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Some Chaotic Properties of $\mathbb{G} -$ Average Shadowing Property

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Abstract

Let $(\mathcal{M}, d)$ be a metric $\mathbb{G}$ −space and $\phi : \mathcal{M} \to \mathcal{M}$ be a continuous map. The notion of the $\mathbb{G}$ -average shadowing property ($\mathbb{G}$ ASP) for a continuous map on $\mathbb{G}$ –space is introduced and the relation between the $\mathbb{G}$ ASP and average shadowing property(ASP)is investigated. We show that if ϕ has $\mathbb{G}$ASP, then ϕ^m has $\mathbb{G}$ASP for every $m \in \mathbb{N}$. We prove that if a map ϕ be pseudo-equivariant with dense set of $\mathbb{G}_\phi$ −periodic points and has the $\mathbb{G}$ ASP, then ϕ is weakly $\mathbb{G}$ −mixing. We also show that if ϕ is a $\mathbb{G}$ –expansive pseudo-equivariant homeomorphism that has the $\mathbb{G}$ASP and ϕ is topologically $\mathbb{G}$ −mixing, then ϕ has a $\mathbb{G}$ -specification. We obtained that the identity map ϕ on $\mathcal{M}$ has the $\mathbb{G}$ ASP if and only if the orbit space $\mathcal{M}/\mathbb{G}$ of $\mathcal{M}$ is totally disconnected. Finally, we show that if ϕ is a pseudo-equivariant map, and the trajectory map $\Psi : \mathcal{M} \to \mathcal{M}/\mathbb{G}$ is a covering map, then ϕ has the $\mathbb{G}$ASP if and only if the induced map $\breve{\phi} : \mathcal{M}/\mathbb{G} \to \mathcal{M}/\mathbb{G}$ has $\mathbb{G}$ASP.

Keywords: Shadowing ; Average shadowing; G-average shadowing; Topologically G-mixing; Weakly G-mixing ; G-specification.

بعض الخصائص الفوضوية لخاصية معدل التظليل في فضاء - $\mathbb{G}$

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الخلاصة

ليكن $(\mathcal{M},d)$ فضاء . $\mathbb{G}$ متريً ، $\phi : \mathcal{M} \to \mathcal{M}$ دالة مستمرة. قدمنا مفهوم خاصية معدل التظليل في فضاء . $\mathbb{G}$ (GASP) لدالة مستمرة على فضاء . $\mathbb{G}$ وتحقيق العلاقة بين $\mathbb{G}$ASP وخاصية معدل التضليل (ASP). لقد أثبتنا إذا كانت الدالة ϕ تمتلك $\mathbb{G}$ ASP ، فإن ϕ^m تمتلك $\mathbb{G}$ASP لكل $m \in \mathbb{N}$. لقد أثبتنا إذا كانت الدالة ϕ هي التكافؤ الكاذب مع مجموعة كثيفة من النقاط الدورية وتمتلك $\mathbb{G}$ASP ، فإن ϕ خلط ضعيف في فضاء . $\mathbb{G}$. أيضًا ، بيّنا إذا كانت الدالة ϕ هي دالة اتساع جي وتمتلك خاصية $\mathbb{G}$ASP و ϕهي خلط تبولوجي في فضاء . $\mathbb{G}$ فان ϕ تمتلك تخصيص جي . وبينا كذلك الدالة الذاتية ϕ على $\mathcal{M}$ تمتلك $\mathbb{G}$ASP إذا وفقط اذا فضاء المسار $\mathcal{M}/\mathbb{G}$ لـ $\mathcal{M}$ منفصل تمامًا. أخيرًا ،بينا إذا كانت الدالة ϕ هي التكافؤ الكاذب ، ودالة المدار $\Psi : \mathcal{M} \to \mathcal{M}/\mathbb{G}$ عبارة عن دالة تغطية ، فإن ϕ تمتلك $\mathbb{G}$ ASP إذا وفقط إذا كانت الدالة المستحدثة $\breve{\phi} : \mathcal{M}/\mathbb{G} \to \mathcal{M}/\mathbb{G}$ تمتلك GASP.

Introduction

The concept of shadowing property is one of the influential notions in the theory of dynamical systems. In 1967 The shadowing property (SP) was introduced by Anosov [1] and the concept of average shadowing property (ASP) was introduced by Blank for investigating chaotic dynamical systems [2]. In 1960, the notion of $\mathbb{G}$ −space was introduced by R. S. Palais [3]. The $\mathbb{G}$ −pseudo-

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trajectory tracing property on a metric $\mathbb{G}$ −space ($\mathbb{G}$PTTP) was introduced by Shah and Das. They studied various properties of such maps and obtained features for the identity map to have $\mathbb{G}$PTTP. Also, they showed that a pseudo-equivariant map $\phi : \mathcal{M} \to \mathcal{M}$ has $\mathbb{G}$PTTP if and only if the induced map $\hat{\phi} : \mathcal{M}/\mathbb{G} \longrightarrow \mathcal{M}/\mathbb{G}$ has PTTP such that $\mathcal{M}$ be metric $\mathbb{G}$ −space and ϕ is continuous map [4]. The $\mathbb{G}$ −shadowing property ($\mathbb{G}$ SP) for the map ϕ was introduced by Shah who observed through the examples that $\mathbb{G}$ −shadowing relies on the action of a group $\mathbb{G}$ acting on $\mathcal{M}$. Also, she studied $\mathbb{G}$ −shadowing for the shift map on the contrary limit space produced by the map ϕ [5].

In section 1 of this paper., we study the ASP for continuous maps on $\mathbb{G}$ −spaces ($\mathbb{G}$ ASP). In section 2, we prove some similar results on the ASP in the metric space with some chaotic properties and we put sufficient conditions to prove these results on $\mathbb{G}$ −spaces.

