ON THE EXISTENCE OF THE SOLUTION TO THE HAMILTON–JACOBI EQUATION BY USING THE DUAL DYNAMICAL PROGRAMMING

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Abstract

Properties of the value function and dual value function for an optimal control problems of Lagrange and Bolza are described. A main theorem is proved, this theorem deals with the existence of a maximum solution to the Hamilton – Jacobi equation for the Lagrange problem, with satisfies the Lipschitz condition by using the dual dynamic programming method. Finally gives an example which illustrates the value of the main theorem.

Keywords: Lagrange problem, optimal control, Hamilton – Jacobi equation, dynamic programming, dual value function, Bolza problem.

الخلاصة

تم إعطاء وصف لخواص دالة القيمة ودالة القيمة المواجهة لمشاكل السيطرة المثلى للاكرانج وبــولزا. استخدمت طريقة البرمجة الديناميكية المواجهة لإثبات النظرية الأساسية التي تتعامل مع وجود حــل أعظــم لمعادلة هاملتن -- جاكوبي لمشكلة لاكرانج للسيطرة المثلى والذي يحقق شرط ليبشتز. وأخيرا" تــم إعطــاء مثال يوضح قيمة النظرية الأساسية

Introduction

The problem we consider in this paper consists of minimizing the optimal control problem of Lagrange.

$$J(x,u) = \int_a^b L(t, x(t), u(t))\,dt, \qquad (1)$$

where the absolutely continuous trajectory $x : [a, b] \to R^n$ and the Lebesgue measurable control function $u : [a, b] \to R^m$ are subject to the non-linear controlled state-space system

$$\dot{x}(t) = f(t, x(t), u(t)), \text{ a.e. in } [a, b], \qquad (2)$$

$$u(t) \in U(t),\ t \in [a, b], \qquad (3)$$

$$x(a) = c \qquad (4)$$

Here $f : [a, b]\times R^n \times R^m \to R^n$, $L : [a, b]\times R^n \times R^m \to R$, are given functions, c is point in R^n, $U(t)$ is the set of controls with the initial condition $x(a) = c$ which is defined as :

$U(t) = \{u(t)$ measurable; such that $t \in [a, b]$ and $u(t) \in K$, where K is a compact subset of R^m }

Throughout this paper we shall assume the following hypothesis:

$(t, x, u) \to f(t, x, u)$ and $(t, x, u) \to L(t, x, u)$ are continuous and bounded functions in $[a, b]\times R^n \times K$; they are Lipschitz functions with respect to t, x, u.

(Z)

For the above problem if we replacing (1) by:

$$J(x,u) = \int_a^b L(t, x(t), u(t))\,dt + \ell(x(b)), \qquad (5)$$

where $\ell : R^n \to R \cup \{+\infty\}$, then we get that the optimization problem which is called a Bolza problem. And it is not difficult to check that these optimization problems (1) and (5) are equivalent so that each can be formulated as one of the other (see , [1] , [2]).

It is well-known (see, [3], [1], [2], [4]) that in classical dynamic programming (briefly, CDP) the whole family of problems with fixed initial points is considered. For one problem the initial point is fixed, but when a family of problems with different initial points is considered, the solution to these problems are dependent on their initial points. This dependence is called the value function. The CDP method describes the properties of this function, e.g. presents the necessary and sufficient conditions for the optimality of solutions.

According to [1] and [2], in the CDP the sufficient condition for optimality of the solution to the Lagrange (or Bolza) problem is expressed as the solution to the Hamilton-Jacobi (briefly, H-J) equation (see, Theorem 2) of this paper.

For the Bolza problem (5), the author in [6] suggested the nonclassical approach for the dynamic programming. He defined the dual value function for the Bolza problem, and used it in [7] to study the properties of the classical value function for problem (5) directly. This method does not require that the classical value function is differentiable. The method in [7] focuses on the construction of a new function, which guarantees the sufficient conditions of optimality of the Bolza problem, and it is completely in the spirit of the dynamic programming technique.

In [8] the problem considered is that of approximation numerical minimization of the non-linear control problem of Bolza (or Lagrange), starting from the CDP method of Bellman, an ε-value function is defined as an approximation for the value function being a solution to the H-J equation. Local optimally conditions and Lipschitzian solution to the H-J equation discussed in [9].

From all the above, it can be seen that the solution to the H-J equation for the problem is essential in the study of optimality. Therefore, it is found to be a reasonable justification to accomplish the study of this paper.

The aim of this paper is to study the existence solution to the H-J equation for the Lagrange problem (1) – (4), by using the nonclassical approach to dynamic programming (the dual dynamic programming) in [7]. Thus, in section 4, it shall be proven that the dual value function $(t, p) \to S_D(t, p)$, $(t, p) \in P \subset R^{n+2}$ for the problem (1) – (4) is a maximum solution to the H-J equation and that is satisfies the Lipschitz condition.

Properties of the Value Function and Dual Value Function

In this section the properties of the classical value function and dual value function for the Lagrange problem (1) – (4), and Bolza problem (5) are described (see, [1], [2], [7] and section 1 of this paper).

Definition 1. For the problem (1) – (4), a pair $x(.)$, $u(.)$ is admissible if it satisfies (2), (3) and $t \to L(t, x(t), u(t))$ is summable; then the corresponding trajectory $t \to x(t)$ will be called admissible.

Let $T \subset [a, b] \times R^n$ be a set with a non-empty interior, covered by graphs of admissible trajectories, i.e., for every $(t_0, x_0) \in T$ there exists an admissible pair $x(\cdot)$, $u(\cdot)$, defined in $[t_0, b]$, such that $x(t_0) = x_0$ and $(s, x(s)) \in T$ for $s \in [t_0, b]$.

Definition 2. Function $(t, x) \to S(t, x)$ defined in T is called the classical value function for the problem
(1) – (4) if,

$$S(t,x) = inf\left\{\int_t^b L(s,x(s),u(s))ds\right\},$$

where the infimum is taken over admissible pairs $x(s)$, $u(s)$, $s \in [t, b]$ whose trajectories start at $(t, x) \in T$ and their graphs are contained in T.

If only the value function $(t, x) \to S(t, x)$ is differentiable in the open set $Q_0 \subset T$, then it satisfies the partial differential equation of dynamic programming known as the H-J equation,

$$S_t(t, x) + H(t, x, S_x(t, x)) = 0,\ (t, x) \in Q_0$$

with the boundary condition, $S(b, x) = 0$, $(b, x) \in Q_0$, where the Hamiltonian is given by

$$H(t, x, y) = y f(t, x, u(t,x)) + L(t, x, u(t,x)),$$

and $t \to u(t,x)$ is an optimal control.

