Iraqi Journal of Science

Iraqi Journal of Science

SOME RESULTS ON (σ,τ)-DERIVATION IN PRIME RINGS

Abdul-Ruhman Hameed and Kassim Abdul-Hameed
Department of Mathematics, College of Science, University of Baghdad. Baghdad -Iraq.

Abstract

Let R be a prime ring and d: $R \to R$ be a (σ,τ)-derivation of R. U be a left ideal of R which is semiprime as a ring .In this paper we proved that if d is a nonzero endomorphism on R ,and $d(R) \subset Z(R)$,then R is commutative ,and we show by an example the condition d is an endomorphism on R can not be excluded. Also , we proved the following.

(i) If $Ua \subset Z(R)$ (or $aU \subset Z(R)$) , for $a \in R$), then $a=0$ or R is commutative.

(ii) If d is a nonzero on R such that $d(U)a \subset Z(R)$ (or $ad(U) \subset Z(R)$ for $a \in Z(R)$, then either

$a=0$ or $\sigma(U)+\tau(U) \subset Z(R)$.

(iii) If d is a nonzero homomorphism on U such that $d(U)a \subset Z(R)$(or $ad(U) \subset Z(R)$) for

$a \in R$, then $a=0$ or $\sigma(U)+\tau(U) \subset Z(R)$.

بعض النتائج على مشتقة- (τ,σ) في الحلقات الاولية

الخلاصة

لتكن *R* حلقة أولية ولتكن *d*:R → R مشتقة -(τ,σ) من R .ولتكن U مثالي يساري من R تمثل حلقة شبه اولية . في هذا البحث أثبتنا انه اذا كان d دالة متشاكلة غير صفرية على الحلقــة R , وأن d(R) ⊂ Z(R) , فان R تكون أبدالية ,كذلك أثبتنا بواسطة مثال ان الشرط كون d دالة متشاكلة على الحلقة *R* لا يمكن الاستغناء عنه. أيضا أثبتنا النتائج التالية:

1.إذا كان *Ua* ⊂ Z(R) او *aU* ⊂ Z(R) لكل a ∍ R , فان أما a =0 , أو R أبدالية .

2.إذا كان d دالة غير صفرية على الحلقة R بحيث ان *d*(*U*)*a* ⊂ Z(R) (او *ad*(*U*) ⊂ Z(R)) لكل a ∍ Z (R) فانه أما a =0 أو τ(*U*)+σ(*U*) ⊂ Z(R).

3.إذا كانت d دالة متشاكلة غير صفرية على U بحيث أن *d*(*U*)*a* ⊂ Z(R) (أو *ad*(*U*) ⊂ Z(R)) لكل a ∍ Z(R), فان أما a =0 أو τ(*U*)+σ(*U*) ⊂ Z(R).

Introduction

Let $d : R \to R$ be an additive mapping. If $d(xy)=d(x)\sigma(y)+ \tau(x)d(y)$ for all $x , y \in R$, then d is called a (σ,τ)-derivation of R,where $\sigma,\tau:R \to R$ be two mappings on R [1].

On the other hand we said that d is an endomorphism or anti-endomorphism respectively if $d(xy)=d(x)d(y)$ or $d(xy)=d(y)d(x)$ for all $x , y \in R$

Recall that a ring R is a prime if $aRb=0$, $a,b \in R$,implies that either $a=0$ or $b=0$[2] .

Also,we recall that a ring R is a semi- -prime if $aRa=0, a \in R$ implies that $a=0$[2].

Neset Aydin and Oznur Golbasi proved that if R is a prime ring and d is a (σ,τ)-derivation of R , where $\sigma,\tau:R \to R$ be two automorphisms on R,see[3]. Then

(i) If d is an endomorphism on R, then $d=0$.

(ii)If d is an anti-endomorphism on R, then $d=0$.

(ii) If U is a nonzero left ideal of R which is a semiprime as a ring.If $Ua=0(aU=0)$ for $a\in R$, then $a=0$.

(iii) If U is a nonzero left ideal of R which is a semiprime as a ring such that $d(U)=0$, then $d=0$.

So, we generalized some of above results.

In this paper we considered R is a prime ring, U a left ideal of R and d is a (σ,τ)-derivation of R,where $\sigma,\tau:R\to R$ be two automorphisms on R.

Also, we used the identities in this paper as follows: For all $x,y,z\in R$.

(i) $[xy,z] = x[y,z] + [x,z]y$

$[x,yz] = [x,y]z + z[x,z]$

(ii) $[xy,z]_{\sigma,\tau}= x[y,\sigma(z)]+ [x,z]_{\sigma,\tau}y$

$= x[y,z]_{\sigma,\tau} +[x,\tau(z)]y$

Results

Theorem (2.1)[2]

Let R be a prime ring. If d is a (σ,τ)- derivation of R which is an endomorphism on R, then $d=0$.

