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On Analytical Solutions of Operator Equation Constructed by The Adjointable Operator

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Abstract
Recently, Operator Equation Theory (OET) has a leading demonstrated potentiality applicable in numerous scientific ranges of engineering, physical and mathematical. In a Hilbert C*-module, OET has enhanced by expanding upon extensive research. In this study, for the general situation of adjointable operators, the solvability of the operator equation $\mathcal{P}^*X\Phi^* + \Phi Y\mathcal{P} = \Omega$, where X and Y are unknown operators, are investigated based on Moore-Penrose inverse. Necessary and sufficient conditions for founding a solution to this equation are proposed. Moreover, by utilizing matrix approaches, four general expressions for the solutions are derived depending on the states of the operators $\mathcal{P}$ and Φ involved in the equation.

Keywords: Operator, Hilbert C*-module, Operator equation, Invertibility, Adjoint operator, Self-adjoint, Moore-Penrose invers.

حول الحلول التحليلية لمعادلة المؤثر التي تم انشاؤها بواسطة المؤثر القابل للترافق

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الخلاصة:
حديثا اثبتت نظرية معادلة المؤثر (OET) إمكانية تطبيقها في العديد من المجالات العلمية منها الهندسية والفيزيائية والرياضية. في وحدة هلبرت- *C (Hilbert C*-module)، تم تعزيز نظرية معادلة المؤثر من خلال التوسع في البحث المكثف. في هذه الدراسة، بالنسبة للوضع العام للمؤثرات المترافقة، يتم التحقيق في قابلية حل معادلة المؤثر $\mathcal{P}^*X\Phi^* + \Phi Y\mathcal{P} = \Omega$ بناءً على معكوس مور-بينروز (Moore-Penrose invers)، حيث X, Y مؤثرين مجهولين. وسوف يتم اقتراح الشروط الضرورية والكافية لوجود حل لهذه المعادلة، علاوة على ذلك وباستخدام تقنية المصفوفة يتم اشتقاق أربع تعبيرات عامة للحلول اعتمادًا على حالات المؤثرين $\mathcal{P}, \Phi$ المتضمنين في المعادلة.

1. Introduction

The realm of Matrix Equations has intrinsic role in various domains of engineering and mathematics. In system theory, the matrix equation formulated by $\mathcal{P}XD + FX^T\Phi = C$ has

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been vastly utilized, for instance, to eigen structure assignment [1, 2]. Following, the Φ revival of Operator Equations on Hilbert space or Hilbert C*-module, which is the extended formula of M-Eqs. Indeed, the Hilbert C*-module is a nature generalized formula of C*-algebras and Hilbert space, was explored by Kaplansky [3] in 1953. The Operator Equations has contributed to algebraic theory, particularly in non-abelian geometry and quantum groups and KK-theory, see [4-9]. There are a variety of previous studies that paved the way or contributed to presenting this study. Interestingly, the mathematicians Baksalary and Kala [10] in 1979 provided the requisite and sufficient stipulations for the existence of a solution to equation $\mathcal{P}X - Y = \Omega$ and constructed the general formula of the solution. In 1998, the researcher Xu, Wei, and Zheng [11] presented the requisite and sufficient stipulations for the existence and uniqueness of the solution to equation $\mathcal{P}X\Phi + CYD = \Omega$. They investigated the general formula of the solution. Then, Wang, Zhang and Yu [12] in 2008 expanded attempts made investigating and deduced the requisite and sufficient stipulations for the existence of real and imaginary solutions to aforementioned equation. In 2009, Dehghan and Hajarian [13] deduced the reflexive solutions to the Matrix Equation $\mathcal{P}X\Phi + CYD = \Omega$ by providing the requisite and sufficient stipulations for the existence of these solutions. Subsequently, Karizaki and Djordjevic [14] in 2016 provided solutions to the Operator Equation $\mathcal{P}X\Phi - SYQ = \Omega$, on a Hilbert C*-module, $ran(\mathcal{P}) = ran(S)$ and $ran(\Phi^*) = ran(Q^*)$ are closed. For more information, see [15-26].

In this sequel, F and D are Hilbert modules over the same C*-algebra. Denote by $B(F, D)$ the set including the adjointable operators defined on F to D. For case $F = D$, $B(F, D)$ coincides with $B(F)$. $\mathcal{P}, \Phi, \Omega$ represents known operators. For $\mathcal{P} \in B(F, D)$, let $Ker(\mathcal{P})$ and $R(\mathcal{P})$ represent the zero-space and range, respectively.

The following several principles are required in this study.

Definition 1.1: [27, 28] Let F and D be vector spaces over the same filed then the operator $\mathcal{P}: F \longrightarrow D$ called invertibility if there exists an operator $\Phi: D \longrightarrow F$ where $\mathcal{P}\Phi = \Phi\mathcal{P} = I$, where I is an identity operator.

Definition 1.2: [29, 31] Let F and D be Hilbert spaces and $\mathcal{P}: F \longrightarrow D$ be a liner operator then the operator $\mathcal{P}^*: D \longrightarrow F$ is called adjoint of the operator $\mathcal{P}$ if $< \mathcal{P}x, y > = < x, \mathcal{P}^*y >$ for each $x \in F$, $y \in D$.

Definition 1.3: [32] An operator $\mathcal{P}: F \longrightarrow D$ where F and D are Hilbert spaces called self-adjoint if $\mathcal{P} = \mathcal{P}^*$.

Definition 1.4: [33, 34] Let $\mathcal{P} \in B(F, D)$,the range of $\mathcal{P}$, is denoted by $R(\mathcal{P})$ such that $R(\mathcal{P}) = \{\mathcal{P}x: x \in F\}$.

Definition 1.5: [35] Let $\mathcal{P} \in B(F, D)$,the kernel of $\mathcal{P}$, is denoted by $ker(\mathcal{P})$ such that $ker(\mathcal{P}) = \{x \in F: \mathcal{P}x = 0\}$.