One can be notice that for the considered problem
(1) – (4) the above H-J equation can be re-written in the following way:

$$\frac{\partial}{\partial t}S(t,x) + \min_{u \in K}\left\{\frac{\partial}{\partial x}S(t,x)f(t,x,u) + L(t,x,u)\right\} = 0,$$

$(t, x) \in Q_0$.

One of the most important properties of the classical value function is stated in Theorem 1.

Theorem1. If the functions $(t, x, u) \to f(t, x, u)$ and $(t, x, u) \to L(t, x, u)$ satisfy assumptions **(Z)** for the problem (1) – (4), then the value function $(t, x) \to$

$S(t, x)$ satisfies a Lipchitz condition and is the solution to the H-J equation :

$$\frac{\partial}{\partial t}S(t,x)+\min_{u\in K}\left\{\frac{\partial}{\partial x}S(t,x)f(t,x,u)+L(t,x,u)\right\}=0,$$

for a.e. $(t, x) \in T$, (6)

with the boundary condition $S(b, x) = 0$, $(b, x) \in T$.

Proof. see([2, Ch. IV , Th. 4.2]).■

According to [1] and [2], in the CDP the sufficient condition for optimality of the solution to the considered problem is expressed as the solution to the H-J equation so that following Theorem 2 holds.

Theorem 2. Let $(t, x) \to G(t, x)$ be a solution of the class $C^1(T)$ to the H-J equation

$G_t(t, x) + H(t, x, G_x(t, x)) = 0$, $(t, x) \in Q_0$

with the boundary condition, $G(b, x) = 0$, $(b, x) \in Q_0$, where $Q_0 \subset T$ is an open set, the Hamiltonian is given by the formula,

$H(t, x, y) = y f(t, x, u(t,x)) + L(t, x, u(t,x))$,

and $t \to u(t,x)$ is an optimal feedback control.

If $x = x(t)$ and a pair $x(\cdot)$, $u(\cdot)$, defined in $[a, b]$, $x(a)= c$, is admissible and such that

$$\frac{\partial}{\partial t}G(t,x(t))+\frac{\partial}{\partial x}G(t,x(t))f(t,x(t),u(t))+L(t,x(t),u(t))=0,$$

then the pair $x(\cdot)$, $u(\cdot)$, is optimal, and also $G(t, x) = S(t, x)$, $(t, x) \in Q_0$, where $S(.,.)$ is the value function.

Proof. See ([2 , Ch. IV, Th. 4.4]).■

It can be seen that some regularity of the function $(t,x)\to G(t,x)$, being the solution to the H-J equation, is required, i.e. it must be at least a Lipschitz function (see. Th.1).

Note1. The above Theorems 1,2 are holds also for the problem of Bolza (5), if only we addition to the assumptions **(Z)**, function $x \to \ell(x)$ is a Lipschitz function with respect to x, and replacing the boundary conditions by $S(b,x)=\ell(x)$, and $G(b,x)=\ell(x)$.

Now for the definition of the dual value function for the Lagrange problem (1) – (4), let us suppose that $T \subset R^{n+1}$ be as defined above, denotes a set covered by the graph of all admissible trajectories for the problem (1) - (4), and $P \subset R^{n+2}$ be a set of variables $(t, y^0, y) = (t, p)$, $t \in [a,b]$, $y^0 \le 0$, with non-empty interior and take a function $x(t, p)$ defined in P such that $(t, x(t,p)) \in T$, $(t,p) \in P$, and assume that it is satisfies the following :

$x(t,p)$, $(t,p) \in P$, is a Borel measurable, locally bounded, Lipschitz function and such that for each admissible trajectory $x(t)$ lying in T there exist an absolutely continuous function $p(t)=(y^0, y(t))$ lying in P such that $x(t)=x(t,p(t))$, also if all trajectories $x(t)$ start at same (t_0, x_0) then all the corresponding $P(t)$ have the same first coordinate y^0.

Definition 3. function $(t,p)\to S_D(t,p)$ defined in $P \subset R^{n+2}$ is called the dual value function for the problem (1) – (4) if,

$$S_D(t,p)=\inf\left\{-y^0\int_t^b L(s,x(s),u(s))ds\right\},\quad y^0\le 0, \quad (7)$$

where the infimum is taken over admissible $x(s)$, $u(s)$, $s \in [t,b]$ whose trajectories start $(t, x(t,p))$ and their graphs contained in T.

Note 2. Let $(t,p)\to S_D(t,p)$ be as in (7) with T and $x(t,p)$ defined above, then we see that

$S_D(t, p) = -y^0 S(t, x(t, p))$, $(t, p) \in P$. Thus it is natural to expect that the dual value function has properties analogous to the classical value function (see, [2], [7]).

Therefore since $(t, x, u) \to f(t, x, u)$ and $(t, x, u) \to L(t, x, u)$ and are Lipschitz functions in $[a,b]\times R^n \times K$, and since $x(t,p)$, $(t,p)\in P$, $t \in [0,b]$, is Lipschitz function then we deduce that

$f(t, x(t,p), u)$ and $L(t, x(t,p), u)$ are Lipschitz function in $T \times K$.

Now since $(t, x) \to S(t, x)$ is a Lipschitz function (see, Th. 1), we see that the dual value function $S_D(t, p) = -y^0 S(t, x(t, p))$, $(t, p) \in P$, $t \in [0,b]$ is a Lipschitz function for $(t,p)\in P$, and it is a solution to the H-J equation

$-y^0 S_t(t, x(t, p)) + \min\{-y^0 S_x(t, x(t, p)) f(t, x(t, p), u) - y^0 L(t, x(t, p), u) : u \in K\} = 0$ a.e., $(t, x(t, p)) \in T$,

$t \in (0, b)$ (8)

with the boundary condition

$-y^0 S(b, x(b, p(b))) = 0$, for all $(b, x(b, p(b))) \in T$ (9)

Definitions and Auxiliary Results

This section presents some definitions and lemmas which will be used in the proof of the main theorem (Theorem 4) in this paper.

Definition 4 [10]. Suppose that f and g are two (complex-valued) functions on R^n, then we define their convolution to be the function $f*g$ given by:

$$f*g(x) = \int_{R^n} f(x-y)g(y)dy. \qquad (10)$$

It is easy to see (by a change of variable) that $f*g = g*f$.

One has to be careful to make sure that (10) makes sense. One way is to require $f \in L^p(R^n)$ and $g \in L^q(R^n)$, in which case the integral in (10) is well defined for all x by Hölder's inequality, let $1/p + 1/q = 1$ with

$1 \le p < \infty$, $q > 1$, and $f \in L^p(R^n)$ and $g \in L^q(R^n)$, then

$(fg)(x) = f(x)\,g(x)$, is in $L^1(R^n)$ and

$$\left|\int_{R^n} fg\,d\mu\right| \le \int_{R^n} |f||g|\,d\mu \le \|f\|_p \|g\|_q,$$

where $\|f\|_p$ is L_p-norm of f.

