EXTENDED GENERALIZED GAMMA DISTRIBUTION

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Abstract
New 6-parameters extended form of the Kobayashi's generalized gamma function is defined, several of its properties and recurrence formulas are derived. Based on this new function we introduce a new 6-parameters with a model function extended generalized gamma distribution. Many new distributions are shown to be particular cases of this suggested distribution, including some familiar known distributions. In terms of the model function statistical properties; reliability and hazard functions, and estimation of some parameters of the distribution are studied. Also, the form of the distribution is considered under various forms of the model function.
We believe that this new distribution model is reasonable for accommodate multivarious applications, since it have wide variety of shapes. Especially, for the life distribution of a component where the presence of the displacement and intensity parameters is vitally important.
Keywords: Generalized gamma function, generalized gamma distribution, Displacement and intensity parameters, Moments and maximum likelihood methods, Reliability and hazard functions.

الخلاصة
يهدف هذا البحث إلى اقتراح أنموذج توزع جديد بدلالة ست معلمات ودالة أنموذج يمكن بحد ذاتها صياغتها بدلالة معلمات أخرى , ويطلق على هذا الأنموذج أنموذج توزع كاما المعمم بالتمدد . يعبر هذا الأنموذج عن الكثير من نماذج التوزعات المستمرة في حالاته الخاصة مما يسمح بتمثيل العديد من الظواهر المختلفة , خصوصاً في حقل المعولية حيث تشكل معلمتي الإزاحة والشدة أهمية كبرى . وتم دراسة عدد من الخصائص والمميزات النظرية لهذا التوزع , وتم كذلك دراسة دوال المعولية والإخفاق وسبل تخمين معالم هذا الأنموذج .

Introduction

Several statistical distribution models have proved today to be of considerable interest in different applied science work. One of the well-known such models is the gamma distribution model. This distribution appears help to explain wider varity of phenomenons, especially in the field of reliability, queeing theory, etc.
Gamma distribution model is an old most extensively model, dating back to the time of Laplace transformations [1] in 1836; it was defined as the random variable X whose probability density function pdf with 1-parameter is of the form

$$f_X(x;p) = \frac{1}{\Gamma(p)} x^{p-1} e^{-x}, \text{ for } x, p > 0,$$

where $\Gamma(p)$ is the gamma function defined by

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x}\, dx; \text{ for } p > 0.$$

The gamma function (also called Euler function) is one of the important special functions occurring in many branches of mathematical physics and it is investigated in detail in a number of literatures.
Since that time many generalization of this distribution were considered by introducing new parameters. Amorose [1] at the end of the 19^{th} century as well as Karl Pearson [2] 1890-1900 system of distributions (type-III) managed to

end with 2-parameters gamma distribution model whose pdf is given by

$$f_x(x;p,\lambda)=\frac{\lambda^p}{\Gamma(p)}x^{p-1}e^{-\lambda x}\text{; for } x,p,\lambda>0,$$

where the parameter p and λ are represents the shape and scale parameters of the distribution, respectively.

Latter the generalized gamma distribution was suggested by Stacy [3] in 1962 which gives highly flexible form, its pdf is defined with 3-parameters as follows

$$f_x(x;p,\lambda,k)=\frac{k\lambda^p}{\Gamma(p)}x^{kp-1}e^{-\lambda x^k}\text{; for } x,p,\lambda,k>0,$$

where both p and k represents shape parameters, while λ is a scale parameter.

By varying the parameters of this distribution we can obtain large number of known distributions such as: exponential, one or two parameters gamma, half normal, chi-squares, Weibull and many more distributions (see Stacy and Mihram [4], 1967).