Definition 1.6: [36] Let $\mathcal{P} \in B(F, D)$, the Moore-Penrose invers of $\mathcal{P}$ indicated by $\mathcal{P}^+$ such that $\mathcal{P}^+$is unique in $B(F, D)$ and fulfills:

$$\mathcal{P}\mathcal{P}^+\mathcal{P} = \mathcal{P},\ \mathcal{P}^+\mathcal{P}\mathcal{P}^+ = \mathcal{P}^+,\ (\mathcal{P}\mathcal{P}^+)^* = \mathcal{P}\mathcal{P}^+,\ (\mathcal{P}^+\mathcal{P})^* = \mathcal{P}^+\mathcal{P}.$$

The existence of the Moore-Penrose bounded inverse of a continuous operator between two Hilbert C*-modules is guaranteed if and only if the operator has a closed range. For more about the properties and applications of Moore-Penrose invers, see [37, 38]. The operator $\mathcal{P} \in B(F, D)$ is called regular if there exists $\Phi \in B(D, F)$ such that $\mathcal{P}\Phi\mathcal{P} = \mathcal{P}$. It is clear that regular operators are almost regular and that regular operator have close range [39].

Theorem 1.7: [40] Let $\Phi \in B(F, D)$ and $\mathcal{P} \in B(Z, D)$ be invertibility operators and $\Omega \in B(D)$, F, D, Z be Hilbert C*-modules. Then the next assumptions are comparable: (i) There is a solution $X \in B(F, Z)$ to $\mathcal{P}X\Phi^* + \Phi X^*\mathcal{P}^* = \Omega$. (ii) Ω is self-adjoint. If (i) or (ii) are available, therefore any solution to $\mathcal{P}X\Phi^* + \Phi X^*\mathcal{P}^* = \Omega$. When $X \in B(F,Z)$ represents in the form $X = \frac{1}{2}\mathcal{P}^{-1}\Omega(\Phi^*)^{-1} - \mathcal{P}^{-1}Z(\Phi^*)^{-1}$, where $Z \in B(D)$satisfies $Z^* = -Z$.

For the case Φ is invertible and $\mathcal{P}$ is regular, the following theorem gives an explicit solution to the equation $\mathcal{P}X\Phi^* + \Phi X^*\mathcal{P}^* = \Omega$.

Theorem 1.8: [40] Suppose $\Phi \in B(F,D)$is an invertibility and $\mathcal{P} \in B(Z,D)$ is regular and $\Omega \in B(D)$, such that F, D, Z are Hilbert C*-modules. After that, next claims are comparable: (i) There exists $X \in B(F,Z)$ is a solution to the operator equation $\mathcal{P}X\Phi^* + \Phi X^*\mathcal{P}^* = \Omega$. (ii) Ω is self-adjoint and $(I - \mathcal{P}\mathcal{P}^+)\Omega(I - \mathcal{P}\mathcal{P}^+) = 0$. If (i) or (ii) are available Thus, the solution to $\mathcal{P}X\Phi^* + \Phi X^*\mathcal{P}^* = \Omega$. It is written in order:$X = \frac{1}{2}\mathcal{P}^+\Omega\mathcal{P}\mathcal{P}^+(\Phi^*)^{-1} + \mathcal{P}^+Z\mathcal{P}\mathcal{P}^+(\Phi^*)^{-1} + \mathcal{P}\Omega(1 - \mathcal{P}\mathcal{P}^+)(\Phi^*)^{-1} + (I - \mathcal{P}\mathcal{P}^+)Y(\Phi^*)^{-1}$, where $Z \in B(D)$ achieves $\mathcal{P}^*(Z + Z^*)\mathcal{P} = 0$ and $Y \in B(D,Z)$ is random.

For $\mathcal{P}$ and Φ both close their regions, the next theorem studies the equation $\mathcal{P}X\Phi^* + \Phi X^*\mathcal{P}^* = \Omega$.

Theorem 1.9: [41] Let $\Phi, \Omega \in B(F)$, $\mathcal{P} \in B(D,F)$. So that Φ is self-adjoint and both $\mathcal{P}$ and Φ possess regulars. Let F, D, Z be Hilbert C*-modules, $\Omega\Phi^+\Phi = \Omega$ and $\mathcal{P}^+\Phi^+\Phi = \mathcal{P}^+$. The next claims are therefore comparable: (i) There is $X \in B(F,D)$ a solution to the operator equation $\mathcal{P}X\Phi^* + \Phi X^*\mathcal{P}^* = \Omega$. (ii) $\Omega = \Omega^*$and $(I - \mathcal{P}\mathcal{P}^+)\Omega(I - \mathcal{P}\mathcal{P}^+) = 0$. If (i) or (ii) are available, thus the solution to $\mathcal{P}X\Phi^* + \Phi X^*\mathcal{P}^* = \Omega$. It is written in order:$X = \mathcal{P}^+\Omega\Phi^+ - \frac{1}{2}\mathcal{P}^+\Omega\mathcal{P}\mathcal{P}^+\Phi^+ + \mathcal{P}^+Z\mathcal{P}\mathcal{P}^+\Phi^+ + V - \mathcal{P}^+\mathcal{P}V\Phi\Phi^+$, where $Z \in B(F)$ achieves $\mathcal{P}^*(Z + Z^*)\mathcal{P} = 0$, $V \in B(F,D)$ is random.

2. Main results

This section investigates and provides four formulations for general solutions along with their prerequisites and conditions. Regarding the operator equation

$$\mathcal{P}^*X\Phi^* + \Phi Y\mathcal{P} = \Omega. \tag{1}$$

And it is considered a generalization of the equation that appeared in [41].

For $\mathcal{P}$ and Φ are invertible, the following theorem discusses the state solution of Equation (1).

Theorem 2.1: Let $\mathcal{P} \in B(B(F,D)$ and $\Phi \in B(D,F)$ be invertible and $\Omega \in B(F)$, where F, D be Hilbert C*-modules, then the operator Equation (1) has a solution $(X,Y) \in B(D) \times B(D)$, in this case any solution to Equation (1) in this situation is shown as follows:

$$X = \frac{1}{2}(\mathcal{P}^*)^{-1}\Omega(\Phi^*)^{-1} - (\mathcal{P}^*)^{-1}K(\Phi^*)^{-1}$$
$$Y = \frac{1}{2}(\Phi)^{-1}\Omega(\mathcal{P})^{-1} + (\Phi)^{-1}K(\mathcal{P})^{-1},$$

where $K \in B(F)$ is arbitrary.

Proof: Suppose$M = \begin{bmatrix} \mathcal{P}^* & 0 \\ 0 & \mathcal{P}^* \end{bmatrix} : \mathcal{D} \oplus \mathcal{D} \longrightarrow \mathcal{F} \oplus \mathcal{F}, N = \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix} : \mathcal{D} \oplus \mathcal{D} \longrightarrow \mathcal{F} \oplus \mathcal{F}, \hat{X} = \begin{bmatrix} 0 & X \\ Y^* & 0 \end{bmatrix} : \mathcal{D} \oplus \mathcal{D} \longrightarrow \mathcal{D} \oplus \mathcal{D}, L = \begin{bmatrix} 0 & \Omega \\ \Omega^* & 0 \end{bmatrix} : \mathcal{F} \oplus \mathcal{F} \longrightarrow \mathcal{F} \oplus \mathcal{F}$.

