GENERALIZATION OF THE RULE COUNTING OF THE FREE VARIABLES IN DOUBLE-EVEN PANDIAGONAL MAGIC SQUARES

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Abstract

In this paper, we generalize the rule of counting the number of the free variables in the double – even pandiagonal magic squares; our method is not based on the direct computation of the solution of the linear system. Instead, We deduce this rule by applying the theorems and methods of linear algebra , finally put algorithm of the solution .

الخلاصة

في هذا البحث تم تعميم القانون الذي يحسب من خلاله عدد المتغيرات الحرة في المربعات السحرية القطرية الزوجية المضاعفة، إن طريقة التعميم المستخدمة لا تعتمد على حل نظام المعادلات الخطية، وإنما من تطبيق عدة طرق ونظريات الجبر الخطي الأخرى في الحساب، أخيراً تم وضع خوارزمية للحل.

1. Introduction

Magic squares have turned up throughout history, some in a mathematical context, others in philosophical or religious contexts. According to legend, the first magic square was discovered in China by an unknown mathematician sometime before the first century A.D. It was a magic square of order three thought to have appeared on the back of a turtle emerging from a river. Other magic squares surfaced at various places around the world in the centuries following their discovery. Some of the more interesting examples were recorded in Europe during the 1500s. Cornelius Agrippa wrote De Occulta Philosophia in 1510. In it he describes the spiritual powers of magic squares and produces some squares of orders from three up to nine. His work, although infuential in the mathematical community, enjoyed only brief success, for the counter-reformation and the witch hunts of the Inquisition began soon thereafter: Agrippa himself was accused of being allied with the devil. Although this story seems outlandish now, we cannot ignore the strange mystical ties magic squares seem to have with the world and nature surrounding us, above and beyond their mathematical signicance.

Despite the fact that magic squares have been studied for a long time, they are still the subject of research projects. These include both mathematical-historical research, such as the discovery of unpublished magic squares of Benjamin Franklin [1], and pure mathematical research, much of which is connected with the algebraic and combinatorial geometry of polyhedra (see, for example , [2] , and [3]). Aside from mathematical research, magic squares naturally continue to be an excellent source of topics for "popular" mathematics books .

A pandiagonal magic square is a n×n matrix

a_1	a_2	………	a_n
a_{n+1}	a_{n+2}	………	a_{2n}
…………	…………	……… ….	……… ….
a_{n^2-n+1}	a_{n^2-n+2}	…...	a_{n^2}

Where n>3,and the entries $a_1, a_2, ..., a_{n^2}$ are distinct real numbers satisfying the following system of equations:

$a_1 + a_2 + ... + a_n = S$

$a_{n+1} + a_{n+2} + ... + a_{2n} = S$

....

....

$a_{n^2-n+1} + a_{n^2-n+2} + .. + a_{n^2} = S$ ...(1 – a)

$a_1 + a_{n+1} + ... + a_{n^2-n+1} = S$

$a_2 + a_{n+2} + ... + a_{n^2-n+2} = S$

.....

.....

$a_n + a_{2n} + ... + a_{n^2} = S$ ...(1 – b)

$a_1 + a_{n+2} + ... + a_{n^2} = S$

$a_2 + a_{n+3} + ... + a_{n^2-n+1} = S$

......

.....

$a_n + a_{n+1} + ... + a_{n^2-1} = S$ …(1 – c)

$a_1 + a_{2n} + ... + a_{n^2-n+2} = S$

$a_2 + a_{n+1} + ... + a_{n^2-n+3} = S$

....

....

$a_n + a_{2n-1} + ... + a_{n^2-n+1} = S$ …(1 – d)

where S is a real constant (the so-called magic sum). In this system the group (1 – a) represents the summation of the entries in each row of the matrix. The group (1 – b) represents the summation of the entries in each column of the matrix. The group (1 – c) represents the summation of the entries in each right (extended) diagonal of the matrix. The group (1 – d) represents the summation of the entries in each left (extended) diagonal of the matrix [4]. We will prove that the linear system (1) will have a solution, which contains

n*n – 4*n + 4

free parameters, if n is even.