Harter [5] in 1967 suggest the same form of Stacy 3-parameter gamma distribution except he add a location parameter η to the model which is given by

$$f_x(x,p,\lambda,k,\eta)=\frac{k\lambda^p}{\Gamma(p)}\cdot(x-\eta)^{kp-1}e^{-\lambda(x-\eta)^k}\text{; for } x>\eta.$$

An interesting form of generalization is the generalized 2-parameter gamma distribution proposed by Lajako [6] in 1977, he defined first a model function $\alpha(x)$ which is continuous monotone increasing and differentiable in the interval $(a,b)\in R$ such that $\alpha(x)\in R^+$; $x\in(a,b)$, $\alpha(a)=0$, $\alpha(x)\to\infty$ as $x\to b$, and $\alpha'(x)=\frac{d}{dx}\alpha(x)\neq 0$. The pdf of this generalization is

$$f_x(x;p,\lambda,\alpha(x))=\frac{\lambda^p\alpha'(x)}{\Gamma(p)}\alpha^{p-1}e^{-\lambda\alpha(x)}\text{; for } x\in(a,b)$$

Other generalization on the lines of Stacy are proposed notably by Bradley [7] in 1988, Strivastava [8] in 1989, Lee and Gross [9] in 1991, and others. They all mainly introduced in order to extend the scope of the generalized 3-parameters gamma distribution such that the hazard function of these generalized distributions can have a wide variety of shapes including bathtub and monotonicity, which offers reasonable models in survival analysis.

Although many industrial components required a life-time distribution with hazard function has the property of bathtub and that of monotonicity, there are many situations the effect of all parameters do not start in the beginning, some of them start playing their role after sometimes this is known as displacement parameter. For example in a new machine system the corrosion problem will start after certain interval of time. In order to study the displacement effect a power is introduced to displaced parameter to observe the intensity of the effect which required another parameter known as intensity parameter.

To overcome these difficulties, Kobayashi [10] in 1991 has introduced a new type of generalized gamma function as follows

$$\Gamma_r(k,n)=\int_0^\infty x^{k-1}[x+n]^{-r}e^{-x}dx\text{; for } r,k,n>0.$$

This function is useful in many problems of diffraction theory and corrosion problems in new machines. However, this function has not been used in statistics until 1996 where Agrawal and Kalla [11] proposed a new generalization of gamma distribution by considering a modified form of the Kobayashi's gamma function given by

$$\Gamma_r(k,n,\lambda)=\int_0^\infty x^{k-1}[x+n]^{-r}e^{-\lambda x}dx\text{; for } r,k,n,\lambda>0$$
$$=\Gamma_r(k,\lambda n)\cdot\lambda^{r-k}$$

The new pdf of this generalization of gamma distribution with 4-parameters is given by

$$f_X(x;k,n,\lambda,r)=\frac{\lambda^{k-r}}{\Gamma_r(k,n\lambda)}x^{k-1}[x+n]^{-r}e^{-\lambda x}\text{; for } x>0$$

Where k,n,λ,r are respectively shap, displacement, scale and intensity parameters.

With slightly modify this distribution Agrawal and Kalla had shown that some well-known distributions are its particular cases.

Different forms of this new generalization of gamma distribution were also introduced by

Ghitany [12] 1998, and Galue et al [13] in 2001. however, Agrawal and Al-Saleh in [12] 2001, unlike the others, they used Kobayashi's exact form of the generalized gamma function to define a new generalization of gamma distribution whose pdf is given by

$$f_X(x;k,n,\lambda,r)=\frac{\lambda^{k-r}}{\Gamma_r(k,r)}x^{k-1}\left[x+\frac{n}{\lambda}\right]^{-r}e^{-\lambda x};\text{ for } x>0$$

Where $k,n,\lambda>0$ and $r=\delta-1;\ \delta\geq 0$. They have shown that the hazard function of this distribution is monotonic and bathtub shape.
In this paper we shall extend the Kobayashi proposed function into 6-parameters, and define different classes of distributions based on it. Several properties will be discussed, including the reliability, hazard functions and the problem of estimation of the parameters.