Theorem 3. Let j be in $L^1(R^n)$ with $\int_{R^n} j = 1$. For $\varepsilon > 0$, we define $j_\varepsilon(x) = \varepsilon^{-n} j(x/\varepsilon)$, so that $\int_{R^n} j_\varepsilon = 1$ and $\|j_\varepsilon\|_1 = \|j\|_1$. Let $f \in L^p(R^n)$ for some $1 \le p < \infty$ and define the convolution $f_\varepsilon = j_\varepsilon * f$, then

$f_\varepsilon \in L^p(R^n)$ and $\|f_\varepsilon\|_p \le \|j\|_1 \|f\|_p$

$f_\varepsilon \to f$ uniformly in $L^p(R^n)$ as $\varepsilon \to 0$,

If $j \in C_c^\infty(R^n)$, then $f_\varepsilon \in C^\infty(R^n)$ and $D^\alpha f_\varepsilon = (D^\alpha j_\varepsilon) * f$

where D^α (α is a nonnegative integer) denote the multi-derivative.

Proof . See [10, p.58] ■

Definition 5. Let us define the set W as follows:

$W = \{H(t,p) = -y^0 w(t, x(t,p)) \mid$ is a Lipschitz for t, p; $(t,p) \in P$, $t \in [0, b]$, $(t, x(t,p)) \in T$; with the boundary condition

$H(b, p(b)) = -y^0 w(b, x(b, p(b))) \le 0$, for all $x(t,p) \in T$, $(t,p) \in P$; and

$-y^0 \frac{\partial}{\partial t} w(t, x(t,p)) + \min\{-y^0 \frac{\partial}{\partial x} w(t, x(t,p)) f(t, x(t,p), u) - y^0 L(t, x(t,p), u) : u \in K\} \ge 0$ a.e., $(t, x(t,p)) \in T$, $(t,p) \in P$. (11) $\}$.

And we define on the set W the following partial ordering:

$H \le \hat{H} \Leftrightarrow H(t,p) \le \hat{H}(t,p)$; $(t,p) \in P$, $t \in [0, b]$, $\forall\ H, \hat{H} \in W$.

Note 3. From the definition of the function $S_D(t, p)$, $(t,p) \in P$ in (7), and (8), (9), we observe that the dual value function $S_D(t,p) = -y^0 S(t, x(t,p))$, $(t,p) \in P$ belongs to the set W of all Lipschitz solutions to the H-J equation (11), when there exists $x(t) = x(t, p(t))$, $(t,p) \in P$, lying in T, as a multiplied solution for the Lagrange problem (1) – (4).

Now let us formulate and prove three Lemmas which will simplify and shorten the proof of the Main Theorem (Theorem 4) in this paper that the dual value function $S_D(t,p)$, $(t,p) \in P$ defined in (7) is a maximum element of the above set W.

To formulate these Lemmas, let us assume that $t_0 < b$ and consider $\delta > 0$ such that the interval $[t_0 + \delta, b - \delta]$ has a nonempty interior. Now let $x_0(t_0) = x_0(t_0, p_0(t_0))$ be arbitrary and let it belong to T, $u(.) \in U(t)$.

Since $(t, x, u) \to f(t, x, u)$ and $(t, x, u) \to L(t, x, u)$ satisfy assumptions **(Z)** (see, section 1), and since $x(t,p)$, $(t,p) \in P$ is bounded and Lipschitz, then the functions, $f(t, x(t,p), u)$ and $L(t, x(t,p), u)$ are bounded and Lipschitz with respect to t, $x(t,p)$, u in $T \times K$, when $(t,p) \in P$. Therefore the response of the system $t \to x(t) = x(t, p(t))$, $t \in [t_0, b]$ with $x_0(t_0) = x_0(t_0, p_0(t_0))$, lying in T is bounded, i.e.,

$x(t, p(t)) \in Q$, for all $(t, p(t)) \in \hat{Q}$, $t \in [t_0, b]$,

where Q and $\hat{Q}$ are compact subsets of T and P respectively.

Now we define a set $\overline{Q}$ as follows: $\overline{Q} = \hat{Q} + B_1(R^{n+2})$,

where $B_1(R^{n+2})$ is the sphere centered at the origin having a radius of 1.

For a shorter and simpler definition, we propose the following notations:

$$\begin{aligned} \tilde{f}(t,p,u) &= f(t, x(t,p), u) \\ \tilde{L}(t,p,u) &= L(t, x(t,p), u), \end{aligned} \qquad (12)$$

since $f : [a, b] \times R^n \times K \to R^n$ and $L : [a, b] \times R^n \times K \to R$ are Lipschitz, satisfies assumptions **(Z)**, and $x(t,p)$ is a Lipschitz function for $(t,p) \in P$, then we deduce that $\tilde{f}(t,p,u)$ and $\tilde{L}(t,p,u)$ are also Lipschitz functions in $P \times K$.

And since $H(t,p) = -y^0 w(t, x(t,p))$, $(t,p) \in P$ then the H-J equation (11) becomes

$H_t(t,p) + \min\{H_x(t,p)\tilde{f}(t,p,u) - y^0 \tilde{L}(t,p,u)$

$: u \in K\} \geq 0$ a.e., $(t, p) \in P$, $t \in [0, b]$ (13)

with the boundary condition
$H(b, p(b)) \leq 0$ for all $(b, p) \in P$.

Note 4. We need in the proof of the Main Theorem of this paper to construct a new function $(t, p) \to H_2^{\varepsilon}(t,p)$, $(t, p) \in \hat{Q}$, $t \in [t_0 + \delta, b - \delta]$, which is sufficiently regular and satisfies the inequality (13). So an arbitrary function $(t, p) \to H(t, p)$ of the set W can be chosen and modified in a few steps of construction until the resulting function $(t, p) \to H_2^{\varepsilon}(t,p)$ satisfies the inequality (13).

Thus for this fact suppose that the function $(t, p) \to H(t, p)$ be any function in the set W. We may construct a new function $(t, p) \to H_1(t, p)$ by shifting the function $(t, p) \to H(t, p)$, as follows:

$$H_1(t,p) = H(t,p) + \alpha(t - b) \qquad (14)$$

where α is a positive number which is close to zero.