It is clear that L is self-adjoint. Therefore, using Theorem 1.7, there is a solution like $\hat{X}$ for equation $M\hat{X}N^* + N\hat{X}^*M^* = L$, from which we will obtain

$$\begin{bmatrix} \mathcal{P}^* & 0 \\ 0 & \mathcal{P}^* \end{bmatrix} \begin{bmatrix} 0 & X \\ Y^* & 0 \end{bmatrix} \begin{bmatrix} \Phi^* & 0 \\ 0 & \Phi^* \end{bmatrix} + \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix} \begin{bmatrix} 0 & Y \\ X^* & 0 \end{bmatrix} \begin{bmatrix} \mathcal{P} & 0 \\ 0 & \mathcal{P} \end{bmatrix}$$
$$\begin{bmatrix} 0 & \mathcal{P}^*X\Phi^* + \Phi Y\mathcal{P} \\ \mathcal{P}^*Y^*\Phi^* + \Phi X^*\mathcal{P} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \Omega \\ \Omega^* & 0 \end{bmatrix} = \mathrm{L},$$

from this, we conclude $\mathcal{P}^*X\Phi^* + \Phi Y\mathcal{P} = \Omega$. That is (X, Y) represents a solution to this equation. It is clear that M and N are reversible. Referring to Theorem 1.7 and utilizing the data that has emerged, we find that the solution to equation $M\hat{X}N^* + N\hat{X}^*M^* = L$ is as follows

$$\hat{X} = \frac{1}{2}M^{-1}L(N^*)^{-1} - M^{-1}Z(N^*)^{-1}, \tag{2}$$

where $Z \in B(F \oplus F)$ satisfied $Z^* = -Z$. If we take $Z=\begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix}$, and after substituting each value with its equivalent in Equation (2), produces

$$\begin{bmatrix} 0 & X \\ Y^* & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & (\mathcal{P}^*)^{-1}\Omega(\Phi^*)^{-1} \\ (\mathcal{P}^*)^{-1}\Omega^*(\Phi^*)^{-1} & 0 \end{bmatrix}$$
$$-\begin{bmatrix} (\mathcal{P}^*)^{-1}Z_1(\Phi^*)^{-1} & (\mathcal{P}^*)^{-1}Z_2(\Phi^*)^{-1} \\ (\mathcal{P}^*)^{-1}Z_3(\Phi^*)^{-1} & (\mathcal{P}^*)^{-1}Z_4(\Phi^*)^{-1} \end{bmatrix}.$$

from this data and by utilizing the properties of matrices, leads to

$$X = \frac{1}{2}(\mathcal{P}^*)^{-1}\Omega(\Phi^*)^{-1} - (\mathcal{P}^*)^{-1}Z_2(\Phi^*)^{-1} \tag{3}$$
$$Y^* = \frac{1}{2}(\mathcal{P}^*)^{-1}\Omega^*(\Phi^*)^{-1} - (\mathcal{P}^*)^{-1}Z_3(\Phi^*)^{-1} \tag{4}$$
$$(\mathcal{P}^*)^{-1}Z_1(\Phi^*)^{-1} = 0, (\mathcal{P}^*)^{-1}Z_4(\Phi^*)^{-1}. \tag{5}$$

We have that both $\mathcal{P}^*$ and Φ^* are invertible, so by utilizing Equation (5), it becomes clear that $Z_1 = 0, Z_4 = 0$. Additionally, we have $Z^* = -Z$, and this relationship provides us with the relationship $Z_3^* = -Z_2$. Therefore, when we assume that $K = Z_2$, both (3) and (4) become as the following form, which represents a solution to Equation (1).

$$X = \frac{1}{2}(\mathcal{P}^*)^{-1}\Omega(\Phi^*)^{-1} - (\mathcal{P}^*)^{-1}K(\Phi^*)^{-1}$$
$$Y = \frac{1}{2}(\Phi)^{-1}\Omega(\mathcal{P})^{-1} + (\Phi)^{-1}K(\mathcal{P})^{-1}.$$

If Φ is a regular and $\mathcal{P}$ is invertible, the following theorem gives another solution of Equation (1).

Theorem 2.2: Let $\mathcal{P} \in B(F, D)$, $\Phi \in B(D, F)$ and $\Omega \in B(F)$, where F, D be Hilbert C*-modules, $\mathcal{P}$ be invertible and Φ is regular, such that $\Phi\Phi^+\Omega = \Omega$, then the statements that follow are interchangeable:

(i) There is a solution $(X, Y) \in B(D) \times B(D)$ to Equation (1),

(ii) $(I - \Phi\Phi^+)\Omega(I - \Phi\Phi^+) = 0$.

Any solution to Equation (1) takes the following form if (i) or (ii) are fulfilled.

$$X = \frac{1}{2}\mathcal{P}^{*^{-1}}\Phi\Phi^+\Omega\Phi^{*^+} - \mathcal{P}^{*^{-1}}\Phi\Phi^+K\Phi^{*^+} + \mathcal{P}^{*^{-1}}(I - \Phi\Phi^+)\Omega\Phi^{*^+}$$
$$+\mathcal{P}^{*^{-1}}U^*(I - \Phi^+\Phi).$$
$$Y = \frac{1}{2}\Phi^+\Omega\Phi\Phi^+\mathcal{P}^{-1} + \Phi^+K\Phi\Phi^+\mathcal{P}^{-1} + \Phi^+\Omega(I - \Phi\Phi^+)\mathcal{P}^{-1} + (I - \Phi^+\Phi)W\mathcal{P}^{-1},$$

where $K \in B(F), W, U \in B(F, D)$.