The Proposed Extended Generalized Gamma Function

Consider the following function, with $\theta=(k,m,n,r,\lambda,p)$:

$$\Lambda(\theta)=\Lambda(k,m,n,r,\lambda,p)=\int_0^\infty x^{k-1}\left[x^m+n\right]^{-r}e^{-\lambda x^p}dx$$

for $r\in R$, $k,m,n,\lambda,p>0$. This 6-parameter function can be regarded as an extension of the Kobayashi's generalized gamma function, since

$$\Lambda(k,1,n,r,1,1)=\Gamma_r(k,n)=\int_0^\infty x^{k-1}[x+n]^{-r}e^{-x}dx\ ;$$

for $k,n,r>0$

Also, this function is reduced to the well known gamma function when

$$\Lambda(k,1,n,0,1,1)=\Gamma_0(k,n)=\Gamma(k)=\int_0^\infty x^{k-1}e^{-x}dx;\text{ for } x>0$$

To investigate the properties of the function $\Lambda(\theta)$, we first consider the problem of the existence of the function.

Theorem 2.1: The improper integral function $\Lambda(\theta)$ is

$$0<\Lambda(\theta)<\infty$$

Proof. For all $k,n,m,r>0$, we have for $x>0$

$$0<x^{k-1}\left[x^m+n\right]^{-r}<x^{k-r\,m-1}$$

thus, for $\lambda,p>0$

$$0<\Lambda(\theta)<\int_0^\infty x^{k-r\,m-1}e^{-\lambda x^p}dx\ .$$

but,

$$\int_0^\infty x^{k-r\,m-1}e^{-\lambda x^p}dx=\frac{\lambda^{\frac{r\,m-k}{p}}}{p}\Gamma\left(\frac{k-r\,m}{p}\right)<\infty;\text{ for } k-r\,m>0$$

hence,

$$0<\Lambda(\theta)<\infty;\text{ for } r>0 \text{ whenever } \frac{k}{m} \text{ is}$$

large.

For all $r<0$; $k,m,n,\lambda,p>0$, $x>0$, we have, using binomial formula

$$0<\left[x^m+n\right]^{-r}\leq\left[x^m+n\right]^S=\sum_{i=1}^{S}\binom{S}{i}n^{S-i}x^{mi}\ ,$$

where $S=-r$ when r is a negative integer, $S=[-r]+1$ when $\left[x^m+n\right]\geq 1$, and $S=[-r]$ otherwise. Thus

$$0<\Lambda(\theta)\leq\sum_{i=1}^{S}\binom{S}{i}n^{S-i}\frac{\lambda^{\frac{k+im}{p}}}{p}\Gamma\left(\frac{k-im}{p}\right)<\infty$$

which proves the theorem.

Theorem 2.2: $\Lambda(\theta)$ can be written as follows

1. $\Lambda(\theta)=\dfrac{1}{p}\Lambda\left(\dfrac{k}{p},\dfrac{m}{p},n,r,\lambda,1\right)$,
2. $\Lambda(\theta)=\dfrac{\lambda^{\frac{mr-k}{p}}}{p}\Lambda\left(\dfrac{k}{p},\dfrac{m}{p},n\lambda^{\frac{m}{p}},r,1,1\right)$,
3. $\Lambda(\theta)=\dfrac{1}{m}\Lambda\left(\dfrac{k}{m},1,n,r,\lambda,\dfrac{p}{m}\right)$

Proof. Using the substitutions $y=x^p$, $y=\lambda x^p$ and $y=x^m$, respectively, the results follows
The second form of this theorem shows that the parameters (k,m,n,r) in $\Lambda(\theta)$ are essential, while the others (λ,p) can be regarded as index parameters. Also, when $p\to\infty$, or $m\to\infty$, we have $\Lambda(\theta)\to 0$.

