be non–empty cylinder sets in $X_i \times X_j$, then $\exists\, U_1, U_2 \subset X_i$ are non-empty cylinder sets in X_i , and $\exists V_1, V_2 \subset X_j$ are non-empty cylinder sets in X_j , such that $W_1 = U_1 \times V_1$ and $W_2 = U_2 \times V_2$. Since σ is strongly blending , thus $\exists\, k_1 > 0 \ni \sigma^{k_1}(U_1) \cap \sigma^{k_1}(U_2)$ contain open set , then $\exists\, N_1 \subset X_i \ni N_i \subset \sigma^{k_1}(U_1) \cap \sigma^{k_1}(U_2)$, and $\exists\, k_2 > 0 \ni \sigma^{k_2}(V_1) \cap \sigma^{k_2}(V_2)$ contain open set , that is , $N_2 \subset X_j \ni N_2 \subset \sigma^{k_2}(V_1) \cap \sigma^{k_2}(V_2)$. Let $k = k_1 + k_2$, we get

$$(\sigma\sigma)^k(W_1) \cap (\sigma\sigma)^k(W_2) = \Big(\sigma^k(U_1) \times \sigma^k(V_1)\Big) \cap \Big(\sigma^k(U_2) \times \sigma^k(V_2)\Big)$$

$= \Big(\sigma^k(U_1) \cap \sigma^k(U_2)\Big) \times \Big(\sigma^k(V_1) \cap \sigma^k(V_2)\Big) \supset \sigma^k(N_2) \cap \sigma^k(N_1)$. It is easy to show that $\sigma^{k_1}(N_2) \cap \sigma^{k_2}(N_1)$ is an open set , then $\sigma\sigma$ is strongly blending .

Conversely , let U_1 and U_2 be non-empty cylinder sets in X_i , $\exists U = U_1 \times X_j$,$V = V_1 \times X_j$ $\ni$ $\exists k > 0 \ni (\sigma\sigma)^k U \cap \sigma\sigma^k$ contain open set ,$(\sigma\sigma)^k\big(U_1 \times X_j\big) \cap (\sigma\sigma)^k\big(V_1 \times X_j\big) = \big(\sigma^k U_1 \cap \sigma^k V_1\big) \times \big(\sigma^k X_j \cap \sigma^k X_j\big)$ contain open set , since $\big(\sigma^k U_1 \cap \sigma^k V_1\big)$ is an open set , and $\big(\sigma^k X_j \cap \sigma^k X_j\big)$ is a non-empty set then f is strongly blending.

5. Some properties in the product Space

In this section , we prove the results on the product maps defined on any metric spaces .

Theorem 5.1

Let X and Y be compact metric space , the map $\mathcal{H}: X \to X$ and $\mathcal{L}: y \to y$ be continuous maps , the product $\mathcal{H} \times \mathcal{L}$ is Lyapunove ε –unstable if and only if at least one $\mathcal{H}$ or $\mathcal{L}$ is Lyapunove ε –unstable .

Proof :

Let $\mathcal{H}$ be Lyapunove ε –unstable . Given $\varepsilon > 0$. let $P = (s, t) \in X \times Y$ be any point and N is any neighborhood of P, then there is an open neighborhood U of s in X and V of t in Y such that $U \times V \subset N$. Since $\mathcal{H}$ is Lyapunove ε –unstable , given $\varepsilon > 0$ for this any $\acute{s} \in U$ and an integer $n > 0$ then $d_1(\mathcal{H}^n(s), \mathcal{H}^n(\acute{s})) > \varepsilon$. So for any $\acute{t} \in V$, $\acute{P} = (\acute{s}, \acute{t})$ belongs to N and

$$d\left((\mathcal{H} \times \mathcal{L})^n(P), (\mathcal{H} \times \mathcal{L})^n(\acute{P})\right) = d_1(\mathcal{H}^n(s), \mathcal{H}^n(\acute{s})) + d_2(\mathcal{L}^n(t), \mathcal{L}^n(\acute{t}))$$
$$\geq d_1(\mathcal{H}^n(s), \mathcal{H}^n(\acute{s})) > \square$$

so that $\mathcal{H} \times \mathcal{L}$ is Lyapunove ε -unstable .