2. Double-even pandiagonal magic square of order (4 x 4)

We consider the linear system (1 – a), (1 – b), (1 – c) and (1 – d), where n is of the form 2k (k = 2, 4, …). In this system there are four equations, which are linearly dependent on the other equations. These equations are the same three equations as in the odd squares beside the (n-1)th equation of the group (1 – d), which is the result of subtraction of the sum of all other odd-ranked equations of the group (1 – d) from the sum of all odd-ranked equations of the group (1 – a) [4].

Since k = 2 yields a very special case, we start illustrating it: As in the case of odd squares we consider the matrix of coefficients of the system after removing the previous mentioned equations. we consider the square :

a_1	a_2	a_3	a_4
a_5	a_6	a_7	a_8
a_9	a_{10}	a_{11}	a_{12}
a_{13}	a_{14}	a_{15}	a_{16}

In this case the system (1) takes the form:

$$\begin{aligned} a_1 + a_2 + a_3 + a_4 &= S \\ a_5 + a_6 + a_7 + a_8 &= S \\ a_9 + a_{10} + a_{11} + a_{12} &= S \\ a_{13} + a_{14} + a_{15} + a_{16} &= S \end{aligned} \quad ….. (2 – a)$$

$$
\begin{aligned}
a_1 + a_5 + a_9 + a_{13} &= S \\
a_2 + a_6 + a_{10} + a_{14} &= S \\
a_3 + a_7 + a_{11} + a_{15} &= S \\
a_4 + a_8 + a_{12} + a_{16} &= S
\end{aligned} \qquad \text{.... } (2-b)
$$

$$
\begin{aligned}
a_1 + a_6 + a_{11} + a_{16} &= S \\
a_2 + a_7 + a_{12} + a_{13} &= S \\
a_3 + a_8 + a_9 + a_{14} &= S \\
a_4 + a_5 + a_{10} + a_{15} &= S
\end{aligned} \qquad \text{...}(2-c)
$$

$$
\begin{aligned}
a_1 + a_8 + a_{11} + a_{14} &= S \\
a_2 + a_5 + a_{12} + a_{15} &= S \\
a_3 + a_6 + a_9 + a_{16} &= S \\
a_4 + a_7 + a_{10} + a_{13} &= S
\end{aligned} \qquad \text{...}(2-d)
$$

In the system (2 – a), …, (2 – d), there are three equations, which are linearly dependent on the other equations. These equations are: the last equation of the group (2 – b), the last equation of the group (2 – c) and the last equation of the group (2 – d). Indeed, the last equation of the group (2 – b) is the result of subtraction of the sum of all other equations of the group (2 – b) from the sum of all equations of the group (2 – a). The last equation of the group (2 – c) is the result of subtraction of the sum of all other equations of the group (2 – c) from the sum of all equations of the group (2 – a). The last equation of the group (2 – d) is the result of subtraction of the sum of all other equations of the group (2 – d) from the sum of all equations of the group (2 – a).

In order to prove that there are no other dependent equations we write down the matrix of coefficients of the system after removing the previous mentioned equations:

$$
\begin{pmatrix}
1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1\\
1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0\\
0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0\\
0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0\\
1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1\\
0&1&0&0&0&0&1&0&0&0&0&1&1&0&0&0\\
0&0&1&0&0&0&0&1&1&0&0&0&0&1&0&0\\
1&0&0&0&0&0&0&1&0&0&1&0&0&1&0&0\\
0&1&0&0&1&0&0&0&0&0&0&1&0&0&1&0
\end{pmatrix} \quad \text{.......}(3)
$$