Preliminaries

Let $\mathbb{Z}$ denote the set of integers numbers, $\mathbb{N}$ denotes the set of natural numbers and $\mathcal{N}_0 = \{0\} \cup \mathbb{N}$. A topological group is a triple $(\mathbb{G},\mathcal{T},*)$, where $(\mathbb{G},*)$ is a group and $\mathcal{T}$ is a Hausdorff topology on $\mathbb{G}$ such that the map $\phi: \mathbb{G} \times \mathbb{G} \to \mathbb{G}$ defined by $\phi(m,y) = my^{-1}$ is continuous. By a $\mathbb{G}$ −space , we mean a triple $(\mathcal{M}, \mathbb{G}, \theta)$, where $\mathcal{M}$ is a Hausdorff space, $\mathbb{G}$ is a topological group, and $\theta: \mathbb{G} \times \mathcal{M} \to \mathcal{M}$ is a continuous action of $\mathbb{G}$ on $\mathcal{M}$ satisfying $\theta(e,m) = m$ and $\theta(g_1,\theta(g_2,m)) = \theta(g_1 g_2, m)$, where e is the identity of $\mathbb{G}$, $m \in \mathcal{M}$, and $g_1, g_2 \in \mathbb{G}$. An action θ of $\mathbb{G}$ on $\mathcal{M}$ is called trivial if $\theta(g,m) = m$, $\forall g \in \mathbb{G}$ and $m \in \mathcal{M}$.

For $m \in \mathcal{M}$, the set $\mathbb{G}(m) = \{\theta(g,m): g \in \mathbb{G}\}$ is called the $\mathbb{G}-$ trajectory of $m \in \mathcal{M}$. We will denote $\theta(g,m)$ by gm. For $S \subseteq \mathcal{M}$, let $gS = \{gs : s \in S\}$ be a subset S of a $\mathbb{G}$ −space and $\mathcal{M}$ is called $\mathbb{G}$ −invariant if $\theta(\mathbb{G} \times S) \subseteq S$. For $m \in \mathcal{M}$, the related $\mathbb{G}_\phi$- trajectory of m is presented by the set $\mathbb{G}_\phi(m) = \mathbb{G}\left(\mathrm{O}_\phi(m)\right) = \{g\phi^i(m): g \in \mathbb{G},\ i \in \mathcal{N}_0\}$. If $\mathcal{M}, Y$ are $\mathbb{G}$ −spaces, then a continuous map $h: \mathcal{M} \to Y$ is called equivariant map if $h(gm) = gh(m)$ for each g in $\mathbb{G}$ and each m in $\mathcal{M}$. In case an equivariant map is a homeomorphism, then h^{-1} is also equivariant. The quotient space $\frac{\mathcal{M}}{\mathbb{G}} = \{\mathbb{G}(m): m \in \mathcal{M}\}$, having $\mathbb{G}$ −orbits as its members, is called the orbit space of $\mathcal{M}$, and the quotient map $\psi : \mathcal{M} \to \mathcal{M}/\mathbb{G}$, taking m to $\mathbb{G}(m)$, is called the trajectory map. The map h is said to be pseudo-equivariant if $h(\mathbb{G}(m)) = \mathbb{G}(h(m))$, $\forall m \in \mathcal{M}$. Clearly, every equivariant map is a pseudo-equivariant map but the converse needs not to be true [6]. We introduce the definitions that we will need in this paper and recall some fundamental definitions. In this paper, we denote the metric $\mathbb{G}-$ space, on which there is a topological group $\mathbb{G}$ with metric d, by $(\mathcal{M}, d)$. Also, by the map $\phi : \mathcal{M} \to \mathcal{M}$, we mean $\phi: (\mathcal{M}, d) \to (\mathcal{M}, d)$. By $(\mathcal{M}, d)$ being a compact metric $\mathbb{G}-$ space, we mean a compact metric $\mathbb{G}-$ space on which there is a compact topological group $\mathbb{G}$ with metric d. If A and B are two non-empty subsets of $\mathcal{M}$, then $N_g(A \cap B) = \{i \in \mathbb{N} : g\,\phi^i(A) \cap B \neq \emptyset\} \neq \emptyset$, $g \in \mathbb{G}$.

Definition 2.1.[7]

Let $(\mathcal{M}, d)$ be a compact metric space and let $\phi : \mathcal{M} \to \mathcal{M}$ be a continuous map. A sequence $\{m_i, i \in \mathbb{Z}\}$ is called trajectory of ϕ, if $\forall i \in \mathbb{Z}$, we have $m_{i+1} = \phi(m_i)$ and we called it a δ - pseudo - trajectory of ϕ, $\forall i \in \mathbb{Z}$. We have $d(\phi(m_i), m_{i+1}) \leq \delta$, and the map ϕ has the shadowing property if $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$, such that every δ-pseudo-trajectory $\{m_i, i \in \mathbb{Z}\}$ is ε- shadowed by the trajectory $\{\phi^i(m), i \in \mathbb{Z}\}$ for some $z \in \mathcal{M}$, that is, $\forall i \in \mathbb{Z}$, thus we have $d\left(\phi^i(z), m_i\right) \leq \varepsilon$.

A sequence $\{m_i, i \in \mathbb{Z}\}$ in $\mathcal{M}$ is called a δ - average pseudo- trajectory of $\mathcal{M}$ if $\exists\, N \in \mathbb{N}$ and $N = N(\delta)$, such that $\forall\, n \geq N$, and $k \in \mathbb{N}$, then

$$\frac{1}{n}\sum_{i=0}^{n-1} d(\phi(m_{i+k}),\, m_{i+k+1}) < \delta ,$$

The map ϕ has the ASP if $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$, such that every δ - average-pseudo - trajectory $\{m_i, i \in \mathbb{Z}\}$ is ε - shadowed in average by the trajectory of some point $z \in \mathcal{M}$, that is

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(\phi^i(\mathrm{z}), m_i) < \varepsilon.$$

Definition 2.2. [5]

Let $(\mathcal{M},d)$ be metric $\mathbb{G}-$space and let $\phi:\mathcal{M}\to\mathcal{M}$ be continuous map. For a positive real number δ, a sequence of points $\{m_i: a<i<b\}$ in $\mathcal{M}$ is called $(\delta,\mathbb{G})-$pseudo- trajectory for ϕ, if $\forall i,\ a<i<b-1$, $\exists\, g_i\in\mathbb{G}$ such that $d(g_i\,\phi\,(m_i),\ m_{i+1})<\delta$.

For a given $\varepsilon>0$, a $(\delta,\mathbb{G})-$pseudo-trajectory $\{m_i: a<i<b\}$ for ϕ is called $\varepsilon-$shadowed by a pointe m of $\mathcal{M}$, if $\forall i, a<i<b$, $\exists\, p_i\in\mathbb{G}$ such that $d\big(\phi^i(m),\ p_i m_i\big)<\varepsilon$. The map ϕ has the $\mathbb{G}-$shadowing property if $\forall\,\varepsilon>0$, $\exists\,\delta>0$ such that for each $(\delta,\mathbb{G})-$pseudo- trajectory for ϕ is $\varepsilon-$shadowed by a pointe of $\mathcal{M}$. Note that if ϕ is bijective then we take $-\infty<a<b<\infty$. Also, when ϕ is not bijective then we take $0\le a<b<\infty$.