Proof: $(i) \Longrightarrow (ii)$ Suppose the Equation (1) has solution $(X, Y) \in B(D) \times B(D)$. Then we have

$$(I - \Phi\Phi^+)\Omega(I - \Phi\Phi^+) = (I - \Phi\Phi^+)(\mathcal{P}^*X\Phi^* + \Phi Y\mathcal{P})(I - \Phi\Phi^+)$$
$$= (I - \Phi\Phi^+)\mathcal{P}^*X\Phi^*(I - \Phi\Phi^+) + (I - \Phi\Phi^+)\Phi Y\mathcal{P}(I - \Phi\Phi^+) = 0.$$

$(ii) \Longrightarrow (i)$, Suppose $(I - \Phi\Phi^+)\Omega(I - \Phi\Phi^+) = 0$ and let $N = \begin{bmatrix} \mathcal{P}^* & 0 \\ 0 & \mathcal{P}^* \end{bmatrix} : \mathcal{D} \oplus \mathcal{D} \longrightarrow \mathcal{F} \oplus \mathcal{F}, M = \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix} : \mathcal{D} \oplus \mathcal{D} \longrightarrow \mathcal{F} \oplus \mathcal{F}, \hat{X} = \begin{bmatrix} 0 & X^* \\ Y & 0 \end{bmatrix} : \mathcal{D} \oplus \mathcal{D} \longrightarrow \mathcal{D} \oplus \mathcal{D}, L = \begin{bmatrix} 0 & \Omega^* \\ \Omega & 0 \end{bmatrix} : \mathcal{F} \oplus \mathcal{F} \longrightarrow \mathcal{F} \oplus \mathcal{F}$. Clear L is self-adjoint also,

$$(I-MM^{+})L(I-MM^{+})=\begin{bmatrix}0 & (I-\Phi\Phi^{+})\Omega^{*}(I-\Phi\Phi^{+})\\ (I-\Phi\Phi^{+})\Omega(I-\Phi\Phi^{+}) & 0\end{bmatrix}=0,$$

from this data and referring to Theorem 1.8, $\hat{X}$ represents a solution to equation $M\hat{X}N^{*}+N\hat{X}^{*}M^{*}=L$, and from this equation, consists

$$M\hat{X}N^{*}+N\hat{X}^{*}M^{*}=\begin{bmatrix}0 & \Phi X^{*}\mathcal{P}+\mathcal{P}^{*}Y^{*}\Phi^{*}\\ \Phi Y\mathcal{P}+\mathcal{P}^{*}X\Phi^{*} & 0\end{bmatrix}=\begin{bmatrix}0 & \Omega^{*}\\ \Omega & 0\end{bmatrix}=L,$$

from this, we conclude $\mathcal{P}^{*}X\Phi^{*}+\Phi Y\mathcal{P}=\Omega$, That is, (X,Y) represents a solution to this equation. Referring to Theorem 1.8 and utilizing the data that has emerged, we find that the solution to equation $M\hat{X}N^{*}+N\hat{X}^{*}M^{*}=L$ is as follows

$$\hat{X}=\frac{1}{2}M^{+}LMM^{+}(N^{*})^{-1}+M^{+}ZMM^{+}(N^{*})^{-1}$$
$$+M^{+}L(I-MM^{+})(N^{*})^{-1}+(I-M^{+}M)S(N^{*})^{-1}, \quad (6)$$

where $Z\in B(F\oplus F)$ satisfied $M^{*}(Z+Z^{*})M=0$ and $S\in B(F\oplus F,D\oplus D)$ is arbitrary. If take $Z=\begin{bmatrix}Z_1 & Z_2\\ Z_3 & Z_4\end{bmatrix}$, $S=\begin{bmatrix}S_1 & S_2\\ S_3 & S_4\end{bmatrix}$ and substituting each value with its equivalent in Equation (6), leads to

$$\begin{bmatrix}0 & X^{*}\\ Y & 0\end{bmatrix}=\frac{1}{2}\begin{bmatrix}0 & \Phi^{+}\Omega^{*}\Phi\Phi^{+}\mathcal{P}^{-1}\\ \Phi^{+}\Omega\Phi\Phi^{+}\mathcal{P}^{-1} & 0\end{bmatrix}+\begin{bmatrix}\Phi^{+}Z_1\Phi\Phi^{+}\mathcal{P}^{-1} & \Phi^{+}Z_2\Phi\Phi^{+}\mathcal{P}^{-1}\\ \Phi^{+}Z_3\Phi\Phi^{+}\mathcal{P}^{-1} & \Phi^{+}Z_4\Phi\Phi^{+}\mathcal{P}^{-1}\end{bmatrix}$$
$$+\begin{bmatrix}0 & \Phi^{+}\Omega^{*}(I-\Phi\Phi^{+})\mathcal{P}^{-1}\\ \Phi^{+}\Omega(I-\Phi\Phi^{+})\mathcal{P}^{-1} & 0\end{bmatrix}$$
$$+\begin{bmatrix}(I-\Phi^{+}\Phi)S_1\mathcal{P}^{-1} & (I-\Phi^{+}\Phi)S_2\mathcal{P}^{-1}\\ (I-\Phi^{+}\Phi)S_3\mathcal{P}^{-1} & (I-\Phi^{+}\Phi)S_4\mathcal{P}^{-1}\end{bmatrix},$$

this, relying on the properties of matrices, it bears fruit

$$X^{*}=\frac{1}{2}\Phi^{+}\Omega^{*}\Phi\Phi^{+}\mathcal{P}^{-1}+\Phi^{+}Z_2\Phi\Phi^{+}\mathcal{P}^{-1}+\Phi^{+}\Omega^{*}(I-\Phi\Phi^{+})\mathcal{P}^{-1}+(I-\Phi^{+}\Phi)S_2\mathcal{P}^{-1} \quad (7)$$
$$Y=\frac{1}{2}\Phi^{+}\Omega\Phi\Phi^{+}\mathcal{P}^{-1}+\Phi^{+}Z_3\Phi\Phi^{+}\mathcal{P}^{-1}+\Phi^{+}\Omega(I-\Phi\Phi^{+})\mathcal{P}^{-1}+(I-\Phi^{+}\Phi)S_3\mathcal{P}^{-1} \quad (8)$$
$$\Phi^{+}Z_1\Phi\Phi^{+}\mathcal{P}^{-1}+(I-\Phi^{+}\Phi)S_1\mathcal{P}^{-1}=0 \quad (9)$$
$$\Phi^{+}Z_4\Phi\Phi^{+}\mathcal{P}^{-1}+(I-\Phi^{+}\Phi)S_4\mathcal{P}^{-1}=0, \quad (10)$$

and after multiplying Equations (9) and (10) by $\mathcal{P}$ from the right and Φ from the left, provides

$$\Phi\Phi^{+}Z_1\Phi\Phi^{+}=0 \text{ and } \Phi\Phi^{+}Z_4\Phi\Phi^{+}=0,$$

and since we have $M^{*}(Z+Z^{*})M=0$, this gives us the following data.

