More Results on Almost Noetherian Domains

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Abstract

In this paper we prove some theorems, the first states: If *R* is an almost Noetherian domain, then the following statements are equivalent: ١- *R* is an almost Dedekind domain.
٢- *A*(*B*∩*C*)= *AB*∩*AC*, for all ideals *A*, *B* and *C* of R. ٣- (*A*+*B*)(*A*∩*B*)= *AB*, for all ideals *A* and *B* of *R* and the second states: If *R* is an almost Noetherian domain which is not a field, then the following statements are equivalent: ١- *R* is a valuation domain. ٢- The nonunits of *R* form a nonzero principal ideal of *R*. ٣- *R* is integrally closed and has exactly one nonzero proper prime ideal. In addition to the above some other results are proved.

نتائج اكثر حول الساحات على الاكثر نويثيرية

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الخلاصة

فى هذا البحث نبرهن بعض المبرهنات حيث تنص الاولى : اذاكانت *R* ساحة على الاكثر نويثيرية فان الشروط الاتية تكون متكافئة: ١- تكون *R* ساحة على الاكثر ديديكندية. ٢- *A*(*B*∩*C*)= *AB*∩*AC* لكل المثاليات (*A*, *B*, *C*) فى *R*. ٣- (*A*+*B*)(*A*∩*B*)= *AB* لكل المثاليات (*A*, *B*) فى *R* وتنص المبرهنة الثانية: اذاكانت *R* ساحة على الاكثر نويثيرية والتى ليست حقلا فان الشروط الاتية تكون متكافئة:
١- تكون *R* ساحة تقييمية. ٢- اللاوحدات فى *R* تشكل مثالى اساسى غير صفرى فى *R*. ٣- تكون *R* مغلق تكامليا وتمتلك تماما مثاليا اوليا تاما ولا صفريا. بالاضافة الى النتائج اعلاه لقد تم البرهنة على بعض النتائج الاخرى.

١. Introduction

Let R be a commutative ring with identity ١ and S is a nonempty subset of R, then S is called a multiplicative system in R if $0 \notin S$ and $a, b \in S$ implies that $ab \in S$ [١]. Define a relation (~) on $R \times S$ as follows: For $(a,r),(b,s) \in R \times S$, we let $(a,r) \sim (b,s)$ if and only if there exists $t \in S$ such that $t(as - br) = 0$. It is easy to show that (~) is an equivalence relation on $R \times S$. Then, denote the equivalence class of (a,r) by $\frac{a}{r}$ [٢] (some times this equivalence class denoted by (*a*, *r*) [١]) and denote the set of all equivalence classes of $R \times S$ relative to the equivalence relation (~) by R_S, that is, we let

$R_S = \{\frac{a}{r} : (a, r) \in R \times S\}$. Next, we define (+) and (.) on R_S as follows:

$$\frac{a}{s} + \frac{b}{t} = \frac{at + bs}{st}$$

and

$$\frac{a}{s} . \frac{b}{t} = \frac{ab}{st},$$

for all

$$\frac{a}{s}, \frac{b}{t} \in R_S .$$

It can be shown that these operations are well-defined and that (R_S, +, .) forms a commutative ring with identity (In fact, $\frac{s}{s}$ is the identity element of R_S, for all $s \in S$) and this ring is called the total quotient ring of R [٢] (this ring is also denoted by $S^{-1}R$ and called the localization of R at the multiplicative system S [٣]). Next, we mention to the following facts: If R is an integral domain, then R_S is a field (and hence an integral domain). If A is an ideal of R, then the set $\{\frac{a}{s} : a \in A, s \in S\}$ forms an ideal of R_S and is denoted by AR_S [٢] and if A' is an ideal of R_S, then there exists an ideal A of R such that $A' = AR_S$ [٤]. It is known that if P is a prime ideal of R, then R-P forms a multiplicative system in R and thus R_{R-P} is the total quotient ring of R and for the sake of simplicity we denote R_{R-P} just by R_P, so that $R_P = \{\frac{a}{m} : a \in R, m \notin P\}$. Also, in this case PR_P is the unique prime ideal of R_P and hence it is the only maximal ideal of R_P which means that R_P is a local ring with PR_P as its unique maximal ideal, so that if M is a maximal ideal of R, then it is prime and thus the total quotient ring of R is $R_M = \{\frac{a}{m} : a \in R, m \notin M\}$.