Definition 2.3.

Let $(\mathcal{M},d)$ be metric $\mathbb{G}-$space and let $\phi:\mathcal{M}\to\mathcal{M}$ be continuous map. For a positive real number δ, a sequence of points $\{m_i: a<i<b\}$ in $\mathcal{M}$ is called $(\delta,\mathbb{G})-$average pseudo- trajectory for ϕ if $\forall i,\ a<i<b-1$, $\exists\, g_i\in\mathbb{G}$ and there exists a positive integer $N=N(\delta)$ such that $\forall n\ge N$, and $k\in\mathbb{N}$, then

$$\frac{1}{n}\sum_{i=0}^{n-1} d(\,g_i\,\phi(m_{i+k}),\ m_{i+k+1})<\ \delta.$$

The map ϕ has the $\mathbb{G}$ ASP if $\forall\,\varepsilon>0$ and there is $\delta>0$ such that every $(\delta,\mathbb{G})-$ average pseudo-trajectory $\{m_i: a<i<b\}$ is $\varepsilon-$shadowed in $\mathbb{G}-$average by a point m of $\mathcal{M}$, if $\forall i$, $\exists\, g_i\in\mathbb{G}$ such that

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} d(\phi^i(m),g_i m_i)<\varepsilon.$$

Note that if ϕ is bijective then we take $-\infty<a<b<\infty$. Also, when ϕ is not bijective then we take $0\le a<b<\infty$.

Definition 2.4. [8]

Let $(\mathcal{M},d)$ be metric $\mathbb{G}-$space and let $\phi:\mathcal{M}\to\mathcal{M}$ be continuous map, then ϕ is called $\mathbb{G}-$transitive if $\forall A,B\ne\emptyset$, and A,B are open subsets of $\mathcal{M}$, $\exists i\in\mathbb{N}$, and $g\in\mathbb{G}$, such that the set $N_g(\,A\cap B\,)=\{i\in\mathbb{N}: g\,\phi^i(A)\cap B\ne\emptyset\}\ne\emptyset$. We say that a homeomorphism ϕ is totally $\mathbb{G}-$transitive if ϕ^i is $\mathbb{G}-$transitive, $\forall i\ge 1$.

Definition 2.5. [9]

Let $(\mathcal{M},d)$ be metric $\mathbb{G}-$space and $\phi:\mathcal{M}\to\mathcal{M}$ be a homeomorphism map, then ϕ is called topologically $\mathbb{G}-$mixing if $\forall A,B\ne\emptyset$, and A,B are open subsets of $\mathcal{M}$, $\exists\, k\in\mathbb{Z}$ such that $\forall\, n\ \ge\ k$, $\exists\, g_k\in\mathbb{G}$ satisfying $g_k\phi^k(A)\cap B\ne\emptyset$.

Definition 2.6.[9]

Let $(\mathcal{M},d)$ be metric $\mathbb{G}-$space and $\phi:\mathcal{M}\to\mathcal{M}$ be a continuous map, then ϕ is called weakly $\mathbb{G}-$mixing if $\phi\times\phi$ is $\mathbb{G}\times\mathbb{G}-$transitive, that means, $\forall\ A\times B,\ E\times D\ne\emptyset$ of are open subsets of $\mathcal{M}\times\mathcal{M}$, $\exists(g,p)\in\mathbb{G}\times\mathbb{G}$ and $k\in\mathbb{N}$, such that,

$$((\,g,p)(\phi\times\phi)^k(A\times B))\cap(E\times D)\ne\emptyset.$$

If $\exists\, N>0$, such that $\forall m,y\in\mathcal{M}$, and $\forall n\ge N$, there exists $(\delta,\mathbb{G})-$pseudo-trajectory from m to y of length exactly n, then the map ϕ is $(\delta,\mathbb{G})-$chain mixing. The map ϕ is chain mixing if it is $\delta-$chain mixing for every $\delta>0$.

Main Results

Proposition 3.1

Let $(\mathcal{M},d)$ be metric $\mathbb{G}-$space, and $\phi:\mathcal{M}\to\mathcal{M}$ be a continuous map. If ϕ has $\mathbb{G}$ ASP, then ϕ^m has $\mathbb{G}$ ASP for every $m\in\mathbb{N}$.

Proof:

Let $m\in\mathbb{N}$, since ϕ has $\mathbb{G}$ ASP, for any $\frac{\varepsilon}{m}>0$, $\exists\,\delta>0$, such that every $(\delta,\mathbb{G})-$average pseudo- trajectory is $\frac{\varepsilon}{m}$-shadowed in average by some point in $\mathcal{M}$. Assume that $\{z_i,\ i\in\mathcal{N}_0\}$ is $(\delta,\mathbb{G})-$ average pseudo – trajectory of ϕ^m, that is, $\exists\,\mu=\mu(\delta)>0$, such that

$$\frac{1}{n}\sum_{i=0}^{n-1} d(\,g_i\,\phi^m(z_{i+k}),\ z_{i+k+1})<\delta,\quad \text{for all}\ \ n\ge\mu,\qquad k\in\mathcal{N}_0\ \text{and}\ g_i\in\mathbb{G}.$$

We write $x_{nm+j} = \phi^{j}(z_n)$ for $0 \leq j < m$, $n \in \mathcal{N}_0$, that is,
$\{x_i, i \in \mathcal{N}_0\} = \{z_0, \phi(z_0), \ldots, \phi^{m-1}(z_0), z_1, \phi(z_1), \ldots, \phi^{m-1}(z_1), \ldots\}$.

We have $\frac{1}{n}\sum_{i=0}^{n-1} d(g_i \phi^{m}(x_{i+k}), x_{i+k+1}) < \delta$, for all $n \geq \mu$ and $k \in \mathbb{Z}_+$.

