ON THE RANGE OF THE MAP N_{AB}

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Abstract

Let H be an infinite dimensional separable complex Hilbert space and B(H) be the Banach algebra of all bounded linear operators on H.
In this paper we introduce a mapping $N_{AB}: B(H) \to B(H)$. By $N_{AB}(T)=AT-T^*B$, $T\in B(H)$.
We study some properties of it , and we study surjectivity of this mapping when A is pseudonormal operator whose spectrum satisfies certain properties if the analytic function f(A) that belongs to the $\left(RangeN_{AA}\right)^*$ then f(A) is the zero function .Also we generalize some results for the Jordan * derivation J_A and the derivation D_A when A is normal operator and prove it when A is pseudonormal operator.

الخلا صة

ليكن B(H)جبر بناخ لكافة المؤثرات الخطية المقيدة المعرفة على فضاء هلبرت القابل للفصل و غير منتهي على حقل الاعداد العقدية . في هذا البحث نعرف الدالة $N_{AB}: B(H) \to B(H)$. بالصيغة
$N_{AB}(T)=AT-T^*B$, $\in B(H)$. .
حيث تم دراسة بعض خواص هذه الدالة ،ودراسة مدى هذه الدالة عندما يكون المؤثر A من صنف المؤثرات السوية الكاذبة(الشبه سوية) وطيفه يحقق موصفات معينة فأذا كانت الدالة التحليلة f(A) تقع في مدى الدالة N_{AA}^* فأنها تكون الدالة الصفرية كذ لك قمنا بتعميم بعض النتائج الخاصة بمدى كل من الدالةالاشتقاقية J_A من النمط جوردان * والدالةالاشتقاقية D_A لصنف المؤثرات السوية الكاذبة(الشبه سوية)
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Introduction

Let H be a separable complex Hilbert space and B(H) be the Banach algebra of all bounded linear operators on H as is customary , let $\sigma(A)$, $\sigma_p(A)$,and $\sigma_{ap}(A)$ denote the spectrum of the operator A, the set of all eigenvalue of the operator A and approximate eigenvalue of the operator A respectively. The operator A is called normal if $A^*A = AA^*$, T is called a dominant operator if for each $\lambda\in ¢$ there exists a number $M_\lambda > 0$ such that
$\|(T^*-\bar{\lambda})x\| \le M_\lambda \|(T-\lambda)x\|$ for all $x \in H$ [1].
Furthermore if the constants M_λ are bounded by a positive number M, then T is called M-hyponormal operator [2] , and if M=1 , T is called hyponormal operator [3] , A is called a *- paranormal operator if $\|A^*x\|^2 \le \|A^2x\|$ for every unit vector x in H [4] .
Finally the operator T is called a pseudonormal operator if $Tx = \lambda x$ for some $x \in H$, $\lambda\in ¢$, then $T^*x = \bar{\lambda}$ i.e., if x is an eigenvector for T with the eigenvalue λ then x is an eigenvector for T* with eigenvalue $\bar{\lambda}$. [5,p59]. In this paper we introduce a mapping $N_{AB}: B(H) \to B(H)$, by $N_{AB}(T)=AT-T^*B$, $T\in B(H)$.

We study some properties of it and its surjectivity. Also we generalize some theorems when A is a normal operator , pseudonormal operator , *- paranormal ,dominant operator , M-hyponormal operator , and hyponormal operator.

We also recall the definition of derivation mapping $D_A(T)$: B(H)→B(H) defined by $D_A(T)$=TA-AT, T ∈ B(H). [6]. Also the maping J_A(T)=TA-AT* is called Jordan *-derivation [7] .We study conditions under which range of J_A , D_A and N_{AB} contain analytic operators when A is a pseudonormal operator.

Propositoin (1)

1- The mapping N_{AB} is bounded in the sense that it maps bounded sets in B(H) in to bounded sets in B(H) and it is clear that it is not a linear mapping.

2- $\left(RangeN_{AB}\right)^* = RangeN_{B^* A^*}$

3-Let S denote the set of all normal operators defined on H . If A*=B then $\left(RangeN_{AB}\right) \subseteq$ S.

4-let S_1 denote the set of all skew- Hermition operators defined on H. If A*=B then $\left(RangeN_{AB}\right) \subseteq S_1$. Moreover if A is an invertible operator then $S_1 = \left(RangeN_{AB}\right)$.

proof (1)

It is clear that

$$\|N_{AB}(X)\| = \|AX - X^*B\| \le \|AX\| + \|X^*B\|$$

$$\le \| X \| \; [\| A \| + \| B \|].$$

Hence if M is arbitrary bounded subset of B(H) we have its image N_{AB}(M) is bounded.