Theorem 2.3: $\Lambda(\theta)$ satisfies the following recurrence relations

1. $\Lambda(\theta)=\frac{1}{\lambda p}\left[\left(\frac{k}{p}-1\right)\Lambda\left(\frac{k}{p}-1,\frac{m}{p},n,r,\lambda,1\right)-\frac{r\,m}{p}\Lambda\left(\frac{k+m}{p}-1,\frac{m}{p},n,r+1,\lambda,1\right)\right]$

2. $\Lambda(\theta)=\frac{\lambda^{r\,m-k}}{p}\left[\left(\frac{k}{p}-1\right)\Lambda\left(\frac{k}{p}-1,\frac{m}{p},n\lambda^{\frac{m}{p}},r,1,1\right)-\frac{r\,m}{p}\Lambda\left(\frac{k+m}{p}-1,\frac{m}{p},n\lambda^{\frac{m}{p}},r+1,1,1\right)\right]$

Proof. This follows by applying integration by parts to the form (1) and (2) of $\Lambda(\theta)$ given in theorem 2.2, respectively.

Theorem 2.4: when $m=p$, we have

$$\Lambda(\theta)=\frac{\lambda^{\frac{mr-k}{m}}}{m}\Lambda\left(\frac{k}{m},1,n\lambda,r,1,1\right)=\frac{\lambda^{\frac{mr-k}{m}}}{m}\Gamma\left(\frac{k}{m}-r\right);$$

for $\frac{k}{m}\succ r$ and for small $n\lambda$.

Proof. From theorem 2.2 (2) and [1] the proof follows.

This theorem shows that, when $m=p$

$$\lim_{k\to\infty}\Lambda(\theta)=\begin{cases}0 & ;\text{ for }\lambda>1\\ \infty & ;\text{ for }\lambda<1\end{cases}$$

Theorem 2.5: When $n=0$, we have

$$\Lambda(\theta)=\frac{\lambda^{\frac{rm-k}{p}}}{p}\Gamma\left(\frac{k-rm}{p}\right);\text{ for }\frac{k}{m}>r.$$

Proof. Using the transformation $y=\lambda x^{p}$, the proof follows

From this theorem, since $\Gamma(2)=\Gamma(1)=1$, we have when $n=0$,

$$\Lambda(\theta)=\begin{cases}\frac{\lambda}{p} & ;\text{ for }k-r\,m=p\\ \frac{\lambda^2}{p} & ;\text{ for }k-r\,m=2p\end{cases}$$

This result is an important special case of $\Lambda(\theta)$, since it is related to the extended generalized Weibull distribution which will be given in the next section.

Theorem 2.6.

1. When $p=0$,

$$\Lambda(\theta)=\frac{e^{-\lambda}n^{\frac{k-rm}{m}}}{m}\cdot B\left(r-\frac{k}{m},\frac{k}{m}\right);\text{ for }r-\frac{k}{m}>0$$

2. When $\lambda=0$,

$$\Lambda(\theta)=\frac{n^{\frac{k-rm}{m}}}{m}\cdot B\left(r-\frac{k}{m},\frac{k}{m}\right);\text{ for }r-\frac{k}{m}>0.$$

Where $B(\cdot,\cdot)$ is the beta function.

Proof. By setting $y=\frac{1}{\left(\frac{x^m}{n}+1\right)}$, the proof follows.

This theorem shows that beta function is related to a particular case of the function $\Lambda(\theta)$.

Theorem 2.7: when $k=p$, we have

$$\lim_{r\to\infty}\Lambda(\theta)=\begin{cases}0 & ;\text{ for }\lambda<1\\ \infty & ;\text{ for }\lambda>1\end{cases}$$

Proof. By substituting $k=p$ in the theorem 2.2 (2), and taking $r\to\infty$.

Theorem 2.8: when $r=0$, we have

$$\Lambda(\theta)=\frac{1}{p\lambda^{\frac{k}{p}}}\Gamma\left(\frac{k}{p}\right)$$

Proof. By setting $y=\lambda x^{p}$, the proof follows

Associated with the function $\Lambda(\theta)$ another function defined by

$$\Lambda_t(\theta)=\int_0^t x^{k-1}\left[x^m+n\right]^{-r}e^{-\lambda x^p}dx$$

This function will be called the incomplete extended generalized gamma function.