$$\Phi^{*}{Z_2}^{*}\Phi=-\Phi^{*}Z_3\Phi \text{ and } \Phi^{*}{Z_3}^{*}\Phi=-\Phi^{*}Z_2\Phi, \quad (11)$$

and since $ker\ (M^{+})=ker\ (M^{*})$, this means that $M^{+}(Z+Z^{*})M=0$, and this leads us to the following

$$\Phi^{+}Z_3\Phi=-\Phi^{+}{Z_2}^{*}\Phi \text{ and } \Phi^{+}Z_2\Phi=-\Phi^{+}{Z_3}^{*}\Phi. \quad (12)$$

Therefore, taking $Z_3=K, S_2=U, S_3=W$ and using Equations (11) and (12). Equations (7) and (8) will become as follows and represent a solution to the Equation (1)

$$X=\frac{1}{2}\mathcal{P}^{*-1}\Phi\Phi^{+}\Omega\Phi^{*+}-\mathcal{P}^{*-1}\Phi\Phi^{+}K\Phi^{*+}+\mathcal{P}^{*-1}(I-\Phi\Phi^{+})\Omega\Phi^{*+}$$
$$+\mathcal{P}^{*-1}U^{*}(I-\Phi^{+}\Phi)$$
$$Y=\frac{1}{2}\Phi^{+}\Omega\Phi\Phi^{+}\mathcal{P}^{-1}+\Phi^{+}K\Phi\Phi^{+}\mathcal{P}^{-1}+\Phi^{+}\Omega(I-\Phi\Phi^{+})\mathcal{P}^{-1}+(I-\Phi^{+}\Phi)W\mathcal{P}^{-1}.$$

If Φ is invertible and $\mathcal{P}$ is a regular, the following theorem provides solution to Equation (1).

Theorem 2.3: Let $\mathcal{P}\in B(F,D)$, $\Phi\in B(D,F)$ and $\Omega\in B(F)$, where F, D be Hilbert C*-modules, Φ be invertible and $\mathcal{P}$ be regular, such that $\Omega\mathcal{P}^{+}\mathcal{P}=\Omega$, then the statements that follow are interchangeable:

(i) There is a solution $(X,Y)\in B(D)\times B(D)$ to Equation (1),

(ii) $(I-\mathcal{P}^{+}\mathcal{P})\Omega(I-\mathcal{P}^{+}\mathcal{P})=0$.

Any solution to Equation (1) takes the following form if (i) or (ii) are fulfilled

$$X = \frac{1}{2}\mathcal{P}^{*+}\Omega\mathcal{P}^{+}\mathcal{P}\Phi^{*-1} + \mathcal{P}^{*+}K\mathcal{P}^{+}\mathcal{P}\Phi^{*-1} + \mathcal{P}^{*+}\Omega(I-\mathcal{P}^{+}\mathcal{P})\Phi^{*-1} + (I-\mathcal{P}\mathcal{P}^{+})U\Phi^{*-1}$$
$$Y = \frac{1}{2}\Phi^{-1}\mathcal{P}^{+}\mathcal{P}\Omega\mathcal{P}^{+} - \Phi^{-1}\mathcal{P}^{+}\mathcal{P}K\mathcal{P}^{+} + \Phi^{-1}(I-\mathcal{P}^{+}\mathcal{P})\Omega\mathcal{P}^{+} + \Phi^{-1}W^{*}(I-\mathcal{P}\mathcal{P}^{+}),$$
where $K \in B(F), W, U \in B(F,D)$