We then prove that this matrix has full rank. To establish this we consider the transpose of this matrix(1):

$$
\begin{pmatrix}
1&0&0&0&1&0&0&1&0&0&1&0\\
1&0&0&0&0&1&0&0&1&0&0&1\\
1&0&0&0&0&0&1&0&0&1&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&1&0&0&0&0&0&0&1\\
0&1&0&0&0&1&0&1&0&0&0&0\\
0&1&0&0&0&0&1&0&1&0&0&0\\
0&1&0&0&0&0&0&0&0&1&1&0\\
0&0&1&0&1&0&0&0&0&1&0&0\\
0&0&1&0&0&1&0&0&0&0&0&0\\
0&0&1&0&0&0&1&1&0&0&1&0\\
0&0&1&0&0&0&0&0&1&0&0&1\\
0&0&0&1&1&0&0&0&0&0&0&0\\
0&0&0&1&0&1&0&0&0&1&1&0\\
0&0&0&1&0&0&1&0&0&0&0&1\\
0&0&0&1&0&0&0&1&0&0&0&0
\end{pmatrix}
$$

and prove that this matrix has full rank(1). In order to do this we study the equation of linear dependence between the columns of the last matrix.

We prove that this matrix has full rank by studying the equation of linear dependence between the columns of the matrix:

$$\mathbf{b}_1 * \mathbf{C}_1 + \mathbf{b}_2 * \mathbf{C}_2 + \ldots + \mathbf{b}_{12} * \mathbf{C}_{12} = \mathbf{0}$$

where $\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_{12}$ denotes the columns of the matrix and $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{12}$ are real numbers.

The component wise form of the last equation is:

$$\begin{aligned} \mathbf{b}_1 + \mathbf{b}_5 + \mathbf{b}_8 + \mathbf{b}_{11} &= \mathbf{0} \\ \mathbf{b}_1 + \mathbf{b}_6 + \mathbf{b}_9 + \mathbf{b}_{12} &= \mathbf{0} \\ \mathbf{b}_1 + \mathbf{b}_7 + \mathbf{b}_{10} &= \mathbf{0} \\ \mathbf{b}_1 &= \mathbf{0} \end{aligned} \quad \ldots..(4-1)$$

$$\begin{aligned} \mathbf{b}_2 + \mathbf{b}_5 + \mathbf{b}_{12} &= \mathbf{0} \\ \mathbf{b}_2 + \mathbf{b}_6 + \mathbf{b}_8 &= \mathbf{0} \\ \mathbf{b}_2 + \mathbf{b}_7 + \mathbf{b}_9 &= \mathbf{0} \\ \mathbf{b}_2 + \mathbf{b}_{10} + \mathbf{b}_{11} &= \mathbf{0} \end{aligned} \quad \ldots..(4-2)$$

$$\begin{aligned} \mathbf{b}_3 + \mathbf{b}_5 + \mathbf{b}_{10} &= \mathbf{0} \\ \mathbf{b}_3 + \mathbf{b}_6 &= \mathbf{0} \\ \mathbf{b}_3 + \mathbf{b}_7 + \mathbf{b}_8 + \mathbf{b}_{11} &= \mathbf{0} \\ \mathbf{b}_3 + \mathbf{b}_9 + \mathbf{b}_{12} &= \mathbf{0} \end{aligned} \quad \ldots(4-3)$$

$$\begin{aligned} \mathbf{b}_4 + \mathbf{b}_5 + \mathbf{b}_9 &= \mathbf{0} \\ \mathbf{b}_4 + \mathbf{b}_6 + \mathbf{b}_{10} + \mathbf{b}_{11} &= \mathbf{0} \\ \mathbf{b}_4 + \mathbf{b}_7 + \mathbf{b}_{12} &= \mathbf{0} \\ \mathbf{b}_4 + \mathbf{b}_8 &= \mathbf{0} \end{aligned} \quad \ldots...(4-4)$$

We conclude from the last equation of the group (4 – 1) that $\mathbf{b}_1 = \mathbf{0}$. Substituting this value in the other equations of the group (4 – 1) and adding up leads to:

$$\sum_{j=5}^{12} \mathbf{b}_j = \mathbf{0} \ldots\ldots\ldots\ldots(5)$$

Since the summation of the equations in (4 – 2) yields

$$4\mathbf{b}_2 + \sum_{j=5}^{12} \mathbf{b}_j = \mathbf{0}$$

we deduce from (5) that $\mathbf{b}_2 = \mathbf{0}$. In the same manner we obtain: $\mathbf{b}_3 = \mathbf{b}_4 = \mathbf{0}$.