Theorem 2.9: The partial derivatives of the function $\Lambda(\theta)$ with respect to each parameter of θ are

1. $\frac{\partial}{\partial k}\Lambda(\theta)=\underset{10}{\Lambda}(\theta)$
2. $\frac{\partial}{\partial m}\Lambda(\theta)=-r\underset{10}{\Lambda}(k+m,m,n,r+1,\lambda,p)$
3. $\frac{\partial}{\partial n}\Lambda(\theta)=-r\Lambda(k,m,n,r+1,\lambda,p)$
4. $\frac{\partial}{\partial r}\Lambda(\theta)=-\underset{mn}{\Lambda}(\theta)$
5. $\frac{\partial}{\partial \lambda}\Lambda(\theta)=-\Lambda(k+p,m,n,r,\lambda,p)$
6. $\frac{\partial}{\partial p}\Lambda(\theta)=-\lambda\underset{10}{\Lambda}(k+p,m,n,r,\lambda,p)$

Where,

$$\underset{uv}{\Lambda}(\theta)=\int_0^\infty x^{k-1}\left[x^m+n\right]^{-r}\ln\left[x^u+v\right]e^{-\lambda x^p}dx$$

Another feature of this function $\Lambda(\theta)$ it is produce several important new sub-types of the extended generalized gamma function, simply by reducing the number of parameters of $\Lambda(\theta)$ to less than 6-parameters by assigning proper values to the eliminated parameter.

In the next section, we shall define a new type of distribution based on the function $\Lambda(\theta)$ and study many of its properties.

The Proposed Extended Generalized Distribution and its Statistical Properties

A continuous random variable X is said to have an extended generalized gamma distribution with 6-parameters and a model function $\alpha(x)$, defined in section 1, denoted by $X \sim EGG(\theta,\alpha(x))$, iff its pdf is given by

$$\begin{aligned} f_X(x) &= f_X(x;\theta,\alpha(x)) \\ &= \frac{\alpha'(x)}{\Lambda(\theta)}\alpha^{k-1}\left[\alpha^m(x)+n\right]^{-r}e^{-\lambda\alpha^p(x)}; \end{aligned}$$

for $x\in(a,b)$,

Where k,m,p are shape parameters, λ is a scale parameter, n and r are, respectively, displacement and intensity parameters. From this definition it follows that the pdf of the random variable $Y=\alpha(X)$ will be

$$f_Y(y)=f_Y(y,\theta)=\frac{1}{\Lambda(\theta)}y^{k-1}\left[y^m+n\right]^{-r}e^{-\lambda y^p};\ \text{for } y\in(0,\infty)$$

And the following relation is true for all $x_1,x_2\in(a,b)$, $x_1<x_2$

$$\int_{x_1}^{x_2} f_X(x)dx=\int_{\alpha(x_1)}^{\alpha(x_2)} f_Y(y)dy .$$

The random variable $Y=\alpha(X)\,\square\,EGG(\theta)$ will play an essential role in derivation of many statistical properties of X.

Theorem 3.1: The function $f_X(x)$ is a pdf.

Proof. It implies from the definition of X that $f_X(x)\geq 0$, $\forall x\in(a,b)$. Also we have

$$\int_a^b f_X(x)dx=\int_0^\infty f_Y(y)dy=1$$

The distribution function of X will be defined by

$$F_X(x;\theta,\alpha(x))=\int_0^x f_X(t)dt;\ \text{for } x\in(a,b).$$

Hence, we get for a given $x;x\in(a,b)$

$$F_X(x)=\int_0^{y=\alpha(x)} f_Y(t)dt=F_y(y)=\frac{\Lambda_y(\theta)}{\Lambda(\theta)};\ \text{for } y\in(o,\infty)$$

Theorem 3.2: If $Y=\alpha(X)\sim EGG(\theta)$, then the s-th moment about the origin is

$$\mu_s=EY^s=\frac{\Lambda(k+s,m,n,r,\lambda,p)}{\Lambda(\theta)};\ \text{for } s=1,2,3,...$$

Proof. Since,

$$EY^s=\int_0^\infty\frac{1}{\Lambda(\theta)}y^s\cdot y^{k-1}\left[y^m+n\right]^{-r}e^{-\lambda y^p}dy$$

The proof is completed.