٢. Some Basic Definitions and Some Known Results:

Before proving the main results of this paper, we restate some basic definitions. A commutative ring is called a Noetherian ring if the ideals in R satisfy the ascending chain condition or equivalently, if every ideal of R is finitely generated [٥] and by a Noetherian domain is meant a Noetherian integral domain [٦]. An integral domain is said to be an almost Noetherian domain if R_M is Noetherian for each maximal ideal M of R [٢]. If A is an ideal of a commutative ring R, then we define $A^{-1} = \{x \in R_S : xA \subseteq R\}$ [٧], where S is the set of all nonzero divisors of R and A is called an invertible ideal of R if $AA^{-1} = R$ [٧]. An integral domain R is called an almost Dedekind domain if R_M is Dedekind for each maximal ideal M of R [٢]. A ring R is called an arithmetical ring if for all ideals A, B, C of R we have $A \cap (B+C) = A \cap B + A \cap C$ and by an arithmetical domain is meant an arithmetical ring which is also an integral domain [٢]. A commutative ring is called hereditary if every ideal of R is projective [٣] and it is called semihereditary if every finitely generated ideal of R is projective [٨]. An integral domain R is called a valuation domain if for any ideals A, B of R we have $A \subseteq B$ or $B \subseteq A$ [٢]. A ring s called local if it has only one maximal idea and it is called semilocal if it contains a finite number of maximal ideals. Let R be a subring of a ring R', we say an element $b \in R'$ is integral over R if there exists a positive integer n and $a_0, a_1, \ldots, a_{n-1} \in R$ such that $a_0 + a_1 b + a_2 b^2, \ldots, + a_{n-1} b^{n-1} + b^n = 0$ [٩] and if every element of R' is integral over R we say that R' is integral over R and R is said to be integrally closed in R' if the elements of R are the only elements of R' which are integrally closed over R and R is said to be integrally closed if it is integrally closed in its total quotient ring [٢].

Next, we mention to the following results the proof of which can be found in the pointed references:

Theorem ٢.١: [٨]

If R is a commutative ring with identity and M is a maximal ideal of R, then:

١. $(A \cap B) R_M = AR_M \cap BR_M$, for all ideals A, B of R and
٢. $(A+B) R_M = AR_M + BR_M$, for all ideals A, B of R.

Theorem ٢.٢: [٢]

If R is a commutative ring with identity and A, B are ideals of R, then $A = B$ if and only if $AR_M = BR_M$, for each maximal ideal M of R.

Theorem ٢.٣: [٢]

Let R be a Noetherian domain, then the following conditions are equivalent:

١. R is a Dedekind domain.
٢. $A\,(C\cap B) = AB \cap BC$, for all ideals A, B and C of R.
٣. $(A+B)(C\cap B) = AB$, for all ideals A, B of R.

Theorem ٢.٤: [٨]

If R is an almost Noetherian domain, then the following conditions are equivalent:

١. R is an almost Dedekind domain.
٢. R is an arithmetical domain.
٣. If A, B and C are ideals of R with A nonzero and contained in each maximal ideal of R such that any multiple of A is a prime ideal, then $AB = AC$ implies that $B = C$.

Theorem ٢.٥: [٨]

If R is an almost Noetherian domain, then R is an almost Dedekind domain if and only if R is semihereditary.

Theorem ٢.٦: [٢]

If R is a Noetherian domain which is not a field, then the following statements are equivalent:

١. R is a valuation domain.
٢. Then nonunits of R form a nonzero principal ideal of R.
٣. R is integrally closed and has exactly one nonzero proper prime ideal.