Using our knowledge about $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ and $\mathbf{b}_4$ we are capable of rewriting the system (4) like this:

$$\begin{aligned} \mathbf{b}_5 + \mathbf{b}_8 + \mathbf{b}_{11} &= \mathbf{0} \\ \mathbf{b}_6 + \mathbf{b}_9 + \mathbf{b}_{12} &= \mathbf{0} \\ \mathbf{b}_7 + \mathbf{b}_{10} &= \mathbf{0} \end{aligned} \quad \ldots..(6-1)$$

$$\begin{aligned} \mathbf{b}_5 + \mathbf{b}_{12} &= \mathbf{0} \\ \mathbf{b}_6 + \mathbf{b}_8 &= \mathbf{0} \\ \mathbf{b}_7 + \mathbf{b}_9 &= \mathbf{0} \\ \mathbf{b}_{10} + \mathbf{b}_{11} &= \mathbf{0} \end{aligned} \quad \ldots..(6-2)$$

$$\begin{aligned} \mathbf{b}_5 + \mathbf{b}_{10} &= \mathbf{0} \\ \mathbf{b}_6 &= \mathbf{0} \\ \mathbf{b}_7 + \mathbf{b}_8 + \mathbf{b}_{11} &= \mathbf{0} \\ \mathbf{b}_9 + \mathbf{b}_{12} &= \mathbf{0} \end{aligned} \quad \ldots..(6-3)$$

$$\begin{aligned} \mathbf{b}_5 + \mathbf{b}_9 &= \mathbf{0} \\ \mathbf{b}_6 + \mathbf{b}_{10} + \mathbf{b}_{11} &= \mathbf{0} \\ \mathbf{b}_7 + \mathbf{b}_{12} &= \mathbf{0} \\ \mathbf{b}_8 &= \mathbf{0} \end{aligned} \quad \ldots\ldots..(6-4)$$

From the group (6 – 3) and the group (6 – 4) we conclude that $\mathbf{b}_6 = \mathbf{0}$ and $\mathbf{b}_8 = \mathbf{0}$. When we substitute these values in the system (6) and do some comparisons, e. g. when equating the left side of the first equation in each of the groups (6 – 1), …, (6 – 4) we obtain:

$$\mathbf{b}_9 = \mathbf{b}_{10} = \mathbf{b}_{11} = \mathbf{b}_{12} \quad \ldots\ldots\ldots..(7)$$

Doing the same with the second equation we get:

$$\mathbf{b}_9 + \mathbf{b}_{12} = \mathbf{b}_{10} + \mathbf{b}_{11} = \mathbf{0} \quad \ldots..(8)$$

We rewrite the equations in (8) in the following manner:

$$\mathbf{b}_{12} = -\mathbf{b}_9, \mathbf{b}_{10} = -\mathbf{b}_{11}$$

Using these relations we obtain from (7) the equation $-2\mathbf{b}_9 = \mathbf{0}$. Thus, $\mathbf{b}_9 = \mathbf{0}$. Due to the previous relations between the variables $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{12}$ we conclude that all of them are zero.

Setting all variables $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{16}$ to $\frac{\mathbf{S}}{\mathbf{4}}$ represents a solution of the nonhomogenous linear system (1). Since we have 16 variables and 16 equations, the solution of the nonhomogenous linear system (1) has according to our analysis 16 – 16 + 4 = 4 free parameters .