Since the function $H(t,p) \in W$, $(t,p) \in P$, $t \in [0, b]$, and since $H_{1t}(t,p) = H_t(t,p) + \alpha$, $H_{1x}(t,p) = H_x(t,p)$, $(t,p) \in P$, $t \in [0, b]$, then we see that the function $(t, p) \to H_1(t,p)$ is a Lipschitz function and satisfies the following:

$H_{1t}(t,p) - \alpha + \min\{H_{1x}(t,p)\tilde{f}(t,p,u) - y^0\tilde{L}(t,p,u) : u \in K\} \geq 0$ a.e., $(t, p) \in P$, $t \in (0, b)$.

Thus,

$H_{1t}(t,p) + \min\{H_{1x}(t,p)\tilde{f}(t,p,u) - y^0\tilde{L}(t,p,u) : u \in K\} \geq \alpha$ a.e., $(t,p) \in P$, $t \in (0, b)$. (15)

and this implies that the function $(t, p) \to H_1(t, p)$ belongs to the set W (see, definition 5).

In order to define a new function $(t, p) \to H_2^{\varepsilon}(t,p)$, $(t, p) \in \hat{Q}$, $t \in [t_0 + \delta, b - \delta]$, for arbitrary and fixed $\varepsilon < \min(1, \delta)$, such that $H_2^{\varepsilon}(t,p) \in C_0^{\infty}(\hat{Q}, t \in [t_0 + \delta, b - \delta])$ and it satisfy the inequality (15), we have to define a new function $(t, p) \to H_2^{\varepsilon}(t,p)$ by using the convolution of the function $(t,p) \to H_1(t,p)$ with a function of class $C_0^{\infty}(R^{n+2})$ having a compact support.

So we will define a function $(t,p) \to H_2^{\varepsilon}(t,p)$, $(t,p) \in \hat{Q}$,
$t \in [t_0 + \delta, b - \delta]$, for arbitrary and fixed $\varepsilon < \min(1, \delta)$ by using the convolution of the function $(t, p) \to H_1(t, p)$ with a function $(t, p) \to \rho_\varepsilon(t, p)$ of class $C_0^{\infty}(R^{n+2})$ having a compact support as follows:

$$H_2^{\varepsilon}(t,p) = (H_1 * \rho_\varepsilon)(t,p) \qquad (16)$$

where the function $(t, p) \to H_1(t, p)$ as defined in (14); $\rho_1 : R \times R^{n+1} \to R$ is a function of class $C_0^{\infty}(R^{n+2})$ having a compact support; and satisfies $\int_{R^{n+2}} \rho_1(t,p)\,dt\,dp = 1$, $\rho_\varepsilon(t,p) = (\frac{1}{\varepsilon^{n+2}})\rho_1(\frac{t}{\varepsilon}, \frac{p}{\varepsilon}) \in C_0^{\infty}(R^{n+2})$; supp $\rho_1 \subset B_1(R^{n+2})$, where $B_1(R^{n+2})$ is a sphere centered at the origin having a radius of 1.

Clearly, this function $(t, p) \to H_2^{\varepsilon}(t,p)$ will also be a Lipschitz function, because the function $(H_1 * \rho_\varepsilon)(\,.\,,\,.\,)$ is Lipschitz for t, p. (14)

In order to show that the function $(t, p) \to H_2^{\varepsilon}(t,p)$, $(t,p) \in \hat{Q}$, $t \in [t_0 + \delta, b - \delta]$ satisfies the inequality (15), i.e.,

$$\exists \varepsilon' > 0 \mid \forall \varepsilon \leq \varepsilon', \frac{\partial}{\partial t}H_2^{\varepsilon}(t,p) + \min\{\frac{\partial}{\partial x}H_2^{\varepsilon}(t,p)\tilde{f}(t,p,u) - y^0\tilde{L}(t,p,u) : u \in K\} \geq \frac{\alpha}{2} > 0, \qquad (17)$$

we need to prove some lemmas, so that the proof of the above fact (17) becomes shorter and simpler (see, Theorem 4).

According to the proof of Theorem 4 (the Main Theorem) the fact that the functions $-y^0\tilde{L}(.,.,.)$ and $-y^0(\tilde{L} * \rho_\varepsilon)(.,.,.)$ have values arbitrarily close to each other is needed. Therefore lemma 1 should be proved first. This gives an estimate of the difference between the values of these two functions by arbitrary real number close to zero.

Lemma 1. *Let $\tilde{L}(.,.,.)$ be a function as defined in (12), and $\rho_\varepsilon(\cdot,\cdot)$ be the function of class $C_0^{\infty}(R^{n+2})$ defined above. Then for arbitrary real number α, described during the definition of function $H_1(\cdot,\cdot)$ there exists $\varepsilon' > 0$ such that for all $\varepsilon \leq \varepsilon'$ and for all $(t, p, u) \in \hat{Q} \times K$, $t \in [t_0 + \delta, b - \delta]$ the following inequality holds:*

$$|-y^0\tilde{L}(t,p,u) - (-y^0(\tilde{L} * \rho_\varepsilon)(t,p,u))| < \frac{\alpha}{4}.$$

Proof. For $(t, p, u) \in \hat{Q} \times K$, the following estimation holds:

$|-y^0\tilde{L}(t,p,u) - (-y^0(\tilde{L} * \rho_\varepsilon)(t,p,u))|$

$=|-y^0||\tilde{L}(t,p,u)-(\tilde{L}*\rho_\varepsilon)(t,p,u)|$

$=|-y^0|\int_{B\varepsilon(R^{n+2})}[\tilde{L}(t,p,u)-\tilde{L}(t-s,p-p',u)]$

$\rho_\varepsilon(s,p')\,ds\,dp'|\leq|-y^0|\int_{B\varepsilon(R^{n+2})}|[\tilde{L}(t,p,u)-$

$\tilde{L}(t-s,p-p',u)]\rho_\varepsilon(s,p')\,|\,ds\,dp'\leq$

$|-y^0|\sup_{\substack{u\in K\\(t,p)\in\hat{Q},t\in[t_0+\delta,b-\delta]\\(s,p')\in B_\varepsilon(R^{n+2})}}|\tilde{L}(t,p,u)-$

$\tilde{L}(t-s,p-p',u)|$,

because the function $\tilde{L}(.,.,.)$ is uniformly continuous in the compact set $\overline{Q}\times K$, $t\in[0,b]$

$|-y^0|\sup_{\substack{u\in K\\(t,p)\in\hat{Q},t\in[t_0+\delta,b-\delta]\\(s,p')\in B_\varepsilon(R^{n+2})}}|\tilde{L}(t,p,u)-$

$\tilde{L}(t-s,p-p',u)|\to 0$ as $\varepsilon\to 0$,

and consequently,

$|-y^0\tilde{L}(t,p,u)-(-y^0(\tilde{L}*\rho_\varepsilon)(t,p,u))|\to 0$ as $\varepsilon\to 0$.