Theorem. 3.3: If $Y=\alpha(X)\,\square\,EGG(\theta)$, then its respective mean and variance are

1. $\text{mean}=\dfrac{\Lambda(k+1,m,n,r,\lambda,p)}{\Lambda(\theta)}$,
2. $\text{variance}=\dfrac{\Lambda(\theta)\Lambda(k+2,m,n,r,\lambda,p)-\Lambda^2(k+1,m,n,r,\lambda,p)}{\Lambda^2(\theta)}$

Proof. From theorem 3.2 for $s=1,2$, the proof is followed.

Theorem 3.4: If $X\sim EGG(\theta,\alpha(x))$ then its respective reliability and hazard (or failure rate) functions are

1. $R(x)=R(x;\theta,\alpha(x))=\dfrac{\Lambda(\theta)-\Lambda_{\alpha(x)}(\theta)}{\Lambda(\theta)};\ \text{for } x\in(a,b)$
2. $$\begin{aligned} h(x) &= h(x;\theta,\alpha(x)) \\ &= \frac{1}{R(x)}\alpha'(x)\alpha^{k-1}(x)\left[\alpha^m(x)+n\right]^{-r}e^{-\lambda\alpha^p(x)} \end{aligned}$$

; for $x\in(a,b)$.

Proof. By substitution f_X and F_X in the definitions

$$R(x)=1-F_X(x);\ h(x)=\frac{f_X(x)}{R(x)}$$

Many distributions can be derived as special cases of the extended generalized gamma distribution. This can be done by reducing the number of parameters of X to less than 6-parameters, by assigning proper values for some parameters of θ. For example, the following are some important new types of pdf of 5-parameters extended generalized gamma function.

1. $f_X(x)=\dfrac{\alpha'(x)}{\Lambda(k,m,0,r,\lambda,p)}\alpha^{k-rm-1}(x)e^{-\lambda\alpha^p(x)}$
2. $f_X(x)=\dfrac{\alpha'(x)}{\Lambda(1,m,n,r,\lambda,p)}\left[\alpha^m(x)+n\right]^{-r}e^{-\lambda\alpha^p(x)}$
3. $f_X(x)=\dfrac{\alpha'(x)}{\Lambda(k,m,n,r,0,p)}\alpha^{k-1}(x)\left[\alpha^m(x)+n\right]^{-r}$
4. $f_X(x)=\dfrac{\alpha'(x)}{\Lambda(k,0,n,r,\lambda,p)}\alpha^{k-1}(x)[1+n]^{-r}e^{-\lambda\alpha^p(x)}$
5. $f_X(x)=\dfrac{\alpha'(x)}{\Lambda(k,m,n,0,\lambda,p)}\alpha^{k-1}(x)e^{-\lambda\alpha^p(x)}$
6. $f_X(x)=\dfrac{\alpha'(x)}{\Lambda(k,m,n,r,\lambda,0)}\alpha^{k-1}(x)\left[\alpha^m(x)+n\right]^{-r}e^{-\lambda}$

Notice that, all of these distributions and others are still defined in general, since the model function is yet unspecified. Many $\alpha(x)$ can be proposed, for example if we take

$$\alpha(x)=\left(\frac{x-\eta}{\delta}\right)^{\beta};\ \text{for } x>\mu,\ \delta>0,\ \beta\geq 1,$$

Where μ,δ,β are, respectively, location, scale, and shape parameters. Then the pdf of $X\sim EGG(\theta,\alpha(x))$ becomes with 9-parameters and is given by

$$f_X(x)=\frac{\beta}{\delta\Lambda(\theta)}\left(\frac{x-\eta}{\delta}\right)^{\beta k-1}\left[\left(\frac{x-\eta}{\delta}\right)^{\beta m}+n\right]^{-r}e^{\lambda\left(\frac{x-\eta}{\delta}\right)^{\beta p}}$$

; for $x>\eta$.