Remark (2.2)

We can not exclude the condition d is an endomorphism on R. So, the following example shows.

Example(2.3)

Let $R=\left\{\begin{pmatrix} x & y \\ z & t\end{pmatrix}, x,y,z,t\in Z\right.$,where Z is the number of integers$\Big\}$ be 2×2 matrices with respect to the usual operation of addition and multiplication, then R is a prime ring, see[4].Let $\sigma,\tau:R\to R$ be automorphisms

$$\sigma\begin{pmatrix} x & y \\ z & t\end{pmatrix}=\begin{pmatrix} x & -y \\ -z & t\end{pmatrix},\ \tau\begin{pmatrix} x & y \\ z & t\end{pmatrix}=\begin{pmatrix} x & -y \\ -z & t\end{pmatrix}.$$

Let $d:R\to R$,defined by $d\begin{pmatrix} x & y \\ z & t\end{pmatrix}=\begin{pmatrix} 0 & -y \\ z & 0\end{pmatrix}$, is a (σ,τ)-derivation of R but is not an endomorphism on R, so $d\neq 0$.

Remark(2.4)

From Theorem (2.1), if d is an anti-endomorphism on R, then the previews example stay true.

To prove the first main theorem, we need the following Lemma.

Lemma(2.5)

Let R be a prime ring. If d is a nonzero (σ,τ) - derivation of R which is an endomorphism on R, then $d(R)\subset Z(R)$.

proof

For all $x,y,r\in R$ we have

$[d(xy),r] = [d(x)\sigma(y)+ \tau(x)d(y),r]$

on the other hand

$[d(xy),r] = [d(x)d(y),r]$. So, we have

$[d(x)d(y),r] = [d(x)\sigma(y),r]+[\tau(x)d(y),r]$

$[d(x)d(y),r] =d(x)[d(y),r]+[d(x),r]d(y)$

$=d(x)[\sigma(y),r]+[d(x),r]\sigma(y)+\tau(x)[d(y),r]$

$+[\tau(x),r]d(y)$ for all $x,y,r\in R$ (1)

On the other hand

$d(xy)=d(x)\sigma(y)+\tau(x)d(y)=d(x)d(y)$,for all $x,y\in R$. Substituting xr for x in (1), we get

$d(xr)\sigma(y)+ \tau(xr)d(y)=d(xr)d(y),\ r\in R$

Since d and τ are homomorphism of R, we have

$d(x)d(r)\sigma(y)+ \tau(x)\tau(r)d(y)= d(x)d(r)d(y)$

Expanding the last equation, we have

$d(x)d(r)\sigma(y)+ \tau(x)\tau(r)d(y)=d(x)d(ry)$

$= d(x)d(r)\sigma(y)+d(x)\tau(r)d(y)$

or equivalently,

$0= d(x)\tau(r)d(y)- \tau(x)\tau(r)d(y)$

$=(d(x)- \tau(x))\tau(r)d(y)$.

Since τ is an automorphism of R, we get

$(d(x)- \tau(x))R\,d(y)=0$, for all $x,y\in R$.

Since R is a prime ring, we conclude that

$d(x)=\tau(x)$, for all $x\in R$ or $d=0$. ...(2)

Since d is a nonzero, so we have $d(x)=\tau(x)$. Then, from (1),we get

$d(x)[\sigma(y),r]+[d(x),r]\sigma(y)=0$.

So $[d(x)\sigma(y),r]=0$ for all $x,y,r\in R$. Take zx instead of $x, z\in R$,then

$0=[d(zx)\sigma(y),r]=0$ for all $x,y,r,z\in R$

$0=[d(z)d(x)\sigma(y),r]$

$= d(z)[d(x)\sigma(y),r]+ [d(z),r]d(x)\sigma(y)$

$= [d(z),r]d(x)\sigma(y)$, for all $x,y,r,z\in R$.

Since $d(x)\sigma(y)\in Z(R)$, then

$0=[d(z),r]R\,d(x)\sigma(y)$, for all $x,y,r,z\in R$.

Hence, $0=[d(R),R]R\,d(R)\sigma(R)$.

Since R is a prime ring and $d(R)\neq 0$, then we have $d(R)\subset Z(R)$.

Theorem(2.6)

Let R be a prime ring and let d be a nonzero (σ,τ)-derivation of R such that is an endomorphism on R,then R is commutative.

Proof

By Lemma(2.5), we have $d(R)\subset Z(R)$.