Then $\{x_i, i \in \mathcal{N}_0\}$ is $(\delta, \mathbb{G})$ −average pseudo-trajectory ϕ. So, $\exists\, \omega \in \mathcal{M}$, such that

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d\big(\phi^{i}(\omega), g_i x_i\big) < \frac{\varepsilon}{m}. \qquad (3\text{-}1)$$

Claim: there are infinite $t \in \mathbb{N}$, such that

$$\frac{1}{t}\sum_{i=0}^{t-1} d(\phi^{im}(\omega), g_i x_i) < \varepsilon .$$

Proof of Claim : Assume there is $\mu_0 \in \mathbb{N}$, such that

$$\frac{1}{t}\sum_{i=0}^{t-1} d(\phi^{im}(\omega), g_i x_i) \geq \varepsilon, \qquad \text{for all } t \geq \mu_0.$$

Then $$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d\big(\phi^{i}(\omega), g_i x_i\big) \geq \frac{\varepsilon}{m}.$$

This contracts with $(3\text{-}1)$, then we have:

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d\big(\phi^{im}(\omega), g_i x_{im}\big) < \varepsilon,$$

since $x_{im} = z_i$,

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d((\phi^{m})^{i}(\omega), g_i z_i) < \varepsilon .$$

Thus, have the ϕ^m $\mathbb{G}$ ASP.

Proposition 3.2. [9]

Let $(\mathcal{M}, d)$ be a metric $\mathbb{G}$ −space, $\phi : \mathcal{M} \to \mathcal{M}$ be pseudo-equivariant and totally $\mathbb{G}$ −transitive with a dense set of $\mathbb{G}_{\phi}$ −periodic points, then ϕ is weakly $\mathbb{G}$ −mixing.

Theorem 3.3

Let $(\mathcal{M}, d)$ be a compact metric $\mathbb{G}$ − space and $\phi : \mathcal{M} \longrightarrow \mathcal{M}$ be pseudo-equivariant with dense set of $\mathbb{G}_{\phi}$ −periodic points. If ϕ has the $\mathbb{G}$ ASP, then ϕ is weakly $\mathbb{G}$ −mixing.

Proof:

By Proposition 3.1, if ϕ has the $\mathbb{G}$ ASP then so does ϕ^m for every $m \in \mathbb{N}$. By Proposition 3.2, if ϕ^m is totally $\mathbb{G}$ − transitive for every $m > 0$, then it is weakly $\mathbb{G}$ −mixing. Therefore, it is enough to prove that ϕ is totally $\mathbb{G}$ − transitive.

We must prove that ϕ^m is $\mathbb{G}$ − transitive for some $m > 1$. Assume that ϕ^m is not $\mathbb{G}$ −transitive for some $m > 1$, then $\exists\, \mathcal{D} \subseteq \mathcal{M}$, such that $\mathcal{D} \neq \emptyset$ proper, closed and $\mathbb{G}$ −invariant. Also $\phi^m(\mathcal{D}) \subseteq \mathcal{D}$ and hence $\phi^{ms}(\mathcal{D}) \subseteq \mathcal{D}$ for any $s \geq 1$ such that $\mathrm{int}(\mathcal{D}) \neq \emptyset$, implies that ϕ^{ms} is not $\mathbb{G}$ −transitive for any $s \geq 1$. So, $\forall s \geq 1$, $\exists\, A_s, B_s$ are non-empty open subsets of $\mathcal{M}$, such that $\forall p \in \mathbb{G}$ and $\forall i \geq 1$. We have $(p(\phi^{ms})^{i}(A_s)) \cap B_s = \emptyset$. Note that A_1, B_1 works $\forall s$. Assume that A, B are nonempty open subsets of $\mathcal{M}$ such that $(p\phi^{mk}(A)) \cap B = \emptyset$, $\forall\, p \in \mathbb{G}$ and $\forall k \geq 1$. Since ϕ is pseudo-equivariant, then $A \cap \left(p\phi^{-mk}(B)\right) = \emptyset$, $\forall\, p \in \mathbb{G}$ and $\forall k \geq 1$. Suppose that $\underbrace{\phi \times \phi \times \cdots \times \phi}_{m-times}$ is not $\underbrace{\mathbb{G} \times \mathbb{G} \times \cdots \times \mathbb{G}}_{m-times}$ −transitive. We take into account that $B' = B \times \phi^{-1}(B) \times \ldots \times \phi^{-(m-1)}(B)$ and $A' = A \times A \times \ldots \times A$. Then, $A' \cap \left((p_1, p_2, \ldots, p_m)\,(\phi \times \phi \times \ldots \times \phi)^{-r}(B')\right) = \emptyset$,

$\forall\, (p_1, p_2, \dots, p_m) \in \mathbb{G} \times \mathbb{G} \times \dots \times \mathbb{G}$ and $\forall r \geq 1$, which implies that $\underbrace{\phi \times \phi \times \dots \times \phi}_{m-times}$ is not $\underbrace{\mathbb{G} \times \mathbb{G} \times \dots \times \mathbb{G}}_{m-times} -$ transitive, which implies a contradiction. Thus ϕ^m is $\mathbb{G}-$transitive for every $m \geq 1$ and hence ϕ is totally $\mathbb{G}-$ transitive.
Thus by Proposition 3.2, ϕ is weakly $\mathbb{G}-$mixing.

Definition 3.4.[5]

Let $(\mathcal{M}, d)$ be a metric $\mathbb{G}-$space and $\phi : \mathcal{M} \longrightarrow \mathcal{M}$ be a homeomorphism map that is called positively $\mathbb{G}-$expansive. If there exists real number $\rho > 0$ such that $\forall\, m, y \in \mathcal{M}$ with $\mathbb{G}(m) \neq \mathbb{G}(y)$, there exists an integer number $k \geq 0$ such that $d(\phi^k(u), \phi^k(v)) > \rho$, $\forall\, u \in \mathbb{G}(m)$, and $v \in \mathbb{G}(y)$. ρ is then called a $\mathbb{G}-$expansive constant for ϕ.

Definition 3.5. [5]

Let $(\mathcal{M}, d)$ be a compact metric $\mathbb{G}-$ space and $\phi : \mathcal{M} \longrightarrow \mathcal{M}$ be a homeomorphism map. Then ϕ has $\mathbb{G}-$specification if $\forall\, \varepsilon > 0$, $\exists\, \mathcal{D} = \mathcal{D}(\varepsilon) > 0$ such that for each finite sequence of points $g_1 m_1, g_2 m_2, \dots, g_k m_k \in \mathcal{M}$ for some $g_1, g_2, \dots, g_k \in \mathbb{G}$ and for $2 \leq k \leq j$, picking any sequence of integers $a_1 \leq b_1 < a_2 \leq b_2 < \dots < a_j \leq b_j$ such that $a_k - b_{k-1} \geq \mathcal{D}(2 \leq k \leq j)$ and an integer ℓ with $\ell \geq \mathcal{D}(b_j - a_1)$, $\exists m \in \mathcal{M}$ with $\phi^\ell(m) = g\, m$, $\exists\, g \in \mathbb{G}$ and hold $d(\phi^i(m), \ell_i \phi^i(m_k)) < \varepsilon$ for some $\ell_i \in \mathbb{G}$ and for $a_k \leq i \leq b_k$, $1 \leq k \leq j$.