The following distributions are some particular cases of this distribution.

Case 1 when $n=0$, from theorem 2.5,

$$f_X(x)=\frac{\beta p\lambda^{\frac{k-rm}{p}}}{\delta\,\Gamma\left(\frac{k-rm}{p}\right)}\left(\frac{x-\eta}{\delta}\right)^{\beta(k-rm)-1}e^{-\lambda\left(\frac{k-\eta}{\delta}\right)^{\beta p}};$$

for $r<\dfrac{k}{m}$,

And for $k-rm=p$,

$$f_X(x)=\frac{\beta p\lambda}{\delta}\left(\frac{x-\eta}{\delta}\right)^{\beta p-1}e^{-\lambda\left(\frac{x-\eta}{\delta}\right)^{\beta p}};\ \text{for } x>\eta,$$

this form can be considered as an extended generalized Weibull distribution, and when $\beta p=1$, then

$$f_X(x)=\frac{\lambda}{\delta}e^{-\lambda\left(\frac{x-\eta}{\delta}\right)};\ \text{for } x>\eta,$$

Which is the generalized 3-parameters exponential distribution. For $\beta(k-rm)=1$, then

$$f_X(x)=\frac{\beta p\lambda^{\frac{1}{\beta p}}}{\delta\,\Gamma\left(\frac{1}{\beta p}\right)}e^{-\lambda\left(\frac{k-\eta}{\delta}\right)^{\beta p}};\ \text{for } x>\eta,$$

this form is an extended generalized normal distribution, and when $\beta p=2$, $\lambda=\frac{1}{2}$, then $f_X(x)$ is a half-normal distribution.

Case 2 when $\lambda=0$, from theorem 2.6, we have

$$f_X(x)=\frac{m\beta n^{r-\frac{k}{m}}}{\delta\beta\left(r-\frac{k}{m},\frac{k}{m}\right)}\left(\frac{x-\eta}{\delta}\right)^{\beta k-1}\left[\left(\frac{x-\eta}{\delta}\right)^{\beta m}+n\right]^{-r}$$

; for $x>\eta$,

which can be considered as an extended generalized beta distribution.

Case 3 when $r=0$, from theorem 2.8, we have

$$f_X(x)=\frac{\beta p\lambda^{\frac{k}{p}}}{\delta\,\Gamma\left(\frac{k}{p}\right)}\left(\frac{x-\eta}{\delta}\right)^{\beta k-1}e^{-\lambda\left(\frac{x-\eta}{\delta}\right)^{\beta p}};\ \text{for } x>\eta$$

this can be considered as an extended generalized gamma distribution, and when $\beta k=1$, then $f_X(x)$ can be regarded as extended generalized exponential distribution.

Case 4 when $m=p$, from theorem 2.4, we have

$$f_X(x)=\frac{m\beta\lambda^{\frac{k}{m}-r}}{\delta\,\Gamma\left(\frac{k}{m}-r\right)}\left(\frac{x-\eta}{\delta}\right)^{\beta k-1}\left[\left(\frac{x-\eta}{\delta}\right)^{\beta m}+n\right]^{-r}e^{-\lambda\left(\frac{x-\eta}{\delta}\right)^{\beta m}}$$

; for $x>\eta$

And $\frac{k}{m}>r$, and for small $n\lambda$, which is another form of the generalization of gamma distributions.

Next, we consider the problem of estimation of the parameters $\theta=(k,m,n,r,\lambda,p)$. Let $X_1,X_2,...,X_N$ be a random sample from X, the estimators of θ, denoted by $\hat{\theta}=(\hat{k},\hat{m},\hat{n},\hat{r},\hat{\lambda},\hat{p})$, obtained by the methods of moments and maximum likelihood are given by the following two theorems.

Theorem 3.4: The moment estimators of θ are the solution of the following non-linear systems of 6-equations.