Proof: $(i) \Longrightarrow (ii)$ Suppose the Equation (1) has solution $(X,Y) \in B(D) \times B(D)$. This achieves
$$(I-\mathcal{P}^{+}\mathcal{P})\Omega(I-\mathcal{P}^{+}\mathcal{P}) = (I-\mathcal{P}^{+}\mathcal{P})(\mathcal{P}^{*}X\Phi^{*} + \Phi Y\mathcal{P})(I-\mathcal{P}^{+}\mathcal{P})$$
$$= (I-\mathcal{P}^{+}\mathcal{P})\mathcal{P}^{*}X\Phi^{*}(I-\mathcal{P}^{+}\mathcal{P}) + (I-\mathcal{P}^{+}\mathcal{P})\Phi Y\mathcal{P}(I-\mathcal{P}^{+}\mathcal{P}) = 0.$$
$(ii) \Longrightarrow (i)$ Suppose $(I-\mathcal{P}^{+}\mathcal{P})\Omega(I-\mathcal{P}^{+}\mathcal{P}) = 0$ and let $M = \begin{bmatrix}\mathcal{P}^{*} & 0\\ 0 & \mathcal{P}^{*}\end{bmatrix}: \mathcal{D}\oplus\mathcal{D} \longrightarrow \mathcal{F}\oplus\mathcal{F}$, $N = \begin{bmatrix}\Phi & 0\\ 0 & \Phi\end{bmatrix}: \mathcal{D}\oplus\mathcal{D} \longrightarrow \mathcal{F}\oplus\mathcal{F}$, $\hat{X} = \begin{bmatrix}0 & X\\ Y^{*} & 0\end{bmatrix}: \mathcal{D}\oplus\mathcal{D} \longrightarrow \mathcal{D}\oplus\mathcal{D}$, $L = \begin{bmatrix}0 & \Omega\\ \Omega^{*} & 0\end{bmatrix}: \mathcal{F}\oplus\mathcal{F} \longrightarrow \mathcal{F}\oplus\mathcal{F}$. Clear L is self-adjoint also,
$$(I-MM^{+})L(I-MM^{+}) = \begin{bmatrix}0 & (I-\mathcal{P}^{+}\mathcal{P})\Omega(I-\mathcal{P}^{+}\mathcal{P})\\ (I-\mathcal{P}^{*}\mathcal{P}^{*+})\Omega^{*}(I-\mathcal{P}^{*}\mathcal{P}^{*+}) & 0\end{bmatrix} = 0,$$
from this data and referring to Theorem 1.8, $\hat{X}$ will represent a solution to equation $M\hat{X}N^{*} + N\hat{X}^{*}M^{*} = L$, and from this equation, consists
$$M\hat{X}N^{*} + N\hat{X}^{*}M^{*} = \begin{bmatrix}0 & \mathcal{P}^{*}X\Phi^{*} + \Phi Y\mathcal{P}\\ \mathcal{P}^{*}Y^{*}\Phi^{*} + \Phi X^{*}\mathcal{P} & 0\end{bmatrix} = \begin{bmatrix}0 & \Omega\\ \Omega^{*} & 0\end{bmatrix} = L,$$
from this, produces $\mathcal{P}^{*}X\Phi^{*} + \Phi Y\mathcal{P} = \Omega$, that is $(X\,,Y)$ represents a solution to this equation. Referring to Theorem 1.8 and utilizing the data that has emerged, we find that the solution to equation $M\hat{X}N^{*} + N\hat{X}^{*}M^{*} = L$ is as follows
$$\hat{X} = \frac{1}{2}M^{+}LMM^{+}(N^{*})^{-1} + M^{+}ZMM^{+}(N^{*})^{-1} + M^{+}L(I-MM^{+})(N^{*})^{-1} + (I-MM^{+})S(N^{*})^{-1}, \quad (13)$$
where $Z \in B(F\oplus F)$ satisfied $M^{*}(Z+Z^{*})M = 0$ and $S \in B(F\oplus F, D\oplus D)$ is arbitrary. If take $Z = \begin{bmatrix}Z_1 & Z_2\\ Z_3 & Z_4\end{bmatrix}$ and $S = \begin{bmatrix}S_1 & S_2\\ S_3 & S_4\end{bmatrix}$ and substituting each value with its equivalent in Equation (13), this achieves
$$\begin{bmatrix}0 & X\\ Y^{*} & 0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}0 & \mathcal{P}^{*+}\Omega\mathcal{P}^{*}\mathcal{P}^{*+}\Phi^{*-1}\\ \mathcal{P}^{*+}\Omega^{*}\mathcal{P}^{*}\mathcal{P}^{*+}\Phi^{*-1} & 0\end{bmatrix} + \begin{bmatrix}\mathcal{P}^{*+}Z_1\mathcal{P}^{*}\mathcal{P}^{*+}\Phi^{*-1} & \mathcal{P}^{*+}Z_2\mathcal{P}^{*}\mathcal{P}^{*+}\Phi^{*-1}\\ \mathcal{P}^{*+}Z_3\mathcal{P}^{*}\mathcal{P}^{*+}\Phi^{*-1} & \mathcal{P}^{*+}Z_4\mathcal{P}^{*}\mathcal{P}^{*+}\Phi^{*-1}\end{bmatrix}$$
$$+ \begin{bmatrix}0 & \mathcal{P}^{*+}\Omega(I-\mathcal{P}^{*}\mathcal{P}^{*+})\Phi^{*-1}\\ \mathcal{P}^{*+}\Omega^{*}(I-\mathcal{P}^{*}\mathcal{P}^{*+})\Phi^{*-1} & 0\end{bmatrix} + \begin{bmatrix}(I-\mathcal{P}^{*+}\mathcal{P}^{*})S_1\Phi^{*-1} & (I-\mathcal{P}^{*+}\mathcal{P}^{*})S_2\Phi^{*-1}\\ (I-\mathcal{P}^{*+}\mathcal{P}^{*})S_3\Phi^{*-1} & (I-\mathcal{P}^{*+}\mathcal{P}^{*})S_4\Phi^{*-1}\end{bmatrix},$$
this, relying on the properties of matrices, yield
$$X = \frac{1}{2}\mathcal{P}^{*+}\Omega\mathcal{P}^{*}\mathcal{P}^{*+}\Phi^{*-1} + \mathcal{P}^{*+}Z_2\mathcal{P}^{*}\mathcal{P}^{*+}\Phi^{*-1} + \mathcal{P}^{*+}\Omega(I-\mathcal{P}^{*}\mathcal{P}^{*+})\Phi^{*-1} + (I-\mathcal{P}^{*+}\mathcal{P}^{*})S_2\Phi^{*-1}. \quad (14)$$
$$Y^{*} = \frac{1}{2}\mathcal{P}^{*+}\Omega^{*}\mathcal{P}^{*}\mathcal{P}^{*+}\Phi^{*-1} + \mathcal{P}^{*+}Z_3\mathcal{P}^{*}\mathcal{P}^{*+}\Phi^{*-1} + \mathcal{P}^{*+}\Omega^{*}(I-\mathcal{P}^{*}\mathcal{P}^{*+})\Phi^{*-1} + (I-\mathcal{P}^{*+}\mathcal{P}^{*})S_3\Phi^{*-1}. \quad (15)$$
$$\mathcal{P}^{*+}Z_1\mathcal{P}^{*}\mathcal{P}^{*+}\Phi^{*-1} + (I-\mathcal{P}^{*+}\mathcal{P}^{*})S_1\Phi^{*-1} = 0. \quad (16)$$
$$\mathcal{P}^{*+}Z_4\mathcal{P}^{*}\mathcal{P}^{*+}\Phi^{*-1} + (I-\mathcal{P}^{*+}\mathcal{P}^{*})S_4\Phi^{*-1} = 0, \quad (17)$$
and after multiplying Equations (16) and (17) by Φ^{*} from the right and $\mathcal{P}^{*}$ from the left, gives
$$\mathcal{P}^{+}\mathcal{P}Z_1\mathcal{P}^{+}\mathcal{P} = 0 \text{ and } \mathcal{P}^{+}\mathcal{P}Z_4\mathcal{P}^{+}\mathcal{P} = 0,$$
and since we have $M^{*}(Z+Z^{*})M = 0$, this provides us with the following data

$$\mathcal{P}Z_2\mathcal{P}^* = -\mathcal{P}Z_3{}^*\mathcal{P}^* \text{ and } \mathcal{P}Z_3\mathcal{P}^* = -\mathcal{P}Z_2{}^*\mathcal{P}^*, \quad (18)$$

and since $ker\ (M^+) = ker\ (M^*)$, this means that $M^+(Z+Z^*)M = 0$, and this leads us to the following

$$\mathcal{P}^{*+}Z_3\mathcal{P}^* = -\mathcal{P}^{*+}Z_2{}^*\mathcal{P}^* \text{ and } \mathcal{P}^{*+}Z_2\mathcal{P}^* = -\mathcal{P}^{*+}Z_3{}^*\mathcal{P}^*. \quad (19)$$

Therefore, taking $Z_2 = K, S_2 = U, S_3 = W$, using Equations (18) and (19). Equations (14) and (15) will become as follows and represent a solution to the Equation (1)

$$X = \frac{1}{2}\mathcal{P}^{*+}\Omega\mathcal{P}^+\mathcal{P}\Phi^{*-1} + \mathcal{P}^{*+}K\mathcal{P}^+\mathcal{P}\Phi^{*-1} + \mathcal{P}^{*+}\Omega(I-\mathcal{P}^+\mathcal{P})\Phi^{*-1} + (I-\mathcal{P}\mathcal{P}^+)U\Phi^{*-1}.$$

$$Y = \frac{1}{2}\Phi^{-1}\mathcal{P}^+\mathcal{P}\Omega\mathcal{P}^+ - \Phi^{-1}\mathcal{P}^+\mathcal{P}K\mathcal{P}^+ + \Phi^{-1}(I-\mathcal{P}^+\mathcal{P})\Omega\mathcal{P}^+ + \Phi^{-1}W^*(I-\mathcal{P}\mathcal{P}^+).$$

The next theorem addresses the fourth solution of Equation (1), where both $\mathcal{P}$ and Φ are regulars.