Theorem 3.6

Let $(\mathcal{M}, d)$ be a compact metric $\mathbb{G}-$ space with d being an invariant metric and let $\phi : \mathcal{M} \longrightarrow \mathcal{M}$ is a $\mathbb{G}$–expansive pseudo-equivariant homeomorphism having the $\mathbb{G}$ ASP. If ϕ is topologically $\mathbb{G}-$mixing then ϕ has the $\mathbb{G}$ -specification.

Proof:

Let $\rho > 0$ be a $\mathbb{G}$–expansive constant for ϕ and we choose ε such that $0 < \varepsilon < \frac{\rho}{2}$. Since ϕ has $\mathbb{G}$ ASP, $\exists\, \beta > 0$ such that every $(\beta, \mathbb{G})-$ average pseudo-trajectory for ϕ is ε - shadowed in $\mathbb{G}-$average by the trajectory of some point $m \in \mathcal{M}$. Let $\mathcal{F} = \{A_1, A_2, \dots, A_m\}$ be a finite open cover of $\mathcal{M}$ with $A_i \neq \emptyset$ and diam $A_i < \frac{\beta}{2}$, $\forall\, i,\ i \in \{1, 2, \dots, m\}$. Since ϕ is topologically $\mathbb{G}-$mixing, then for each open sets A_i, A_j there is $\mathcal{D}_{i,j} > 0$, such that $\forall\, n \geq \mathcal{D}_{i,j}$, and there is $g'_n \in \mathbb{G}$ satisfying $A_j \cap g'_n \phi^n(A_i) \neq \emptyset$ $(3-2)$.

Let $\mathcal{D} = \max\{\mathcal{D}_{i,j} : 1 \leq i, j \leq m\}$ and $g_1 m_1, g_2 m_2, \dots, g_k m_k \in \mathcal{D}$, for some $g_1, g_2, \dots, g_k \in \mathbb{G}$ and for $2 \leq j \leq k$, picking any sequence of integers $a_1 \leq b_1 < a_2 \leq b_2 < \dots < a_k \leq b_k$ such that $a_j - b_{j-1} \geq \mathcal{D}(2 \leq j \leq k)$ and an integer p with $p \geq \mathcal{D}(b_k - a_1)$. We define $a_{k+1} = b_{k+1} = p + a_1$, $m_{k+1} = \phi^{a_1 - b_{k+1}}(g_1 m_1)$. We denote by $A(z)$ an open ball A in $\mathcal{F}$ containing z. Since $a_{j+1} - b_j \geq \mathcal{D}$, by $(3-2)$, $\exists\, g'_{a_{j+1}-b_j} \in \mathbb{G}$, such that $A\left(\phi^{a_j+1}(g_{j+1} m_{j+1})\right) \cap g'_{a_{j+1}-b_j} \phi^{a_{j+1}-b_j}\left(A\left(\phi^{b_j}(g_j m_j)\right)\right) \neq \emptyset$, that is,
$\exists\, y_j \in \phi^{a_{j+1}-b_j}\left(A\left(\phi^{b_j}(g_j m_j)\right)\right) \neq \emptyset$ such that $\phi^{a_{j+1}-b_j}(y_j) = k'_{a_{j+1}-b_j}\, y'_j$. We establish a $(\beta, \mathbb{G})-$ average pseudo- trajectory $\{\omega_i : i \in \mathbb{Z}\}$ for ϕ in $\mathcal{M}$, as follows:

$\omega_i = \phi^i(g_j m_j)$ if $a_j \leq i \leq b_j$

$\omega_i = \phi^{i-b_j}(y_j)$ if $b_j \leq j \leq a_{j+1}$

$\omega_{i+p} = \omega_i$, $\forall i \in \mathbb{Z}$

Since ϕ has the $\mathbb{G}$ ASP, $\{\omega_i : i \in \mathbb{Z}\}$ is ε - shadowed in $\mathbb{G}-$average by the trajectory of some point $m \in \mathcal{M}$. Therefore, $\forall i \in \mathbb{Z}$, $\exists\, \ell_i, \ell_{i+p} \in \mathbb{G}$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1-n}^{n-1} d(\phi^i(m), \ell_i\, \omega_i) < \varepsilon, \quad \text{and} \quad \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1-n}^{n-1} d(\phi^{i+p}(m), \ell_{i+p}\, \omega_{i+p}) < \varepsilon,$$

this implies that $\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1-n}^{n-1} d(\phi^i(m), \ell_i\, \omega_i) < \varepsilon$, and $\limsup_{n\to\infty}\frac{1}{n}\sum_{i=1-n}^{n-1} \text{d}\big(\phi^{i+p}(m),\ \ell_{i+p}\,\omega_i\big) < \varepsilon$, which implies that $\forall i \in \mathbb{Z}, \exists\, \ell_1, \ell_k \in \mathbb{G}$, satisfying

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{i=1-n}^{n-1} \text{d}\big(\ell_{i+p}^{-1}\,\phi^{i+p}(m), \ell_i^{-1}\phi^i(m)\big) < 2\,\varepsilon < \varepsilon.$$

But ϕ is a $\mathbb{G}-$expansive homeomorphism. Consequently, $\mathbb{G}(\phi^p(m)) = \mathbb{G}(m)$. Therefore, $\phi^p(m) = gm$, for some $g \in \mathbb{G}$. Also for $a_j \le j$ or $b < b_j$, $\omega_i = \phi^i(g_j m_j)$.

So, $\limsup_{n\to\infty}\frac{1}{n}\sum_{i=1-n}^{n-1} \text{d}\big(\phi^i(m),\ \ell_i\,\omega_i\big) = \limsup_{n\to\infty}\frac{1}{n}\sum_{i=1-n}^{n-1} \text{d}\Big(\phi^i(m), \ell_i\,\phi^i\big(g_j m_j\big)\Big) < \varepsilon$

and $\phi^p(m) = gm$. Thus, ϕ has the $\mathbb{G}$ -specification by Definition 3.5.