Therefore for all $x,y,r\in R$, we have

$0=[d(xr),y]=[d(x)\sigma(r)+\tau(x)d(r),y]$

$=[d(x)\sigma(r),y]+[\tau(x)d(r),y]$
$=d(x)[\sigma(r),y]+[d(x),y]\sigma(r)+\tau(x)[d(r),y]$
$+[\tau(x),y]d(r)$.

Since $d(R)$ in the centre of R, then we have
$0=d(x)[\sigma(r),y]+[\tau(x),y]d(r)$,
for all $x,y,r\in R$.

By (2) from a proof of Lemma (2.5), we have $\tau=d$ on R, then
$d(x)[\sigma(r),y]+[d(x),y]d(r)=0$,
for all $x,y,r\in R$. Also, we have $d(x)[\sigma(r),y]=0$. Therefore,
$d(x)R[\sigma(r),y]=0$, for all $x,y,r\in R$. So, we have $d(R)R[R,R]=0$. By a primeness of R, and $d(R)\neq 0$, then R is commutative.

Remark(2.7)

By an Example(2.3), we can see from the above Theorem that the condition d is an endomorphism on R, can not excluded.

Also, we generalized the following Lemma.

Lemma(2.8)[2]

Let R be a prime ring and U be a nonzero left ideal of R which is a semiprime as a ring. If $Ua=0$(or $aU=0$) for $a\in R$, then $a=0$.

Theorem (2.9)

Let R be a prime ring and let U be a nonzero left ideal of R which is a semiprime as a ring. If $Ua\subset Z(R)$ (or $aU\subset Z(R)$) for $a\in R$, then $a=0$ or R is commutative.

Proof

If $Ua\subset Z(R)$, then for $u\in U$, $x,r\in R$
$0=[xua,r]=x[ua,r]+[x,r]ua$
$=[x,r]ua$. Take xy, $y\in R$, instead of x, then we have $0=[x,r]yua$ and hence,
$0=[x,r]Rua$, for all $u\in U$, $x,r\in R$. Since R is a prime ring, then we have either
R is commutative or $Ua=0$.
If $Ua=0$, then by Lemma (2.8) we have $a=0$.
If $aU\subset Z(R)$, then for $u,v\in U$, $r\in R$
$0=[auv,r]=au[v,r]+[au,r]v$
$=au[v,r]$. Replace r by ry, where $y\in R$. So, we have
$0=au[v,ry]=aur[v,y]+au[v,r]y$.
$=aur[v,y]$. for all $u,v\in U$, $r,y\in R$.
Hence, $aUR[U,R]=0$
Since R is a prime ring, then either $aU=0$ or $U\subset Z(R)$. If $aU=0$, then by Lemma (2.8) we have $a=0$. If $U\subset Z(R)$, then R is commutative.

Now, we generalize the following Theorem.

Theorem(2.10) [2]

Let R be a prime ring, U a nonzero left ideal of R which is semiprime as a ring. If d is a nonzero (σ,τ)-derivation of R such that $d(U)a=0$ (or $ad(U)=0$), then $a=0$.

So, we need the following Lemma.

Lemma (2.11)

Let R be a prime ring and U be a nonzero left ideal of R which is a semiprime as a ring. If d is a (σ,τ)-derivation of R such that $d(U)\subset Z(R)$, then either $d(R)=0$ or $\sigma(U)+\tau(U)\subset Z(R)$.

Proof

Assume $d(U)\subset Z(R)$. Then for all $u\in U$, $x\in R$ we have
$0=[d(xu),r]=[d(x)\sigma(u)+\tau(x)d(u),r]$
$=[d(x)\sigma(u),r]+[\tau(x)d(u),r]$
$=d(x)[\sigma(u),r]+[d(x),r]\sigma(u)+$
$\tau(x)[d(u),r]+[\tau(x),r]d(u)$
So, we have
$0=d(x)[\sigma(u),r]+[d(x),r]\sigma(u)+[\tau(x),r]d(u)$
Therefore, $d(x)\sigma(u)r-d(x)r\sigma(u)+$
$d(x)r\sigma(u)-rd(x)\sigma(u)+\tau(x)rd(u)-r\tau(x)d(u)$
$=d(x)\sigma(u)r-rd(x)\sigma(u)+\tau(x)rd(u)-r\tau(x)d(u)$
$=d(x)[\sigma(u),r]+[\tau(x),r]d(u)$, for all $u\in U$, $x,y\in R$. In special case assume that $x=u$, then
$0=d(u)[\sigma(u),r]+[\tau(u),r]d(u)$
Since $d(U)\subset Z(R)$ So,
$d(u)[\sigma(u)+\tau(u),r]=0$, for all $u\in U$, $r\in R$.
Hence, $d(U)R[\sigma(U)+\tau(U),r]=0$. Since R is a prime ring, then either
$d(U)=0$ or $\sigma(U)+\tau(U)\subset Z(R)$.
If $d(U)=0$, then by Lemma 2[2], we have $d(R)=0$.