Hence, for an arbitrary real number α, there exists $\varepsilon'>0$ such that for all $\varepsilon\leq\varepsilon'$ and for all $(t, p, u)\in\hat{Q}\times K$, $t\in[t_0+\delta, b-\delta]$ the following holds:

$$|-y^0\tilde{L}(t,p,u)-(-y^0(\tilde{L}*\rho_\varepsilon)(t,p,u))|<\frac{\alpha}{4}.$$

The fact that the functions

$\frac{\partial}{\partial x}H_2^\varepsilon(.,.)\tilde{f}(.,.,.)$ and $\left[\left(\frac{\partial}{\partial x}H_1\tilde{f}(.,.,.)*\rho_\varepsilon\right)\right](.,.)$

have values arbitrarily close will be needed in the proof of Theorem 4, so lemma 2 must be proved. This gives an estimate of the difference between the values of these two functions by a real number arbitrary close to zero.

Lemma2. Let $H_1(.,.)$, $H_2^B(.,.)$ and $\rho_\varepsilon(.,.)$ be functions defined in $\hat{Q}$ (see (16) and let $\tilde{f}(.,.,.)$ be a function as defined in (12). Then for an arbitrary real number α described in the definition of $H_1(.,.)$ there exists $\varepsilon'>0$ such that for all $\varepsilon\leq\varepsilon'$ and for all $(t, p, u)\in\hat{Q}\times K$, $t\in[t_0+\delta, b-\delta]$ the following inequality holds:

$$\left|\frac{\partial}{\partial x}H_2^\varepsilon(t,p)\tilde{f}(t,p,u)-\left[\left(\frac{\partial}{\partial x}H_1\tilde{f}(\cdot,\cdot,u)\right)*\rho_\varepsilon\right](t,p)\right|<\frac{\alpha}{4}.$$

Proof. Since the function $H_1(.,.)$ is a Lipschitz function, then it is satisfies the Lipschtiz condition, i.e., $\left|\frac{\partial}{\partial x}H_1(.,.)\right|\leq M$ for some constant $M>0$. Thus

for all $(t, p, u)\in\hat{Q}\times K$, $t\in[t_0+\delta, b-\delta]$, and by using the definitions of $H_2^B(.,.)$ and the convolution, the following holds:

$\left|\frac{\partial}{\partial x}H_2^\varepsilon(t,p)\tilde{f}(t,p,u)-\left[\left(\frac{\partial}{\partial x}H_1\tilde{f}(\cdot,\cdot,u)\right)*\rho_\varepsilon\right](t,p)\right|=$

$\left|\frac{\partial}{\partial x}(H_1*\rho_\varepsilon)(t,p)\tilde{f}(t,p,u)-\left[\left(\frac{\partial}{\partial x}H_1\tilde{f}(\cdot,\cdot,u)\right)*\rho_\varepsilon\right](t,p)\right|$

$=\left|\int_{B_\varepsilon(R^{n+2})}\frac{\partial}{\partial x}H_1(t-s,p-p')\tilde{f}(t,p,u)\right.$

$\left.\rho_\varepsilon(s,p')\,ds\,dp'-\int_{B\varepsilon(R^{n+2})}\frac{\partial}{\partial x}H_1(t-s,p-p')\right.$

$\left.\tilde{f}(t-s,p-p',u)\rho_\varepsilon(s,p')\,ds\,dp'\right|\leq$

$\int_{B\varepsilon(R^{n+2})}\left|\frac{\partial}{\partial x}H_1(t-s,p-p')\right|\left|\tilde{f}(t,p,u)-\right.$

$\left.\tilde{f}(t-s,p-p',u)\right|\rho_\varepsilon(s,p')\,ds\,dp'\leq$

$M\sup_{\substack{u\in K\\(t,p)\in\hat{Q},t\in[t_0+\delta,b-\delta]\\(s,p')\in B_\varepsilon(R^{n+2})}}\left|\tilde{f}(t,p,u)-\tilde{f}(t-s,p-p',u)\right|.$

Since the function $\tilde{f}(.,.,.)$ is uniformly continuous on the compact set $\overline{Q}\times K$, $t\in[0,b]$, then we obtain the last inequality tends to zero as $\varepsilon\to 0$, that is

$$\sup_{\substack{u\in K\\(t,p)\in\hat{Q},t\in[t_0+\delta,b-\delta]\\(s,p')\in B_\varepsilon(R^{n+2})}}\left|\tilde{f}(t,p,u)-\tilde{f}(t-s,p-p',u)\right|\to 0$$

as $\varepsilon\to 0$, and consequently,

$$\left|\frac{\partial}{\partial x}H_2^\varepsilon(t,p)\tilde{f}(t,p,u)-\left[\left(\frac{\partial}{\partial x}H_1\tilde{f}(\cdot,\cdot,u)\right)*\rho_\varepsilon\right](t,p)\right|\to 0$$

as $\varepsilon\to 0$.

Thus, for arbitrary α there exists $\varepsilon'>0$ such that for all $\varepsilon\leq\varepsilon'$ and for all $(t, p, u)\in\hat{Q}\times K$, $t\in[t_0+\delta, b-\delta]$, the following holds:

$$\left|\frac{\partial}{\partial x}H_2^\varepsilon(t,p)\tilde{f}(t,p,u)-\left[\left(\frac{\partial}{\partial x}H_1\tilde{f}(\cdot,\cdot,u)\right)*\rho_\varepsilon\right](t,p)\right|<\frac{\alpha}{4}.$$

In the proof of Theorem 4, the uniform convergence of the sequence $\{H_2^\varepsilon(t,p)\}$ to $H_1(t,p)$ as ε converges to zero, for all $(t,p)\in\hat{Q}$, $t\in[t_0+\delta, b-\delta]$ is also required as is shown in the following result.

Lemma 3. Let $H_1(.,.)$, $H_2^B(.,.)$ and $\rho_\varepsilon(\cdot,\cdot)$ be functions defined in $\hat{Q}$ (see (16)). Then for all $(t, p) \in \hat{Q}$, $t \in [t_0 + \delta, b - \delta]$, we have

$\lim_{\varepsilon \to 0} H_2^\varepsilon(t,p) = H_1(t,p)$,

and this convergence is uniform.