$\Lambda(\hat{k}+s,\hat{m},\hat{n},\hat{r},\hat{\lambda},\hat{p})=\Lambda(\hat{\theta})M_s$; for $s=1,2,...,6$,

Where,

$$M_S=\frac{1}{N}\sum_{i=1}^{N}\alpha^s(x_i),$$

Is the s-th sample moment of the random variable $\alpha(X)$.

Proof. By equating μ_s, given in theorem 3.2, with M_s which concludes the proof of the theorem.

Theorem 3.5: The maximum likelihood estimators of θ are the solution of the following non-linear system of 6-equations.

$$\sum_{i=1}^{N}\ln\alpha(x_i)-\frac{N\,\Lambda_{10}(\hat{\theta})}{\Lambda(\hat{\theta})}=0$$

$$-\hat{r}\sum_{i=1}^{N}\frac{\alpha^{\hat{m}}(x_i)\ln\alpha(x_i)}{\left[\alpha^{\hat{m}}(x_i)+\hat{n}\right]}+\frac{N\hat{r}\,\Lambda_{10}(\hat{k}+\hat{m},\hat{n},\hat{r}+1,\hat{\lambda},\hat{p})}{\Lambda(\hat{\theta})}=0$$

$$-\hat{r}\sum_{i=1}^{N}\frac{1}{\left[\alpha^{\hat{m}}(x_i)+\hat{n}\right]}+\frac{N\hat{r}\,\Lambda(\hat{k},\hat{m},\hat{n},\hat{r}+1,\hat{\lambda},\hat{p})}{\Lambda(\hat{\theta})}=0$$

$$-\sum_{i=1}^{N}\ln\left[\alpha^{\hat{m}}(x_i)+\hat{n}\right]+\frac{N\,\Lambda_{\hat{m}\hat{n}}(\hat{\theta})}{\Lambda(\hat{\theta})}=0$$

$$-\sum_{i=1}^{N}\alpha^{\hat{p}}(x_i)+\frac{N\,\Lambda(\hat{k}+\hat{p},\hat{m},\hat{n},\hat{r},\hat{\lambda},\hat{p})}{\Lambda(\hat{\theta})}=0$$

$$-\hat{\lambda}\sum\alpha^{\hat{p}}(x_i)\ln\alpha(x_i)+\frac{N\,\hat{\lambda}\,\Lambda_{10}(\hat{k}+\hat{p},\hat{m},\hat{n},\hat{r},\hat{\lambda},\hat{p})}{\Lambda(\hat{\theta})}=0$$

Proof. By taking the partial derivative of natural logarithm of the likelihood function with respect of each parameter and the aid of theorem 2.8, the results are obtained by setting each to zero.

For a specific $\alpha(x)$, each of the two systems of equations required a numerical method to obtain a solution. Also, when $\alpha(x)$ contains additional parameters then the number of equations of moments system have to be extended to equal the total number of the parameters, while the maximum likelihood systems new equations have to be added by setting zero the equation of the partial derivative of natural logarithm of the likelihood function with respect to each new parameter.

Conclusion

In terms of a model function $\alpha(x)$ a new distribution with its particular cases is proposed, its reliability and hazard functions as well as the method of moments and maximum likelihood for estimation the parameters are derived. It seems to us that this paper open new horizon for studying the characteristics of these concepts for each suitable chosen $\alpha(x)$, especially the shape of the hazard function and the existence and uniqueness of the solutions of the systems given by both methods of estimations.

The proposed distribution depends on the function $\Lambda(\theta)$; several simplifications of the form of $\Lambda(\theta)$ are possible. Nevertheless $\Lambda(\theta)$ as well as $\Lambda_t(\theta)$ required approximations formulas to be more useful in applications, this is left as an open problem. Also, for further study we suggest the modified form of $\Lambda(\theta)$ given by

$$\int_0^\infty (x-a)^{k-1}\left[bx^m+n\right]^{-r}e^{-\lambda x^p}dx\text{ , where } a,b\in R$$

We believe that the new distribution has wide variety of shapes that help to explain various phenomenons, particularly to accommodate the requirements of industrial developments.

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