3- Generalization of double-even pandiagonal magic square

Now, we treat the general case, we consider the matrix of coefficients of the system after removing the previous mentioned dependent equations. This matrix will be the same as the matrix (3) after deleting the last row.

$$\left(\begin{array}{l}
1 \; \dots \; 1 \;\; 1 \\
\qquad 1 \; \dots \; 1 \;\; 1 \\
\qquad\qquad 1 \; \dots \; 1 \;\; 1 \\
\qquad\qquad\qquad \ddots \\
\qquad\qquad\qquad\quad 1 \; \dots \; 1 \;\; 1 \\
\hline
1 \qquad 1 \qquad 1 \qquad 1 \\
\ddots \qquad \ddots \qquad \ddots \quad \cdots \quad \ddots \\
\quad 1 \;\; 0 \qquad 1 \;\; 0 \qquad 1 \;\; 0 \qquad 1 \;\; 0 \\
\hline
1 \qquad 0\;1 \qquad 0\;\;01 \qquad 0 \cdots 0 \;\; 1 \\
\ddots \qquad \ddots \qquad \ddots \quad \cdots \;\; 1 \\
\qquad\qquad\qquad 1 \qquad \ddots \\
\quad 1 \;\; 0 \qquad 1 \quad 1 \;\; 0 \;\cdots\; 0 \qquad 1 \;\; 0 \;\; 0 \\
\hline
1 \qquad 0 \quad 0 \;\; 1 \quad 0 \quad 1 \;\; 0 \quad 0\;1 \\
\ddots \quad 1 \qquad\qquad 1 \;\; \cdots \\
\quad \ddots \qquad 1 \qquad\qquad \ddots \\
\qquad\qquad \ddots \\
\quad 1 \;\; 0 \quad 1 \;\; 0 \;\; 0 \quad 1\,0\,0\,0 \;\; \cdots \qquad 1
\end{array}\right) \quad \text{.....(9)}$$

We then prove that this matrix has full rank. To establish this we consider the transpose of this matrix(1):

$$\left(\begin{array}{l}
1 \qquad 1 \qquad 1 \qquad 1 \\
\vdots \qquad \ddots \qquad \ddots \qquad \ddots \\
1 \qquad 1 \qquad 1 \qquad 1 \\
1 \qquad 0 \qquad 0 \qquad 0 \\
\hline
1 \qquad 1 \qquad 0 \qquad 0\;1 \\
\vdots \qquad \ddots \qquad 1 \qquad \ddots \\
\qquad\qquad\qquad \ddots \qquad 1 \\
1 \qquad 1 \qquad\qquad\qquad 0 \\
1 \qquad 0 \qquad 1 \qquad 1 \quad 0 \\
\hline
1 \qquad 1 \qquad 0 \;\; 1 \qquad 0\;0\;1 \\
\vdots \qquad \ddots \qquad 0 \qquad \ddots \\
\qquad\qquad\qquad 1 \qquad 1 \\
1 \qquad 1 \qquad \ddots \qquad 1 \\
1 \qquad 0 \qquad 10 \qquad 0\;1 \\
\hline
\ddots \quad \vdots \qquad \vdots \qquad \vdots \\
\hline
\quad 1\;\;1 \qquad 0\;\;1 \qquad 0 \\
\quad \vdots \;\; \ddots \qquad \ddots \qquad 1 \\
\qquad\qquad\qquad 1 \qquad \ddots \\
\quad 1 \qquad 1 \qquad 0 \\
\quad 1 \qquad 0 \qquad 1 \quad 0 \qquad 1
\end{array}\right)$$

and prove that this matrix has full rank[5]. In order to do this we study the equation of linear dependence between the columns of the last matrix.