Proof. By definition of uniformly convergent sequence of functions to prove that this lemma holds, it is sufficient to show that for arbitrary $\gamma > 0$ a $\varepsilon' > 0$ exists such that for every $\varepsilon \le \varepsilon'$ and for all $(t, p) \in \hat{Q}$, $t \in [t_0 + \delta, b - \delta]$ the following holds:

$| H_2^\varepsilon(t,p) - H_1(t,p) | < \gamma$

Now by using the definitions of the function $H_2^B(.,.)$ and the convolution, for all $(t, p) \in \hat{Q}$, $t \in [t_0 + \delta, b - \delta]$, the following holds:

$$| H_2^\varepsilon(t,p) - H_1(t,p) | = | (H_1 * \rho_\varepsilon)(t,p) - H_1(t,p) | =$$

$$\left| \int_{B_\varepsilon(R^{n+2})} [H_1(t-s, p-p') - H_1(t,p)] \rho_\varepsilon(s,p')\, ds\, dp' \right|$$

$$\le \int_{B_\varepsilon(R^{n+2})} \left| [H_1(t-s,p-p') - H_1(t,p)] \rho_\varepsilon(s,p') \right| ds\, dp'$$

$$\le \sup_{\substack{u \in K \\ (t,p) \in \hat{Q}, t \in [t_0+\delta, b-\delta] \\ (s,p') \in B_\varepsilon(R^{n+2})}} | H_1(t-s, p-p') - H_1(t,p) |.$$

Since the function $H_1(.,.)$ is uniformly continuous in the compact set $\bar{Q}$, then we have

$$\sup_{\substack{u \in K \\ (t,p) \in \hat{Q}, t \in [t_0+\delta, b-\delta] \\ (s,p') \in B_\varepsilon(R^{n+2})}} | H_1(t-s, p-p') - H_1(t,p) | \to 0 \text{ as } \varepsilon \to 0.$$

Consequently,

$| H_2^\varepsilon(t,p) - H_1(t,p) | \to 0$ as $\varepsilon \to 0$.

Therefore, for an arbitrary $\gamma > 0$ a $\varepsilon' > 0$ exists such that for all $\varepsilon \le \varepsilon'$ and for all $(t, p) \in \hat{Q}$, $t \in [t_0 + \delta, b - \delta]$ the following holds: $| H_2^\varepsilon(t,p) - H_1(t,p) | < \gamma$.

The Main Theorem

The main result of this work is formulated in Theorem 4, which ensures that the dual value function $S_D(t, p)$, $(t, p) \in P$ is a maximum element of the set W.

Theorem 4. The dual value function $S_D(t, p)$, $(t, p) \in P$, $t \in [0, b]$ for the Lagrange problem (1)-(4), (see, (10) and (6)) is the maximum element of the set W (see, definition 5), that is

$S_D(t, p) \ge H(t, p)$, for all $H(t, p) \in W$, $(t, p) \in P$.

Proof. Suppose that the function $(t, p) \to H(t, p)$, $(t, p) \in P$ is any function in the set W then by using the definition 5 of W, and equation (9), we get

$H(b, p(b)) \le 0 = S_D(b, p(b))$, for all $(t, p) \in P$, $t \in [0, b]$.

Now as stated in section 3, notes 3 and 4, let $t_0 < b$ and consider $\delta > 0$ such that the interval $[t_0 + \delta, b - \delta]$ has a non-empty interior, and let $x_0(t_0) = x_0(t_0, p_0(t_0))$ be an arbitrary belonging to T, and $u(.) \in U(t)$. and let the functions $(t, p, u) \to \tilde{f}(t,p,u)$, $(t, p, u) \to \tilde{L}(t,p,u)$, and $(t, p) \to H_2^\varepsilon(t,p)$ be as defined in (12) and (16) respectively.

we need here to show that the function $(t,p) \to H_2^\varepsilon(t,p)$ satisfies the inequality (17), i.e.,

$$\exists \varepsilon' > 0 \,|\, \forall \varepsilon \le \varepsilon', \frac{\partial}{\partial t} H_2^\varepsilon(t,p) + \min\{ \frac{\partial}{\partial x} H_2^\varepsilon(t,p) \tilde{f}(t,p,u) - y^0 \tilde{L}(t,p,u) : u \in K \} \ge \frac{\alpha}{2} > 0,$$

and this fact implies that the function $(t, p) \to H_2^\varepsilon(t,p)$ also belongs to the W, (see, (11) and (16)).

Now to prove that the above inequality (17) is hold, we have

$$\frac{\partial}{\partial t} H_2^\varepsilon(t,p) + \frac{\partial}{\partial x} H_2^\varepsilon(t,p) \tilde{f}(t,p,u) - y^0 \tilde{L}(t,p,u)$$
$$= -y^0 \tilde{L}(t,p,u) - (-y^0(\tilde{L} * \rho_\varepsilon)(t,p,u) + [(\frac{\partial}{\partial t} H_1 + \frac{\partial}{\partial x} H_1 \tilde{f}(.,.,u) - y^0 \tilde{L}(\cdot,\cdot,u)) * \rho_\varepsilon](t,p) + \frac{\partial}{\partial x} H_2^\varepsilon(t,p) \tilde{f}(t,p,u) - [(\frac{\partial}{\partial x} H_1 \tilde{f}(.,.,u)) * \rho\varepsilon](t,p) \tag{18}$$

In order to find the values of the left side of (18), it is sufficient to find the values of each term of the right side in (18).

From Lemma 1 we know that for an arbitrary positive real number α which is close to zero, there exists $\varepsilon' > 0$ such that for all $(t, p, u) \in \hat{Q} \times K$, $t \in [t_0 + \delta, b - \delta]$, we get:

$| -y^0 \tilde{L}(t,p,u) - (-y^0(\tilde{L} * \rho_\varepsilon)(t,p,u)) | < \frac{\alpha}{4}$.

Moreover, lemma 2 gives: for an arbitrary positive real number α which is close to zero, there exists $\varepsilon' > 0$, such that for all $\varepsilon \leq \varepsilon'$ and for all $(t, p, u) \in \hat{Q} \times K$, $t \in [t_0+\delta, b-\delta]$, we have

$$\left| \frac{\partial}{\partial x} H_2^{\varepsilon}(t,p)\, \tilde{f}(t,p,u) - \left[\left(\frac{\partial}{\partial x} H_1\, \tilde{f}(.,.,u)\right) * \rho_\varepsilon\right](t,p) \right| < \frac{\alpha}{4}.$$

Therefore, by using the values of all terms in the inequality (15), lemmas 1 and 2, we see that, it is possible to estimate the values of the left side in (18) for all $(t, p) \in \hat{Q}$, $t \in [t_0 + \delta, b - \delta]$ as follows:

$$\frac{\partial}{\partial t} H_2^{\varepsilon}(t,p) + \frac{\partial}{\partial x} H_2^{\varepsilon}(t,p)\, \tilde{f}(t,p,u) - y^0 \tilde{L}(t,p,u)$$
$$\geq (\alpha * \rho_\varepsilon)(t,p) - \frac{\alpha}{4} - \frac{\alpha}{4} = \frac{\alpha}{2} > 0,$$

thus we have,

$$\frac{\partial}{\partial t} H_2^{\varepsilon}(t,p) + \frac{\partial}{\partial x} H_2^{\varepsilon}(t,p)\, \tilde{f}(t,p,u) - y^0 \tilde{L}(t,p,u) > \frac{\alpha}{2} > 0. \quad (19)$$

Since the right hand side of the above inequality is independent of $u(\cdot)$, then we see that the inequality (17) is satisfied.