Theorem 2.4: Let $\mathcal{P} \in B(F,D)$, $\Phi \in B(D,F)$ be self-adjoint and $\Omega \in B(F)$, where F, D be Hilbert C*-modules, Φ and $\mathcal{P}$ are regular, such that $\Omega\mathcal{P}^+\mathcal{P} = \Omega$, $\mathcal{P} = \mathcal{P}\Phi\Phi^+$, $\Phi\Phi^+\Omega = \Omega = \Omega\Phi^+\Phi$ then the statements that follow are interchangeable:

(i) There is a solution $(X,Y) \in B(F,D) \times B(D,F)$ to Equation (1),

(ii) $(I-\mathcal{P}^+\mathcal{P})\Omega(I-\mathcal{P}^+\mathcal{P}) = 0$ and $(I-\Phi\Phi^+)\Omega(I-\Phi\Phi^+) = 0$.

Any solution to Equation (1) takes the following form if (i) or (ii) are fulfilled

$$X = \mathcal{P}^{*+}\Omega\Phi^+ - \frac{1}{2}\mathcal{P}^{*+}\Omega\mathcal{P}^+\mathcal{P}\Phi^+ + \mathcal{P}^{*+}K\mathcal{P}^+\mathcal{P}\Phi^+ - U + \mathcal{P}\mathcal{P}^+U\Phi\Phi^+.$$

$$Y = \Phi^+\Omega\mathcal{P}^+ - \frac{1}{2}\Phi^+\mathcal{P}^+\mathcal{P}\Omega\mathcal{P}^+ - \Phi^+\mathcal{P}^+\mathcal{P}K\mathcal{P}^+ - W^* + \Phi^+\Phi W^*\mathcal{P}\mathcal{P}^+,$$

where $K \in B(F), W, U \in B(F,D)$.

Proof: $(i) \Longrightarrow (ii)$ Suppose the Equation (1) has solution $(X,Y) \in B(F,D) \times B(F,D)$. Then we have

$$\begin{aligned}(I-\mathcal{P}^+\mathcal{P})\Omega(I-\mathcal{P}^+\mathcal{P}) &= (I-\mathcal{P}^+\mathcal{P})(\mathcal{P}^*X\Phi^* + \Phi Y\mathcal{P})(I-\mathcal{P}^+\mathcal{P})\\ &= (I-\mathcal{P}^+\mathcal{P})\mathcal{P}^*X\Phi^*(I-\mathcal{P}^+\mathcal{P}) + (I-\mathcal{P}^+\mathcal{P})\Phi Y\mathcal{P}(I-\mathcal{P}^+\mathcal{P}) = 0,\end{aligned}$$

and

$$\begin{aligned}(I-\Phi\Phi^+)\Omega(I-\Phi\Phi^+) &= (I-\Phi\Phi^+)(\mathcal{P}^*X\Phi^* + \Phi Y\mathcal{P})(I-\Phi\Phi^+)\\ &= (I-\Phi\Phi^+)\mathcal{P}^*X\Phi^*(I-\Phi\Phi^+) + (I-\Phi\Phi^+)\Phi Y\mathcal{P}(I-\Phi\Phi^+) = 0.\end{aligned}$$

$(ii) \Longrightarrow (i)$ Suppose $(I-\mathcal{P}^+\mathcal{P})\Omega(I-\mathcal{P}^+\mathcal{P}) = 0$, and $(I-\Phi\Phi^+)\Omega(I-\Phi\Phi^+) = 0$ and let $M = \begin{bmatrix}\mathcal{P}^* & 0\\ 0 & \mathcal{P}^*\end{bmatrix}: \mathcal{D}\oplus\mathcal{D} \longrightarrow \mathcal{F}\oplus\mathcal{F}$, $N = \begin{bmatrix}\Phi & 0\\ 0 & \Phi\end{bmatrix}: \mathcal{F}\oplus\mathcal{F} \longrightarrow \mathcal{F}\oplus\mathcal{F}$, $\hat{X} = \begin{bmatrix}0 & X\\ Y^* & 0\end{bmatrix}: \mathcal{F}\oplus\mathcal{F} \longrightarrow \mathcal{D}\oplus\mathcal{D}$, $L = \begin{bmatrix}0 & \Omega\\ \Omega^* & 0\end{bmatrix}: \mathcal{F}\oplus\mathcal{F} \longrightarrow \mathcal{F}\oplus\mathcal{F}$. Clear L is self-adjoint also,

$$(I-MM^+)L(I-MM^+) = \begin{bmatrix}0 & (I-\mathcal{P}^+\mathcal{P})\Omega(I-\mathcal{P}^+\mathcal{P})\\ (I-\mathcal{P}^*\mathcal{P}^{*+})\Omega^*(I-\mathcal{P}^*\mathcal{P}^{*+}) & 0\end{bmatrix} = 0,$$

from this data and referring to Theorem 1.9, $\hat{X}$ represents a solution to equation $M\hat{X}N^* + N\hat{X}^*M^* = L$, and from this equation, consists

$$M\hat{X}N^* + N\hat{X}^*M^* = \begin{bmatrix}0 & \mathcal{P}^*X\Phi^* + \Phi Y\mathcal{P}\\ \mathcal{P}^*Y^*\Phi^* + \Phi X^*\mathcal{P} & 0\end{bmatrix} = \begin{bmatrix}0 & \Omega\\ \Omega^* & 0\end{bmatrix} = L,$$

from this, produces $\mathcal{P}^*X\Phi^* + \Phi Y\mathcal{P} = \Omega$, that is (X, Y) represents a solution to this equation. Referring to Theorem 1.9 and utilizing the data that has emerged, the solution to equation $M\hat{X}N^* + N\hat{X}^*M^* = L$ is as follows