Lemma 3.7. [5]

Let $\mathcal{M}$ be a compact connected Hausdorff metric space that contains more than one point and let $m, y \in \mathcal{M}$. Then for a continuous map $\phi: \mathcal{M} \to \mathcal{M}$ and $\delta > 0$, there exists a $\delta -$ pseudo- trajectory for ϕ containing m, y in $\mathcal{M}$.

We recall that the topological space $\mathcal{M}$ is called a **totally disconnected** space if $\forall\, m, y \in \mathcal{M}$. There are two sets $A, B \subset \mathcal{M}$ that are disconnection such that $m \in A$ and $y \in B$.

Theorem 3.8

Let $(\mathcal{M}, \text{d})$ be a compact metric $\mathbb{G} -$ space. Then the identity map $\phi: \mathcal{M} \to \mathcal{M}$ has the $\mathbb{G}$ ASP if and only if the orbit space $\mathcal{M}/\mathbb{G}$ of $\mathcal{M}$ is totally disconnected.

Proof: ($\Longrightarrow$)

Assume that the identity map $\phi: \mathcal{M} \to \mathcal{M}$ has the $\mathbb{G}$ ASP. By hypothesis, $\frac{\mathcal{M}}{\mathbb{G}}$ is compact, then it is enough to prove that $\dim(\mathcal{M}/\mathbb{G}) = 0$. Suppose, conversely, that $\dim(\mathcal{M}/\mathbb{G}) \neq 0$. Since $\dim(\mathcal{M}/\mathbb{G}) \ge 1$, so there is a closed connected subset E in $\mathcal{M}/\mathbb{G}$ which has a dimension that is at least one. E is a compact subset of $\mathcal{M}/\mathbb{G}$, since $\mathcal{M}/\mathbb{G}$ is compact. So $\exists\, \mathbb{G}(a) \neq \mathbb{G}(b) \in E$, such that diam $E = \text{d}_1(\mathbb{G}(a), \mathbb{G}(b)) = \gamma$. By compactness of $\mathbb{G}$, there is $y_1 \in \mathbb{G}(a)$ and $y_2 \in \mathbb{G}(b)$ such that $r = \text{d}(y_1, y_2)$. Let $\varepsilon = \frac{\gamma}{3}$. We get a contradiction by exhibiting that for $\forall\, \varepsilon > 0$ there is a $(\delta, \mathbb{G}) -$ average pseudo- trajectory for ϕ which is not ε - shadowed in $\mathbb{G}-$average by the trajectory of some point $m \in \mathcal{M}$.

By Lemma 3.7, there is a$(\delta, \mathbb{G}) -$ average pseudo- trajectory $\{m_i : i \in \mathbb{Z}\}$ for ϕ in $\mathcal{M}$ containing y_1, y_2. Such a $(\delta, \mathbb{G}) -$ average pseudo- trajectory can be obtained as follows: Since E is a compact connected subset of $\mathcal{M}/\mathbb{G}$ by Lemma 3.7, then there is a $\delta -$pseudo- trajectory $\{\mathbb{G}(m_i): i \in Z\}$, for $\breve{\phi}$ containing $\mathbb{G}(a)$ and $\mathbb{G}(b)$. This implies that $\forall i \in \mathbb{Z}$,

$$\frac{1}{n}\sum_{i=0}^{n-1} \text{d}_1\Big(\breve{\phi}\big(\mathbb{G}(m_i)\big), \mathbb{G}(m_{i+1})\Big) < \varepsilon.$$

Since $\mathbb{G}$ is Compact, implies for $\forall i \in \mathbb{Z}$, $\exists\, \ell_i,\ u_i \in \mathbb{G}$ such that,

$\frac{1}{n}\sum_{i=0}^{n-1} \text{d}\,(\ell_i\phi(m_i),\ u_i m_{i+1}) < \delta$ which implies $\frac{1}{n}\sum_{i=0}^{n-1} \text{d}(\,g_i\,\phi(m_i),\ m_{i+1}) < \delta$,

for some $g_i \in \mathbb{G}$, and hence $\{m_i : i \in \mathbb{Z}\}$, is a $(\delta, \mathbb{G}) -$ average pseudo- trajectory for ϕ. Now, $\{\mathbb{G}(m_i): i \in \mathbb{Z}\}$ contains $\mathbb{G}(a)$ and $\mathbb{G}\{b)$. Therefore, for some $k, p \in \mathbb{Z}$, $\mathbb{G}(m_k) = \mathbb{G}(a)$ and $\mathbb{G}(m_p) = \mathbb{G}(b)$. Also, $y_1 \in \mathbb{G}(a)$ and $y_2 \in \mathbb{G}(b)$, implies $g' y_1 = m_k$ and $g'' y_2 = m_p$, for some $g', g'' \in \mathbb{G}$. We take the place of m_k by $g' y_1$ and m_p by $g'' y_2$ in $\{m_i : i \in \mathbb{Z}\}$ and continue to denote the new $(\delta, \mathbb{G}) -$ average pseudo- trajectory, containing y_1 and y_2, by $\{m_i : i \in \mathbb{Z}\}$.

Let $\{m_i : i \in \mathbb{Z}\}$ ε - shadowed in $\mathbb{G}-$average by the point $m \in \mathcal{M}$. So, $\forall\, i \in \mathbb{Z}$, $\exists\, p_i \in \mathbb{G}$, such that

$$\frac{1}{n}\sum_{i=0}^{n-1} \text{d}(\,m, p_i\, m_i) = \limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} \text{d}\,\big(\phi^i(\,m_i), p_i\, m_i\big) < \varepsilon \qquad (3-3).$$

Since $\{m_i : i \in \mathbb{Z}\}$ is a $(\delta, \mathbb{G})$ - average pseudo- trajectory for ϕ containing y_1 and y_2, $\exists\, k, n \in \mathbb{Z}$ such that $m_k = y_1$ and $m_n = y_2$. So, by $(3-3)$ $d(m, p_k\, m_k) < \varepsilon$ and $d(m, p_n\, m_n) < \varepsilon$, which implies that $d_1(\mathbb{G}(m), \mathbb{G}(p_k m_k)) < \varepsilon$ and $d_1(\mathbb{G}(m), \mathbb{G}(p_n\, m_n)) < \varepsilon$, and hence $d_1(\mathbb{G}(a), \mathbb{G}(b)) \le d_1(\mathbb{G}(a), \mathbb{G}(m)) + d_1(\mathbb{G}(m), \mathbb{G}(b)) < \varepsilon + \varepsilon = \frac{2\gamma}{3}$, which is a contradiction. This proves that $\dim(\mathcal{M}/\mathbb{G}) = 0$. Hence, the orbit space $\mathcal{M}/\mathbb{G}$ of $\mathcal{M}$ is totally disconnected.