Conversely, let both $\mathcal{H}$ and $\mathcal{L}$ are not Lyapunove ε –unstable, so that , given $\varepsilon > 0$ there is $s \in X$ such that for any open set $U \subset X$ containing s , then

$$d_1(\mathcal{H}^n(s), \mathcal{H}^n(\acute{s})) < \varepsilon/2$$

for every $\acute{s} \in U$ and positive integer n . Identically , there exists $t \in Y$ such that for any open set $V \subset Y$ containing t , then

$$d_2(\mathcal{L}^n(t), \mathcal{L}^n(\acute{t})) < \varepsilon/2$$

for every $\acute{t} \in V$ and positive integer n , then we get

$$d\left((\mathcal{H} \times \mathcal{L})^n(P), (\mathcal{H} \times \mathcal{L})^n(\acute{P})\right) = d_1(\mathcal{H}^n(s), \mathcal{H}^n(\acute{s})) + d_2(\mathcal{L}^n(t), \mathcal{L}^n(\acute{t})) < \varepsilon$$

For $(\acute{s}, \acute{t}) \in U \times V$. So that $\mathcal{H} \times \mathcal{L}$ is not Lyapunove ε –unstable , contradicting

So either $\mathcal{H}$ or $\mathcal{L}$ is Lypanoveε –unstable.

Theorem 5.2

Let X and Y be any spaces, the map $\mathcal{H}: X \to X$ and $\mathcal{L}: Y \to Y$ are not necessarily continuous maps . $\mathcal{H}$ and $\mathcal{L}$ are locally everywhere onto if and only if $\mathcal{H} \times \mathcal{L}$ is locally everywhere onto .

Proof:

Let $\mathcal{H}: X \to X$ and $\mathcal{L}: Y \to Y$ be locally everywhere onto . Given $W \subset X \times Y$, there are open sets $U \subset X$ and $V \subset Y$ for this $U \times V \subset W$. By assumption there exist n_1 and n_2 such that $\mathcal{H}^n(U) = X$ for $n \geq n_1$ and $\mathcal{L}^n(V) = Y$ for $n \geq n_2$.

For $n \geq n_0 = \max\{n_1, n_2\}$ we get

$$(\mathcal{H} \times \mathcal{L})^n(W) = (\mathcal{H} \times \mathcal{L})^n(U \times V)$$
$$= \mathcal{H}^n(U) \times \mathcal{L}^n(V)$$
$$= X \times Y$$

Which means that $\mathcal{H} \times \mathcal{L}$ is locally everywhere onto .

Conversely: if $\mathcal{H} \times \mathcal{L}$ is locally everywhere onto . To show that $\mathcal{H}$ and $\mathcal{L}$ is locally everywhere onto , let U_1 be a non-empty open subset in X and let V_1 be a non-empty open subset in Y , $\forall\, U = U_1 \times V_1$ are non-empty open subsets of $X \times Y$. Since $\mathcal{H} \times \mathcal{L}$ is locally everywhere onto then $\exists\, k \in \mathbb{N}$ such that

$$X \times Y = (\mathcal{H} \times \mathcal{L})^k(U)$$
$$= (\mathcal{H} \times \mathcal{L})^k(U_1 \times V_1)$$
$$= (\mathcal{H}^k \times \mathcal{L}^k)(U_1 \times V_1)$$
$$= \mathcal{H}^k(U_1) \times \mathcal{L}^k(V_1)$$

Thus $\mathcal{H}^k(U_1) = X$ and $\mathcal{L}^k(V_1) = Y$, then $\mathcal{H}$ and $\mathcal{L}$ are locally everywhere onto .

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