$$\hat{X} = M^+LN^+ - \frac{1}{2}M^+LMM^+N^+ + M^+ZMM^+N^+ + S - M^+MSNN^+, \quad (20)$$

where $Z \in B(F\oplus F)$ satisfied $M^*(Z+Z^*)M = 0$ and $S \in B(F\oplus F, D\oplus D)$ is arbitrary. If take $Z = \begin{bmatrix}Z_1 & Z_2\\ Z_3 & Z_4\end{bmatrix}$ and $S = \begin{bmatrix}S_1 & S_2\\ S_3 & S_4\end{bmatrix}$ and substituting each value with its equivalent in Equation (1), this generates

$$\begin{bmatrix}0 & X\\ Y^* & 0\end{bmatrix} = \begin{bmatrix}0 & \mathcal{P}^{*+}\Omega\Phi^+\\ \mathcal{P}^{*+}\Omega^*\Phi^+ & 0\end{bmatrix}$$

$$-\frac{1}{2}\begin{bmatrix} 0 & \mathcal{P}^{*+}\Omega\mathcal{P}^{*}\mathcal{P}^{*+}\Phi^{+} \\ \mathcal{P}^{*+}\Omega^{*}\mathcal{P}^{*}\mathcal{P}^{*+}\Phi^{+} & 0 \end{bmatrix} + \begin{bmatrix} \mathcal{P}^{*+}Z_1\mathcal{P}^{*}\mathcal{P}^{*+}\Phi^{+} & \mathcal{P}^{*+}Z_2\mathcal{P}^{*}\mathcal{P}^{*+}\Phi^{+} \\ \mathcal{P}^{*+}Z_3\mathcal{P}^{*}\mathcal{P}^{*+}\Phi^{+} & \mathcal{P}^{*+}Z_4\mathcal{P}^{*}\mathcal{P}^{*+}\Phi^{+} \end{bmatrix}$$
$$-\begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} + \begin{bmatrix} \mathcal{P}^{*+}\mathcal{P}^{*}S_1\Phi\Phi^{+} & \mathcal{P}^{*+}\mathcal{P}^{*}S_2\Phi\Phi^{+} \\ \mathcal{P}^{*+}\mathcal{P}^{*}S_3\Phi\Phi^{+} & \mathcal{P}^{*+}\mathcal{P}^{*}S_4\Phi\Phi^{+} \end{bmatrix},$$

this, relying on the properties of matrices, yield

$$X = \mathcal{P}^{*+}\Omega\Phi^{+} - \frac{1}{2}\mathcal{P}^{*+}\Omega\mathcal{P}^{*}\mathcal{P}^{*+}\Phi^{+} + \mathcal{P}^{*+}Z_2\mathcal{P}^{*}\mathcal{P}^{*+}\Phi^{+} - S_2 + \mathcal{P}^{*+}\mathcal{P}^{*}S_2\Phi\Phi^{+}. \quad (21)$$

$$Y^{*} = \mathcal{P}^{*+}\Omega^{*}\Phi^{+} - \frac{1}{2}\mathcal{P}^{*+}\Omega^{*}\mathcal{P}^{*}\mathcal{P}^{*+}\Phi^{+} + \mathcal{P}^{*+}Z_3\mathcal{P}^{*}\mathcal{P}^{*+}\Phi^{+} - S_3 + \mathcal{P}^{*+}\mathcal{P}^{*}S_3\Phi\Phi^{+}. \quad (22)$$

$$\mathcal{P}^{*+}Z_1\mathcal{P}^{*}\mathcal{P}^{*+}\Phi^{+} - S_1 + \mathcal{P}^{*+}\mathcal{P}^{*}S_1\Phi\Phi^{+} = 0. \quad (25)$$

$$\mathcal{P}^{*+}Z_4\mathcal{P}^{*}\mathcal{P}^{*+}\Phi^{+} - S_4 + \mathcal{P}^{*+}\mathcal{P}^{*}S_4\Phi\Phi^{+} = 0, \quad (26)$$

and after multiplying Equations (25) and (26) by Φ from the right and $\mathcal{P}^{*}$ from the left, provide

$$\mathcal{P}^{*}\mathcal{P}^{*+}Z_1\mathcal{P}^{*}\mathcal{P}^{*+} = 0, \mathcal{P}^{*}\mathcal{P}^{*+}Z_4\mathcal{P}^{*}\mathcal{P}^{*+} = 0,$$

from the relationship $M^{*}(Z + Z^{*})M = 0$and because $ker\ (M^{+}) = ker\ (M^{*})$, gives$M^{+}(Z + Z^{*})M = 0$, and this leads us to the following data

$$\mathcal{P}^{*+}Z_3\mathcal{P}^{*} = -\mathcal{P}^{*+}{Z_2}^{*}\mathcal{P}^{*}\ and\ \mathcal{P}^{*+}Z_2\mathcal{P}^{*} = -\mathcal{P}^{*+}{Z_3}^{*}\mathcal{P}^{*}. \quad (27)$$

Therefore, when we assume that $Z_2 = K, S_2 = U, S_3 = W$, using Equation (27), then Equations (21) and (22) will become as follows and represent a solution to the Equation (1)

$$X = \mathcal{P}^{*+}\Omega\Phi^{+} - \frac{1}{2}\mathcal{P}^{*+}\Omega\mathcal{P}^{+}\mathcal{P}\Phi^{+} + \mathcal{P}^{*+}K\mathcal{P}^{+}\mathcal{P}\Phi^{+} - U + \mathcal{P}\mathcal{P}^{+}U\Phi\Phi^{+}$$
$$Y = \Phi^{+}\Omega\mathcal{P}^{+} - \frac{1}{2}\Phi^{+}\mathcal{P}^{+}\mathcal{P}\Omega\mathcal{P}^{+} - \Phi^{+}\mathcal{P}^{+}\mathcal{P}K\mathcal{P}^{+} - W^{*} + \Phi^{+}\Phi W^{*}\mathcal{P}\mathcal{P}^{+}.$$

3. Conclusions

The aim of this work was to provide some necessary and sufficient conditions for the existence of solutions to the operator equation $\mathcal{P}^{*}X\Phi^{*} + \Phi Y\mathcal{P} = \Omega$, in addition to constructing the forms of these solutions under additional assumptions and by using matrix techniques. This technique may also be applicable to studying other equations such as $\mathcal{P}^{*}X + Y\mathcal{P} = \Omega$ and so on.

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