We prove that this matrix has full rank by studying the equation of linear dependence between the columns of the matrix:

$$b_1 * C_1 + b_2 * C_2 + ... + b_{4n-4} * C_{4n-4} = 0 \qquad \text{……(10)}$$

where $C_1, C_2, ..., C_{4n-4}$ denotes the columns of the matrix and $b_1, b_2, ..., b_{4n-4}$ are real numbers. The componentwise form of equation (10) is:

$$\begin{array}{l}
b_1 + b_{n+1} + b_{2n} + b_{3n-1} = 0 \\
b_1 + b_{n+2} + b_{2n+1} + b_{3n} = 0 \\
.... \\
b_1 + b_{2n-2} + b_{3n-3} + b_{4n-4} = 0 \\
b_1 + b_{2n-1} + b_{3n-2} = 0 \\
b_1 = 0
\end{array} \qquad \text{…..(11 – 1)}$$

$$b_2 + b_{n+1} + b_{3n} = 0$$
$$b_2 + b_{n+2} + b_{2n} + b_{3n+1} = 0$$
....
$$b_2 + b_{3n-2} + b_{3n-1} = 0 \quad .....(11-2)$$

…

…

…

$$b_n + b_{n+1} + b_{2n+1} = 0$$
$$b_n + b_{n+2} + b_{2n+2} + b_{3n-1} = 0$$
....
$$b_n + b_{2n} = 0 \quad .....(11-n)$$

We conclude from the last equation of the group (11 – 1) that $b_1 = 0$. Substituting this value in the other equations of the group (11 – 1) and adding up leads to:

$$\sum_{j=n+1}^{4n-4} b_j = 0 \quad ....(12)$$

Since the summation of the equations in (11 – 2) yields

$$nb_2 + \sum_{j=n+1}^{4n-4} b_j = 0$$

Hence, we deduce from (12) that $b_2 = 0$. In the same manner we obtain:

$b_3 = b_4 = ... = b_n = 0$.

Using our knowledge about $b_1, b_2, ..., b_n$ we are capable of rewriting the system (11) like this:

$$b_{n+1} + b_{2n} + b_{3n-1} = 0$$
$$b_{n+2} + b_{2n+1} + b_{3n} = 0$$
....
$$b_{2n-2} + b_{3n-3} + b_{4n-4} = 0$$
$$b_{2n-1} + b_{3n-2} = 0 \quad ..(13-1)$$

$$b_{n+1} + b_{3n} = 0$$
$$b_{n+2} + b_{2n} + b_{3n+1} = 0$$
....
$$b_{3n-2} + b_{3n-1} = 0 \quad .....(13-2)$$

…

…

…

$$b_{n+1} + b_{2n+1} = 0$$
$$b_{n+2} + b_{2n+2} + b_{3n-1} = 0$$
....
$$b_{2n} = 0 \quad .....(13-n)$$

From the group $(13 - \frac{n}{2})$ and the group (13 – n) we conclude that $b_{\frac{3}{2}n} = 0$ and $b_{2n} = 0$.

When we substitute these values in the system (13) and do some comparisons, e. g. when equating the left side of the first equation in each of the groups (13 – 1), …, (13 – n) we obtain:

$$b_{2n+1} = b_{2n+2} = b_{2n+3} + b_{4n-4} = b_{2n+4} + b_{4n-5} = ... = b_{3n-2} + b_{3n+1} = b_{3n} = b_{3n-1} \quad ....(14)$$

Doing the same with the second equation we get:

$$b_{2n+1} + b_{3n} = b_{2n+2} + b_{3n-1} = b_{2n+3} = b_{2n+4} = b_{2n+5} + b_{4n-4} = b_{2n+6} + b_{4n-5} = ... = b_{3n-2} + b_{3n+3} = b_{3n+2} = b_{3n+1} \quad ....(15)$$

Continuing these comparisons till we reach the $\frac{n}{2}$th equation in each of the groups (13 – 1), …, (13 – n) we get:

$$b_{2n+1} + b_{4n-4} = b_{2n+2} + b_{4n-5} = ... = b_{3n-3} + b_{3n-1} = b_{2n} = 0 \quad ....(16)$$

Remember that in the case k = 2 this equation will be also the second equation, and that we have to deal with two equations, only. We rewrite the equations in (16) in the following manner:

$$b_{3n-1} = -b_{3n-2}$$

$$b_{3n} \;\; = -b_{3n-3}$$

....