Proof : (⟸)

Assume that $\mathcal{M}/\mathbb{G}$ is totally disconnected. Then clopen sets form a basis for topology of $\mathcal{M}$. By hypothesis, $\mathbb{G}$ is compact, then we have the possibility of an invariant metric d on $\mathcal{M}$ congruous with topology of $\mathcal{M}$. Let $\varepsilon > 0$ be given and let $\{A_1, A_2, \dots, A_n\}$ be a finite subcover of $\mathcal{M}/\mathbb{G}$ containing clopen sets such that $A_i \cap A_j = \emptyset$ for $i \ne j$ and diam $A_i < \varepsilon$, $\forall i \in \{1, 2, \dots, n\}$.

A set $B_i = \Psi^{-1}(A_i)$, $\forall i$, since A_i is a closed subset of $\mathcal{M}/\mathbb{G}$ and π is a continuous map, $B_i = \Psi^{-1}(A_i)$ is compact, since $B_i \subset \mathcal{M}$, and B_i is a closed. So, $A_i \cap A_j = \emptyset$, implies $\Psi^{-1}(A_i) \cap \Psi^{-1}(A_j) = \emptyset$, implies $B_i \cap B_j = \emptyset$.

Let $\alpha_{ij} = d(A_i, A_j)$ for $\ne j$. Then A_i, A_j is compact, implies $\alpha_{ij} > 0$ for $i \ne j$. Choose α such that $0 < \alpha < \min\{\alpha_{ij} : 1 \le i, j \le n\}$. We must prove that the identity map ϕ has the $\mathbb{G}$ ASP. We prove that every $(\alpha, \mathbb{G})-$ average pseudo- trajectory for ϕ is ε - shadowed in $\mathbb{G}-$average by the trajectory of some point $m \in \mathcal{M}$. Let $S = \{m_i : i \in \mathbb{Z}\}$ be a $(\alpha, \mathbb{G})-$ average pseudo- trajectory for ϕ. Then for $\forall i \in \mathbb{Z}$, $\exists\, g_i \in \mathbb{G}$ such that

$$\frac{1}{n}\sum_{i=0}^{n-1} d(g_i\,\phi(m_i), m_{i+1}) < \alpha \text{ implies to } \frac{1}{n}\sum_{i=0}^{n-1} d(g_i\, m_i, m_{i+1}) < \alpha, \qquad (3-4)$$

Note that if $m_i \in B_k$ then $m_{i+1} \in B_k$. For if $m_{i+1} \in B_j$, $j \ne k$, then B_k is $\mathbb{G}$-invariant $g_i\, m_i \in B_k$ and $m_{i+1} \in B_j$, implies

$$\frac{1}{n}\sum_{i=0}^{n-1} d(g_i\,\phi(m_i), m_{i+1}) \ge \frac{1}{n}\sum_{i=0}^{n-1} d(B_k, B_j) = \alpha_{ij} > \alpha,$$

This is a contradiction with $(3-4)$. Similarly, if $m_i \in B_k$, then $m_{i-1} \in B_k$. For if $m_{i-1} \in B_j$, $j \ne k$, then B_j is $\mathbb{G}$-invariant $g_{i-1}\, m_{i-1} \in B_j$ and $m_{i+1} \in B_j$, implies

$$\frac{1}{n}\sum_{i=0}^{n-1} d(g_{i-1}\, m_{i-1}, m_i) \ge \frac{1}{n}\sum_{i=0}^{n-1} d(B_k, B_j) = \alpha_{ij} > \alpha,$$

This is a contradiction with $(3-4)$. So, $\forall\, i \in \mathbb{Z}$, $m_i \in B_k$. This implies that $\mathbb{G}(m_i) \in A_k$, but diam $A_k < \varepsilon$, so $\forall\, \mathbb{G}(m) \in A_k$ and $\forall\, i \in \mathbb{Z}$,

$$\frac{1}{n}\sum_{i=0}^{n-1} d_1(\mathbb{G}(m), \mathbb{G}(m_i)) < \varepsilon.$$

By hypothesis, G is compact, so $\forall i \in \mathbb{Z}, \exists\, \ell_i,\ u_i \in \mathbb{G}$, such that

$$\frac{1}{n}\sum_{i=0}^{n-1} d(\ell_i m, u_i\, m_i) < \varepsilon.$$

Thus $\forall i \in \mathbb{Z}$, $\exists\, g_i \in \mathbb{G}$ such that

$$\frac{1}{n}\sum_{i=0}^{n-1} d(\phi^i(m_i), g_i\, m_i) < \varepsilon.$$

Hence $S = \{m_i : i \in \mathbb{Z}\}$ is ε - shadowed in $\mathbb{G}-$average by the trajectory of some point $m \in \mathcal{M}$. Since S is an arbitrary $(\alpha, \mathbb{G})-$ average pseudo- trajectory for ϕ, it follow that every $(\alpha, \mathbb{G})-$ average pseudo- trajectory for ϕ is ε - shadowed in $\mathbb{G}-$average by the trajectory of some point $m \in \mathcal{M}$. Hence ϕ has the $\mathbb{G}$ ASP.

Definition 3.9. [5]

Let $\mathcal{M}$ and Y be metric spaces. A continuous onto map $h: \mathcal{M} \to Y$ is called a covering map, if for each $y \in Y$, there exists an open neighborhood B_y of y in Y such that $\phi^{-1}(B_y) = \bigcup_i A_i$,

($i \neq j$, implies $A_i \cap A_j = \emptyset$, where each A_i is open in $\mathcal{M}$ and $h|_{A_i} : A_i \to B_y$ is a homeomorphism).

Theorem 3.10

Let $\phi : \mathcal{M} \to \mathcal{M}$ be a pseudo-equivariant map on a compact metric $\mathbb{G}$ −space $(\mathcal{M}, d)$ and let the orbit map $\Psi : \mathcal{M} \to \mathcal{M}/\mathbb{G}$ be a covering map, then ϕ has the $\mathbb{G}$ ASP iff the induced map $\breve{\phi} : \mathcal{M}/\mathbb{G} \to \mathcal{M}/\mathbb{G}$ has the ASP.