By Theorem 3, we get $H_2^{\varepsilon}(t,p) \in C_0^{\infty}(\hat{Q}, t \in [t_0 + \delta, b - \delta])$ and since $H_2^{\varepsilon}(t,p) = -y^0 w_2^{\varepsilon}(t, x(t,p))$ (see, definition of W), then we obtain

$$\frac{d}{dt} H_2^{\varepsilon}(t,p) = \frac{\partial}{\partial t} H_2^{\varepsilon}(t,p) + \frac{\partial}{\partial x} H_2^{\varepsilon}(t,p)\, \tilde{f}(t,p,u) \quad (20)$$

Hence, by substitution of the above equation (20) in the inequality (19), we get,

$-y^0 \tilde{L}(t,p,u) \geq -\frac{d}{dt} H_2^{\varepsilon}(t,p)$, for all $\varepsilon < \varepsilon'$,

and by taking the integration of both sides of the above inequality, we get that for all $\varepsilon < \varepsilon'$

$$-\int_{t_0+\delta}^{b-\delta} y^0 \tilde{L}(t,p,u)\,dt \geq -\int_{t_0+\delta}^{b-\delta} \frac{d}{dt} H_2^{\varepsilon}(t,p)\,dt$$
$$= -H_2^{\varepsilon}(t,p)\Big|_{t_0+\delta}^{b-\delta} = H_2^{\varepsilon}(t_0+\delta, p(t_0+\delta)) - H_2^{\varepsilon}(b-\delta, p(b-\delta)).$$

By the properties of convolution (see, Theorem 3; approximation by C^{∞}-functions) and by lemma 3. we see that, $H_2^{\varepsilon}(t,p)$ converge to $H_1(t,p)$ uniformly in $\hat{Q}$, $t \in [t_0 + \delta, b - \delta]$. Therefore,

$$-\int_{t_0+\delta}^{b-\delta} y^0 \tilde{L}(t,p,u)\,dt \geq H_1(t_0 + \delta, p(t_0 + \delta)) - H_1(b - \delta, p(b - \delta)) = H(t_0+\delta, p(t_0+\delta)) - H(b-\delta, p(b-\delta)) + \alpha(t_0+2\delta-b)$$

Now, putting $\alpha \to 0$, we obtain

$$-\int_{t_0+\delta}^{b-\delta} y^0 \tilde{L}(t,p,u)\,dt \geq H(t_0 + \delta, p(t_0 + \delta)) - H(b - \delta, p(b - \delta))$$

Hence, by taking the limit $\delta \to 0$, and (12) we have

$$-y^0 \int_{t_0}^{b} L(t, x(t,p), u)\,dt \geq H(t_0, p_0(t_0)) - H(b, p(b)) \geq H(t_0, p_0)$$

(because in the definition of the set W, the boundary condition $H(b, p(b)) \leq 0$).

Since the right hand side of the above inequality is independent of $u(\cdot)$, we observe that

$$\inf_{u(\cdot) \in K} \{-y^0 \int_{t_0}^{b} L(t, x(t,p), u)\,dt\} \geq H(t_0, p_0).$$

Thus, $-y^0 S(t_0, x_0(t_0, p_0)) \geq H(t_0, p_0)$,

and since $S_D(t, p) = -y^0 S(t, x(t, p))$, $(t, p) \in P$, we get that $S_D(t_0, p_0) \geq H(t_0, p_0)$.

Now, since t_0 and $x_0(t_0, p_0) = x_0$ with a suitable function $p_0(t_0) = p_0$ are arbitrary, we see that (19)

$S_D(t, p) \geq H(t, p)$, for all $(t, p) \in P$, $t \in [0, b]$.

Therefore, $S_D(t, p)$ is the maximum element of set W.

Conclusion

According to the properties of the classical value function $(t, x) \to S(t, x)$, and the dual value function $(t, p) \to S_D(t, p) = -y^0 S(t, x(t, p))$ (see, section 2), we observe that, the Main Theorem 4 of this paper identifies that (in the case where there is not a unique solution for problem (1) – (4)) the dual value function which satisfies the Lipschitz condition is an approximate of optimal, when it is a solution to the H-J equation, i.e., if the dual value function $S_D(.,.)$ is evaluated along any admissible trajectory such that it is Lipschitz and satisfying the solution to the H-J equation (8) and (9), then that trajectory is optimal.

Example

To illustrate the importance of the Main Theorem 4, we present the following example.

Consider the optimal control problem:

$$\text{minimize} \int_{-1}^{\pi} (a(t)x^2(t) + b(t)u^2(t))\,dt$$

subject to

$\dot{x}(t) = f(t, x(t), u(t)) = B(t)u(t)$ a.e., in $[-1, \pi]$,

$u(t) \in U(t) = [-1, 1]$, $t \in [-1, \pi]$,

$x(-1) = x(\pi) = 0$, where

$$a(t) = \begin{cases} -1/2, & 0 \leq t \leq \pi, \\ 0, & -1 < t \leq 0. \end{cases}$$

$$b(t) = \begin{cases} 1/2, & 0 \le t \le \pi, \\ 1, & -1 \le t < 0. \end{cases}$$

$$B(t) = \begin{cases} 1, & 0 < t \le \pi, \\ -1, & t \in I_{k_1} \cup I_{k_3}, \\ 0, & t \in I_{k_2} \cup \{-1\}, \end{cases}$$

$$I_{k_j} = (-1 + (\tfrac{1}{2})^{3k+j}, -1 + (\tfrac{1}{2})^{3k+j-1}], j = 1, 2, 3,$$

$$k = 0, 1, \ldots, \bigcup_{k=0}^{\infty} \bigcup_{j=1}^{3} I_{k_j} = (-1, 0].$$

To study the existence of a solution for the H-J Eqs. for the above problem and obtained of the optimal pair for the problem by using Theorem 4, we help ourselves by resolving the maximum principle (the necessary optimality conditions) for the above problem, that is, $x(t)$, $u(t)$, $y(t)$, and $y^0 < 0$ satisfy the following conditions:

$dy(t)/dt = -2y^0 a(t)x(t)$ a.e., $t \in [-1, \pi]$

$\max \{y^0 b(t)u^2 + y(t)B(t)u + y^0 a(t)x^2(t) \mid u \in U(t)\}$ $= y^0 b(t)u^2(t) + y(t)B(t)u(t) + y^0 a(t)x^2(t)$ a.e., $t \in [-1, \pi]$, $y(\pi) \in R^1$, $-y^0 \in [0, \infty)$, $|y(\pi)| + y^0 \neq 0$.