....

$$b_{4n-4} = -b_{2n+1} \qquad \text{………...(17)}$$

Using these relations we obtain from the equations in (14), (15) and similar equations the following linear system:

$$-2b_{2n+1} + b_{2n+3} = 0$$

$$-2b_{2n+3} + b_{2n+7} = 0$$

$$\ldots\ldots$$

$$-2b_{\frac{5}{2}n-3} + b_{3n-5} = 0$$

$$-2b_{\frac{5}{2}n-1} = 0$$

$$-2b_{\frac{5}{2}n+1} + b_{2n+3} = 0$$

$$-2b_{\frac{5}{2}n+3} + b_{2n+7} = 0$$

$$\ldots\ldots$$

$$-2b_{3n-3} + b_{3n-5} = 0$$

………………...(18)

The matrix of coefficients of this system is

$$\begin{pmatrix}
-2 & 1 & 0 & 0 & 0 & & & & & & & 0 & 0 & 0 \\
0 & -2 & 0 & 1 & 0 & & & & & & & 0 & 0 & 0 \\
\ldots\ldots & & & & & & & & & & & & & \\
0 & 0 & 0 & 0 & 0\cdots 0 & -2 & 0 & 0 & 0 & 0\cdots 0 & & & 1 & 0 \\
\hline
0 & 0 & 0 & 0 & 0\cdots 0 & 0 & -2 & 0 & 0 & 0\cdots 0 & & & 0 & 0 \\
\hline
0 & 1 & 0 & 0 & 0\cdots 0 & 0 & 0 & -2 & 0 & 0\cdots 0 & & & 0 & 0 \\
0 & 0 & 0 & 1 & 0\cdots 0 & 0 & 0 & 0 & -2 & 0\cdots 0 & & & 0 & 0 \\
\ldots\ldots & & & & & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & & & & & & & 0 & 1 & -2
\end{pmatrix}$$

which is strictly diagonally dominant [6]. Thus, it is invertible ,and hence, the last linear system has the trivial solution, only. Due to the previous relations between the variables $b_1, b_2, ..., b_{4n-4}$ we conclude that all of them are zero.

Setting all variables $a_1, a_2, ..., a_{n^2}$ to $\frac{\mathbf{S}}{\mathbf{n}}$ represents a solution of the nonhomogenous linear system (1). Since we have n*n variables and 4*n equations, the solution of the nonhomogenous linear system (1) has according to our analysis n*n – 4*n + 4 free parameters .

Algorithm Double-Even PMS

Input

$M_{nX\,n}$= Matrix and represent even Pandiagonal magic squares ,

Output

Free parameters of Pandiagonal magic squares

V= n*n-4*n+4

Step 1:

Delet the equations of linear dependence from system (1)

equation in M_{eq} Do

$M_{eq} \longrightarrow M_q$

End for

Step 2:

Write coefficients remaining of the system (1) as matrix (3) , and prove that this matrix has full rank .

Step 3:

Prove that the columns of matrix linear independent

$M_{nX\,n} \longrightarrow \{ b_1, b_2, b_3, \ldots, b_n \}$, from :

$$b_1 * C_1 + b_2 * C_2 + ... + b_{4n-3} * C_{4n-3} = 0$$

where $C_1, C_2, ..., C_{4n-3}$ denotes the columns of the matrix

and $b_1, b_2, ..., b_{4n-3}$ are real numbers

Steps 4:

Write the component wise of above equations.

Step 5 : do

$\{ b_1, b_2, b_3, \ldots, b_n \} \longrightarrow \{ 0,0,0, \ldots\ldots,0 \}$

End for
End Algorithm

Conclusion

This table provide a convenient way of describing the rule which counting the free parameters in doubly-even pandiagonal magic squares.

PMS Order	n*n-4*n+4	4*n	n*n
4	4	16	16
8	36	32	64
12	100	48	144
16	196	64	256
20	324	80	400
24	494	96	576
28	386	112	784
Etc.	…	…	…

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