Proof: $(\Longrightarrow)$

Assume that ϕ has the $\mathbb{G}$ ASP. We must prove that $\breve{\phi}$ has the ASP. We choose $\varepsilon > 0$. Since Ψ is uniformly continuous, $\exists\, \gamma > 0$, such that $d(m, y) < \gamma$, implies $d_1\left(\Psi(m), \Psi(y)\right) < \varepsilon$. Also, ϕ has the $\mathbb{G}$ ASP, so $\exists\, \mu > 0$, such that every $(\mu, \mathbb{G})$ − average pseudo- trajectory for ϕ is γ - shadowed in $\mathbb{G}$ −average by a point $m \in \mathcal{M}$. Since Ψ is a covering map on a compact space, $\exists\, \delta > 0$, such that $\forall m \in \mathcal{M}$. We find an α_m satisfying $(\Psi|_{A_{\alpha_m}})^{-1}(A_\delta(\Psi(m))) \subset A_\mu(m)$. We must prove that $\breve{\phi}$ has the ASP. We show that every δ − average pseudo- trajectory for $\breve{\phi}$ is ε -shadowed in average by a point of $\mathcal{M}/\mathbb{G}$. Let $\{\mathbb{G}(m_i) : i \in \mathcal{N}_0\}$ is an δ − average pseudo-trajectory for $\breve{\phi}$. Then $\exists\, \alpha_{m_{i+1}}$ such that $m_{i+1} \in \left(\Psi|_{A_{\alpha_{m_{i+1}}}}\right)^{-1}(A_\delta(\Psi(\phi(m_i)))) \subset A_\mu(\phi(m_i))$, implies $\{\mathbb{G}(x_i) : i \in \mathcal{N}_0\}$ is an $(\mu, \mathbb{G})$ − average pseudo- trajectory for ϕ and so is γ − shadowed in average by some point $m \in \mathcal{M}$. Hence, $\forall\, i \in \mathcal{N}_0$, $\exists\, g_i \in \mathbb{G}$, such that :

$$\frac{1}{n}\sum_{i=0}^{n-1} d\left(g_i\, m_i, \phi^i(m_i)\right) < \gamma .$$

Moreover, using uniform continuity of the covering map Ψ, we get :

$$\frac{1}{n}\sum_{i=0}^{n-1} d_1\left(\mathbb{G}\left(\phi^i(m)\right), \mathbb{G}(m_i)\right) < \varepsilon$$

This proves that $\{\mathbb{G}(m_i) : i \in \mathcal{N}_0\}$ is ε - shadowed in average by $\mathbb{G}(m)$. Hence, $\breve{\phi}$ has the ASP.

Proof: $(\Longleftarrow)$

Assume that $\breve{\phi}$ has the ASP. We must prove that ϕ has the $\mathbb{G}$ ASP. We choose $\varepsilon > 0$. Since Ψ is a covering map and $\mathcal{M}$ is compact, then $\exists\, \delta > 0$ such that for $\Psi(m) \in \mathcal{M}/\mathbb{G}$, $\Psi^{-1}(A_\delta(\Psi(m)) = \bigcup_{\alpha \in \wedge} A_\alpha$, where $\forall\, A_\alpha$ in $\mathcal{M}$, $\alpha \in \wedge$, $\alpha \neq \beta$, which leads to $A_\alpha \cap A_\beta = \emptyset$ and that $\Psi|_{A_\alpha} : A_\alpha \to A_\delta(\Psi(m))$ is a homeomorphism. For ε −neighborhood $A_\varepsilon(m)$ of m, consider A_α which contains m. If diam $A_\alpha < \varepsilon$, we have $\Psi^{-1}|_{A_\alpha}\left(A_\delta(\Psi(m))\right) \subset A_\alpha \subset A_\varepsilon(m)$. If diam $A_\alpha \nless \varepsilon$, then choose $A'_\alpha \subset A_\alpha$ such that diam $A'_\alpha < \varepsilon$ and $m \in A'_\alpha$, we have $\Psi^{-1}|_{A'_\alpha}\left(A_\delta(\Psi(m))\right) \subset A'_\alpha \subset A_\varepsilon(x)$. Since $\breve{\phi}$ has the ASP then $\exists\, \mu > 0$, such that every μ − average pseudo- trajectory for $\breve{\phi}$ is δ - shadowed in average by a point of $\mathcal{M}/\mathbb{G}$. Uniform continuity of Ψ implies that $\exists\, \gamma > 0$ such that $d(m, y) < \gamma$ which leads to $d_1\left(\Psi(m), \Psi(y)\right) < \gamma$. To prove that ϕ has the $\mathbb{G}$ ASP, we show that every $(\gamma, \mathbb{G})$ − average pseudo- trajectory for ϕ is ε - shadowed in $\mathbb{G}$ −average by a point of $\mathcal{M}$. Let $\{m_i : i \in \mathcal{N}_0\}$ be a $(\gamma, \mathbb{G})$ − average pseudo- trajectory for ϕ.

This implies that $\forall\, i \in \mathcal{N}_0\ \exists\, p_i \in \mathbb{G}$ such that $\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(p_i h(m_i), m_{i+1}) < \gamma$,

Therefore, $\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d_1\left(\Psi(\phi(m_i)), \Psi(m_{i+1})\right) < \mu$, and hence we have

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d_1\left(\mathbb{G}(\phi(m_i)), \mathbb{G}(m_{i+1})\right) < \mu ,$$

which proves that $\{\mathbb{G}(m_i) : i \in \mathcal{N}_0\}$ is an μ −average pseudo- trajectory for $\breve{\phi}$. Since $\breve{\phi}$ has the ASP, then $\{\mathbb{G}(m_i) : i \in \mathcal{N}_0\}$ is ε - shadowed in average by a point of $\mathcal{M}/\mathbb{G}$.

Suppose that $\mathbb{G}(x)$ and hence $\frac{1}{n}\sum_{i=0}^{n-1} d_1\left(\mathbb{G}\left(\phi^i(m)\right), \mathbb{G}(m_i)\right) < \delta$, $\forall\, i \in \mathcal{N}_0$. But this gives $\Psi\left(\phi^i(m)\right) \subset A_\alpha \Psi(m_i)$, implies $\phi^i(m) \in \Psi^{-1}\left(A_\delta\big(\Psi(m_i)\big)\right) \subset A_\varepsilon(m)$

and therefore $\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d\left(\phi^i(m_i), g_i m_i\right) < \varepsilon, \quad g_i \in \mathbb{G}$.

Hence ϕ has the $\mathbb{G}$ ASP.

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