Thus N_{AB} is bounded .

2- $\left(RangeN_{AB}\right)^*$ = { $(AX - X^*B)^*$: X∈ B(H)}

={$X^*A^* - B^*X$: X∈ B(H)}

let $X_1 = -X$, -$X_1^* = X^*$

thus $\left(RangeN_{AB}\right)^*$={ $B^*X_1 - X_1^*A^*$: X_1∈ B(H)}

3- Let T∈ $\left(RangeN_{AB}\right)$ then there exists an operator X∈ B(H) such that T=AX- X*B hence T*= X* A*- B*X . Now TT*=(AX- X*B) (X* A*- B*X) =AXX*A*-AX B*X- X*B X*A*+ X* B B*X = T*T.

4- Let T ∈(Range N_{AB}) then there exists an operator X∈B(H) such that T=AX- X*B , hence T*= X* A*- B*X=-(B*X- X* A*)=-T and ,$\left(RangeN_{AB}\right) \subseteq S_1$.

Let A∈B(H) be an invertible operator and Let T∈ S_1 ,consider X=(1/2) A^{-1}T then T=AX- X*B.

Lemma (2)

Let A∈B(H) if λ∈ $\sigma_{ap}(A)$ then $\lambda^n \in \sigma_{ap}(A^n)$ for each positive integer n.

Proof See [8.p.71].

Theorem (3)

Let A∈ B(H) be a normal operator and $A^n \in \left(RangeN_{AA}\right)$ for an integer n ≥2 then $(\sigma(A))^{n-1} \subseteq$ iR . If n=1 , then A=0.

proof :

Let λ∈ $\sigma(A)$ then by a theorem in [9.p44] , λ∈ $\sigma_{ap}(A)$, hence there exists a unit vector x∈H

Such that ‖ (Ax- λx) ‖ <∈, $\|x\| = 1$ ……(1)

Since λ∈ $\sigma(A)$ then $\lambda^n \in \sigma(A^n)$, by spectral mapping theorem. And by lemma (2)

$\lambda^n \in \sigma_{ap}(A^n)$ thus $\| (A^n x - \lambda^n x) \|$ ≤q∈ $\|x\| = 1$. Hence $\left|\langle A^n x - \lambda^n x, x\rangle\right|$<q∈.

Thus $\left|\langle A^n x, x\rangle - \lambda^n\right|$<q∈ …… .(2)

But $A^n \in \left(RangeN_{AA}\right)$, hence there exists T∈B(H) such that A^n - (AT-T*A)=0 and thus $\left|\langle A^n x, x\rangle - \langle ATx, x\rangle + \langle T^*Ax, x\rangle\right| = 0$ ….(3)

We can assume from inequality (1) that $\|A\,x - \lambda\,x\|\|T\|$<∈ and

$\left\|A^*x - \bar{\lambda}\,x\right\|\|T\|$<∈, Now

$\left|\langle A\,x\,, T\,x\rangle - \lambda\langle x\,, T\,x\rangle\right|$<∈ and

$\left|\langle Tx, A^*x\rangle - \lambda\langle Tx, x\rangle\right|$<∈ ……….(4)

By adding inequality (2),equation (3),and inequality (4) , we get

$\left|\langle \lambda^n - \lambda\langle (T - T^*)x, x\rangle\right|$< ∈(2+q)

Since $(T - T^*)$ is a skew - Hermitian Thus $\langle (T - T^*)x, x\rangle$ is pure imaginary say z , z∈iR , Thus $\left|\lambda^n - \lambda z\right|$ <∈(2+q) , z∈iR.

Since ∈ is arbitrary, then $\lambda\left(\lambda^{n-1} - z\right)$=0

That is either λ=0 or λ^{n-1}=z , z∈iR

Thus $\lambda^{n-1} \in$ iR and if λ=0 then 0∈iR , and $(\sigma(A))^{n-1} \subseteq$ iR.

Now if n=1 ,then by the same way in above we have $(\lambda - \lambda z) = 0, z \in$ iR thus $\lambda(1 - z) = 0$,

It is obvious that (1-z)≠0 , hence $\lambda = 0$ and $\sigma(A) = \{0\}$, then A is quasinilpotent operator since the only normal quasinilpotent operator is zero operator hence A=0.

Lemma (4)

Let f be an analytic function defined on { z∈₵: $|z|$<r , r>o} and A∈B(H) such that $\|A\|$<r . If $\lambda \in \sigma_p(A)$ then f(λ)∈ $\sigma_p(f(A))$.