Then we calculate from it the following triplets $x(t)$, $u(t)$, and $p(t) = (y^0, y(t))$ as follows:

$y^0 = -e$, $x(t, c_1) = c_1 \sin t$, $y(t, ec_1) = ec_1 \cos t$, $u(t, c_1) = c_1 \cos t$, where $t \in [0, \pi]$, $c_1 \in (-1, 1)$, $e \in (1/2, 3/2)$;

$y^0 = -e$, $x(t, e) = 0$, $y(t, e) = 0$, $u(t, e) = 0$, $t \in [-1, \pi]$;

$y^0 = -e$, $x(t, c_2) = -\frac{c_2}{2} \int_t^0 B^2(s)ds$, $y(t, ec_2) = ec_2$,

$u(t, c_2) = \frac{c_2}{2} B(t)$, $t \in [-1, 0]$, $c_2 \in (-1, 1)$.

For $p(t) = (y^0, y(t))$, $t \in [-1, \pi]$, we define $u(t, y^0, y)$ and $x(t, y^0, y)$ as follows:

$$u(t,y^0,y) = \begin{cases} -\dfrac{yB(t)}{2y^0}, & t \in [-1,0], y^0 \in (-3/2,-1/2), y \in (-3/2,3/2), \\ 0, & t \in [-1,\pi], y^0 \in (-3/2,-1/2), y = 0, \\ -\dfrac{y}{y^0}, & t \in [0,\pi], y^0 \in (-3/2,-1/2), \\ & |y| < \dfrac{3}{2}|\cos t|, y = 0 \ \text{for}\ t = \pi/2 \end{cases}$$

$$x(t,y^0,y) = \begin{cases} \dfrac{y}{2y^0} \displaystyle\int_t^0 B^2(s)ds, & t \in [-1,0], y^0 \in (-3/2,-1/2), y \in (-3/2,3/2), \\ 0, & t \in [-1,\pi], y^0 \in (-3/2,-1/2), y = 0, \\ -\dfrac{y}{y^0} tgt, & t \in [0,\pi], y^0 \in (-3/2,-1/2), \\ & |y| < \dfrac{3}{2}|\cos t|, y = 0 \ \text{for}\ t = \pi/2 \end{cases}$$

Next, define $V(t, y^0, y)$ in the same sets of t and (y^0, y), respectively (see, [9] and section 7), as:

$$V(t,y^0,y) = \begin{cases} -\dfrac{y^2}{4y^0} \displaystyle\int_t^0 B^2(s)ds, \\ 0, \\ \dfrac{y^2}{2y^0} tgt. \end{cases}$$

Now, define $w(t, x(t, p))$ as follows:

$$w(t,x(t,p)) = \begin{cases} \dfrac{y^2}{4y^{02}} \displaystyle\int_t^0 B^2(s)ds, & t \in [-1,0], y^0 \in (-3/2,-1/2), y \in (-3/2,3/2), \\ 0, & t \in [-1,\pi], y^0 \in (-3/2,-1/2), y = 0, \\ -\dfrac{y^2}{2y^{02}} tgt, & t \in [0,\pi], y^0 \in (-3/2,-1/2), \\ & |y| < \dfrac{3}{2}|\cos t|, y = 0 \ \text{for}\ t = \pi/2 \end{cases}$$

It is simple to verify that the functions $H(t, y^0, y) = -y^0\ w\ (t, x(t, p))$, as described above, is a Lipschitz functions in the sets of t and (y^0, y) and they satisfies (11) and the boundary condition in the set W (see, definition 5). Thus the functions $H(t, y^0, y)$ in the sets of t and (y^0, y) described above, are belongs to set W.

And we see that for $p(t) = (y^0, y(t))$, when $t \in [-1, \pi]$, y^0 is any given number in the interval $(-3/2, -1/2)$, and $y = 0$, it is not difficult to check that the dual value function for the above problem $S_D(t, p) = -y^0 S\ (t, x(t, p)) = -y^0\ V_{y0}\ (t, p)$ (see, section 7) which is equal to zero, satisfy (8) and (9), and thus it belongs to the set W.

Therefore, from all above and Theorem 4, we observe that the dual value function $S_D(t, p)$, $t \in [-1, \pi]$ which is equal to zero is the maximum element of the set W, and we fined that $x(t) = 0$ and $u(t) = 0$, $t \in [-1, \pi]$ is an optimal pair.

Future Study

For the Bolza problem (5) (see, section 1) of optimal control in [9, Th.3.1], established that the sufficient conditions for the control to be optimal is that, there exists a function $V(t, p)$, $(t, p) \in R^{n+2}$, $t \in [0, b]$, such that it's a Lipschitz solution to the dual partial differential equation of dynamic programming

and it satisfies the boundary condition $y^0 V_{y0}(b, p) = y^0 \ell(-V_y(b,p))$, $(b,p) \in P \subset R^{n+2}$, and the relation $V(t, p) = V_{y0}(t, p)y^0 + V_y(t, p)y = -S_D(t, p) - x(t, p)y = V_p(t, p)p$, $(t, p) \in P$, where $S_D(.,.)$ is the dule value function of Bolza problem and $x(., .)$ as defined in section 2.

Thus, the future study, is the existence of the solution for the optimal control problem of Bolza, and show that the

function $V(t, p)$, $(t, p) \in P$ is the minimum of the set W' which is defined as:

$W' = \{H'(t, p) \mid$ is Lipschitz for $(t, p) \in P$, $t \in [0, b]$, $H'(t, b) = -S_D(t, p) - x(t, p)y = y^0 H'_{y0}(t, p) + H'_y(t, p)y =$

$$H'_p(t, p)p,$$

with the boundary condition

$y^0 H'_{y0}(b, p) \geq y^0 \ell(-H'_y(b, p))$, $\forall (b, p) \in P$, and $H'_t(t, p) + \max_{u \in K} \{yf(t, -H'_y(t,p), u) + y^0 L(t, -H'_y(t,p), u)\} \leq 0$, a.e., $(t, p) \in P$, $t \in [0, b]$, where K compact subset of R^m $\}$.

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