Theorem(2.12)

Let R be a prime ring and U be a nonzero left ideal of R which is a semiprime as a ring. If d is a nonzero (σ,τ)-derivation of R such that $ad(U)\subset Z(R)$ ($d(U)a\subset Z(R)$) for $a\in Z(R)$, then either $a=0$ or $\sigma(U)+\tau(U)\subset Z(R)$.

Proof

For all $r\in R$ and we have $ad(U)\subset Z(R)$, then
$0=[ad(U),r]=a[d(U),r]+[a,r]d(U)$.
$0=[ad(U),r]=a[d(U),r]$. Since $a\in Z(R)$, we have $0=aR[d(U),r]$, for all $r\in R$.
Since R is a prime ring, then either $a=0$ or $d(U)\subset Z(R)$. If $d(U)\subset Z(R)$, then by Lemma (2.11) and $d(R)\neq 0$, then $\sigma(U)+\tau(U)\subset Z(R)$.
If we have $d(U)a\subset Z(R)$, then for all $r\in R$, we have

$0=[d(U)a,r]=d(U)[a,r]+[d(U),r]a$
$= [d(U),r]a$

Since R is a prime ring then either
$d(U) \subset Z(R)$ or $a=0$.

If $d(U) \subset Z(R)$, then by Lemma (2.11) and $d(R)\neq 0$, we have $\sigma(U)+\tau(U)\subset Z(R)$.

Also , we generalized Theorem (2.10) as following .

Theorem (2.13)

Let R be a prime ring ,U be a nonzero left ideal of R which is a semiprime as a ring. Let d be a nonzero (σ,τ)-derivation and is a homomorphism on U such that if $d(U)a\subset Z(R)$(or $ad(U)\subset Z(R)$), then either $a=0$ or $\sigma(U)+\tau(U)\subset Z(R)$.

proof

Assume that $ad(U)\subset Z(R)$.

Since d is a homomorphism on U , then for all $u,v\in U$, we have

$ad(uv)=ad(u)d(v)\in Z(R)$.So , for all $r\in R$

$0=[ad(u)d(v),r]=ad(u)[d(v),r]+[ad(u),r]d(v)$
$= ad(u)[d(v),r]$ for all $u, v\in U$, $r\in R$

Hence , $ad(U)[d(U),R]=0$.

Since $ad(U)\subset Z(R)$, then we have $ad(U)R[d(U),R]=0$. By a primeness of R , we have either $ad(U)=0$ or $d(U) \subset Z(R)$. If $ad(U)=0$, then by Theorem (2.10) we have $a=0$.If $d(U)\subset Z(R)$, then by Lemma (2.11) $\sigma(U)+\tau(U)\subset Z(R)$.

Assume that $d(U)a\subset Z(R)$.

Since d is a homomorphism on U , then for all $u,v\in U$, we have

$d(uv)a=d(u)d(v)a\in Z(R)$.So , for all $r\in R$

$0=[d(u)d(v)a,r]$
$=d(u)[d(v)a,r]+[d(u),r]d(v)a$
$=[d(u),r]d(v)a$ for all $u, v\in U$, $r\in R$

Hence , $[d(U),R]\, d(U)\, a=0$.

Since $d(U)a\subset Z(R)$, then we have $[d(U),R]R\, d(U)a=0$. By a primeness of R , we have either $d(U)a=0$ or $d(U) \subset Z(R)$.

If $d(U)a=0$, then by Theorem (2.10) we have $a=0$ .If $d(U) \subset Z(R)$, then by Lemma (2.11) $\sigma(U)+\tau(U)\subset Z(R)$.

References

1. Yasuyuki Hirano and Hisao Tominga, **1984,** *Some Commutativity Theorems for prime rings with derivations and deferentially semiprime rings* .Math. J.Okayama Univ. 26. 101-108.
2. Herstein, I.N. **1969,** *Topics in ring theory*.Univ.of Chicago press,Chicago,.
3. Golbasi,O. and Aydin,N. , **2002,** *Some results on endomorphisms of prime ring which are (σ,τ)-derivation*,East Asian Math.J.18.33-41.
4. Aydin,N.KAya,K.Golbasi, **2001.O.,** *Some results for generalized Lie ideals with derivation II*,Applied Mathematics E-Notes 1.24-30.