Moreover if x is an eigenvector corresponding to λ, then x is also an eigenvector for f(A) corresponding to f (λ).

Proof : See [8,p80].

The next theorem is proved in [1,p81] for normal operator A .we prove it for pseudonormal operator .

Theorem(5)

Let A ∈ B(H) be a pseudonormal operator and let f be analytic function defined on a neighborhood B_r of zero such that $\|A\|$ <r , r>0

1-if f(A) ∈ Range(J_A) then for each $\lambda \in \sigma_p(A)$, if λ=0 then f(λ)=0 and in general f(λ)/λ is pure imaginary number.

2-If f(A)∈Range(J_{A^*}) then for each $\lambda \in \sigma_p(A)$, λf(λ) is pure imaginary number .

Proof (1)

Suppose that f(A) ∈ Range(J_A)={TA-AT*; T ∈B(H)}.

Let $\lambda \in \sigma_p(A)$ then there exist x≠0 ∈H , $\|$ x $\|$ =1 ,such that Ax = λx by spectral mapping theorem f(λ) is an eigenvalue for f(A) with the same eigenvector , hence f(A)x= f(λ)x.

Thus <f(A)x-f(λ)x ,x>=0 .

And <f(A)x ,x>-f(λ)=0 ………….......(1)

but f(A) ∈ Range(J_A) hence there is T ∈B(H) such that f(A)-(TA-A T*)=0.Thus

<f(A)x ,x>-<TAx ,x>+<AT*x ,x> = 0…...(2)

Since A is pseudonormal then A*x= $\bar{\lambda}$x ,hence <Ax-λx, T*x>=0 , and

<T*x , A*x- $\bar{\lambda}$x>=0.

Thus <Ax ,T*x>-λ<x ,T*x> = 0 …...(3)

And <T*x ,A*x>-λ<T*x ,x>=0……….(4)

By adding the equations (1), (2) ,(3) and (4) we get f(λ)-λ(<x ,T*x>-<T*x ,x>)=0.

Thus f (λ)-λ(<(T-T*)x ,x>)=0.

This implies f(λ) -λc=0, c ∈ iR . and f(λ)=λc

If λ=0 then f(λ)=0 , other wise f(λ)/λ=c , c ∈iR

In a similar manner one can prove (**2**).

Theorem (6)

let A ∈ B(H) be a pseudonormal operator and let f be analytic function defined on a neighborhood B_r of zero such that $\|A\|$ <r , r>0

If $\sigma_p(A)$ contains a simple closed contour then

1-f(A)∈Range(J_A) if and only if f(x)=cx,c ∈iR

2- f(A) ∈ $Range(J_A)^*$ if and only if f=0

Proof 1 :

Suppose f(A)∈Range(J_A) then by theorem(5) for each $\lambda \in \sigma_p(A)$, f(λ)/λ=c, c ∈iR

Since $|\sigma(A)| \le \|A\|$ and f is analytic on Br and $\|A\|$ <r then f is analytic on $\sigma_p(A)$. Moreover if λ≠0 , f(λ)/λ is analytic on $\sigma_p(A)$.

Hence f(λ)/λ is constant function and for all x, f(x)= xc , c ∈iR.

Conversely let f(x)= cx , hence f(A)=cA we check thet f(A)∈Range(J_A)

Observe that if T=(1/2)cI , then J_A(T)=TA-AT*=(1/2)cIA-A(c\2I)*

=(1/2)cA+(1/2)cA=cA . Hence cA∈Range(J_A)

2- Let f(A) ∈ $Range(J_A)^*$ then by theorem(5) for each $\lambda \in \sigma_p(A)$, λf(λ)=c , c ∈iR

Since λf(λ) is analytic on $\sigma_p(A)$ and $\sigma_p(A)$ contains a simple closed contour , then by lemma in [8,p78] it is constant function , clearly this is possible only if f(λ)=0. Conversely if f=0 then f(A)=0A ,observe that if T=0 then J_{A^*} (T)= $A^*T^* - TA^*$=0, hence f(A) ∈ $Range(J_A)^*$

Remark

we can prove theorem (5) and theorem(6) in a similar manner if we replace J_A by the derivation mapping D_A.

In next theorem we study conditions under which $(RangeN_{AB})$ contains analytic operators when A is a pseudonormal operator .

Theorem (7)

Let A ∈ B(H) be a pseudonormal operator then

1- If f(A) $\in (RangeN_{AB})$ then for each $\lambda \in \sigma(A) \cap \sigma(B)$ such that λ is an eigenvalue of A and B with the same corresponding eigenvector, if λ=0 then f(λ)=0 . Moreover f(λ)/λ is a pure imaginary number.

2- if f(A) $\in (RangeN_{AB})^*$ then for each $\lambda \in \sigma_p(A)$ λf(λ) is a pure imaginary number

Proof (1):

Since λ is an eigenvalue of A and B with the same eigenvector then Ax=λx and Bx=λx .

If f(A) $\in (RangeN_{AB})$ then there exists T∈ B(H) such that f(A)-(AT-T*B)=0 so <f(A)x,x>-<ATx,x>+<T*Bx,x>=0…..(1).

Since $\lambda \in \sigma_p(A)$ then f(A)x=f(λ)x .

This implies that <f(A)x,x>-f(λ)=0......(2)

But A is a pseudonormal operator then A*x= $\bar{\lambda}$x and <Tx ,A*x>-λ<Tx,x> = 0…...(3)

And <Bx Tx>-λ<x,Tx> = 0…...(4)

By adding (1),(2),(3),and (4) we get

f (λ)-λ(<(T-T*)x ,x>)=0 this implies that f (λ)-λc=0 , c∈iR and hence f (λ)=λc

if λ=0 then f(λ)=0. otherwise f(λ)/λ= c, c∈iR.

2- Suppose that f(A) $\in (RangeN_{AB})^* = \{ B^*T - T^*A^* : T \in B(H)\}$

Let $\lambda \in \sigma_p(A)$ then there exist x≠0 ∈H , ‖ x‖ =1 such that Ax = λx . By spectral mapping theorem f(λ) is an eigenvalue for f(A) with the same eigenvector , hence f(A)x= f(λ)x.

Thus <f(A)x-f(λ)x ,x>=0 . Hence

-<f(A)x ,x>+f(λ)=0 ……………......(1)

But f(A) $\in (RangeN_{AB})^*$, hence there exists T ∈B(H), such that f(A)- $B^*T - T^*A^*$ =0.Thus <f(A)x ,x>-< B^* Tx ,x>+<T^*A^* x ,x>= 0…...(2)

Since A is a pseudonormal then A^* x= $\bar{\lambda}$x , hence < A^* x ,Tx>-< $\bar{\lambda}$x ,Tx> = 0 …...(3)

And <Tx ,Bx>- $\bar{\lambda}$ <Tx , x>=0……….(4)

By adding the equations (1), (2) ,(3), and (4) we get f(λ)- $\bar{\lambda}$(<Tx ,x>-<T^* x ,x>)=0.

Thus f (λ)- $\bar{\lambda}$ (<(T - T^*)x ,x>)=0

This implies f(λ) - $\bar{\lambda}$c=0, c ∈iR . and f(λ)= $\bar{\lambda}$c

Now if we multiply the sides of the above equation by λ we arrive at λf(λ)=$|\lambda|^2$ c.

Since $|\lambda|^2$ is real number hence λf(λ) is a pure imaginary number**.**

Theorem(8)

Let A ∈ B(H) be a pseudonormal operator and Let f be analytic function defined on a neighborhood B_r of zero such that $\|A\|$ <r , r>0

If $\sigma_p(A)$ contains a simple closed contour then

1-f(A)∈Range(N_{AA}) if and only if f(x)=cx , c ∈iR .

2- f(A) ∈ $Range(N_{AA})^*$ if and only if f=0

The proof theorem(8) is similar to the proof of theorem(6) and hence is omitted.

References

1. Stampfli J.G., and Wadhwa B.L., **1976** . " *An asymmetric Putnam-Fuglede theorem for dominant operators*" Indiana Univ.math., J 25.359-365.
2. wadha B.L., , **1974**. " *M-hyponormal operators*". Duke math.J.41.655-660
3. Stampfli J.G.,**1965**."*Hyponormal operators and spectral Density*",Trans , Amer. Math Soc, 117.469-476.
4. RTOO C. S., **1998**. "*some class of operators*" Math. J, Toyamess univ., 21.147-152
5. Naji , Sadiq,,**2000,** "*Some generalizations of hyponnormal operators*" M.Sc, Thesis. College of science ,University of Baghdad
6. Stampfli J.G., **1972**." *Derivations on B(H)*" Illinois.J.Math.17.518-524.
7. P. Semrl, **1990**."*On Jordon*- Derivations and an Application*",Colloq. Math,59.241-151.
8. Al-kabbani ,Amina Rasim ,**1999**." *on the range of Jordan *-derivation*" M.Sc, Thesis. College of science ,University of Baghdad
9. Halmos, P.R.,**1982**. "*A* Hilbert *space problem Book*" springer- verlag New York.