Theorem ٢.٧: [٤]

If R is a Noetherian ring and S is a multiplicative system in R, then R_S is also Noetherian.

Theorem ٢.٨:

Let R be a commutative ring with identity, then R is a local ring if and only if the nonunits of R form an ideal.

Theorem ٢.٩:

An almost Noetherian ring which is semilocal is Noetherian and hence an almost Noetherian ring which is local is Noetherian.

٣. The main Results:

Before giving the main results of this paper, we prove some simple results which will help us to prove the main theorems.

Lemma ٣.١:

If R is a commutative ring with identity and M is a maximal ideal of R, then for each positive integr n

١. $\dfrac{\sum_{i=1}^{n} a_i b_i}{m} = \sum_{i=1}^{n} \dfrac{a_i}{m}\dfrac{b_i}{1}$, for all a_i, $b_i \in R$ and $m \notin M$.

٢. $\dfrac{\sum_{i=1}^{n} a_i}{m} = \sum_{i=1}^{n} \dfrac{a_i}{m}$, for all $a \in R$ and $m \notin M$.

Proof:

١. We use mathematical induction on n. For n=١, we have

$$\frac{\sum_{i=1}^{1} a_i b_i}{m} = \frac{a_1 b_1}{m.1} = \frac{a_1}{m}\frac{b_1}{1} = \sum_{i=1}^{1} \frac{a_i}{m}\frac{b_i}{1}.$$

Next, suppose that the result is true for n-١, (where $n \geq$ ٢), that is

$$\frac{\sum_{i=1}^{n-1} a_i b_i}{m} = \sum_{i=1}^{n-1} \frac{a_i}{m}\frac{b_i}{1}$$

and to show the result is true for n.
Now, we have

$$\frac{\sum_{i=1}^{n} a_i b_i}{m} = \frac{\sum_{i=1}^{n-1} a_i b_i + a_n b_n}{m} = \frac{\sum_{i=1}^{n-1} a_i b_i + a_n b_n}{m}\frac{m}{m} =$$

$$\frac{(\sum_{i=1}^{n-1} a_i b_i)m + a_n b_n m}{mm} = \frac{\sum_{i=1}^{n-1} a_i b_i}{m} + \frac{a_n b_n}{m} =$$

$$\sum_{i=1}^{n-1} \frac{a_i}{m}\frac{b_i}{1} + \frac{a_n}{m}\frac{b_n}{1} = \sum_{i=1}^{n} \frac{a_i}{m}\frac{b_i}{1}.$$

Thus the result is proved.

٢. Take b_i =١, for all i in (١) the result follows directely.

Proposition ٣.٢:

If R is a commutative ring with identity and M is a maximal ideal of R, then, $(AB)R_M = AR_M BR_M$, for all ideals A, B of R.

Proof:

Let $y' \in (AB)R_M$, so $y' = \frac{x}{m}$, for some $x \in AB$ and $m \notin M$, so there exists a positive integer n such that $x = \sum_{i=1}^{n} a_i b_i$, for $a_i \in A$ and $b_i \in B$ and then as $1 \notin M$, by using **Lemma ٣.١**, we get that

$$y' = \frac{x}{m} = \frac{\sum_{i=1}^{n} a_i b_i}{m} = \sum_{i=1}^{n} \frac{a_i}{m} \frac{b_i}{1} \in AR_M BR_M .$$

Thus $(AB)R_M \subseteq AR_M BR_M$. Next, if $y' \in AR_M BR_M$, then there is a positive integer k such that

$$y' = \sum_{i=1}^{k} \frac{a_i}{m_i} \frac{b_i}{q_i}, \text{ for } a_i \in A, b_i \in B$$

and

$$m_i \notin M, q_i \notin M .$$

Then we get

$$y' = \sum_{i=1}^{k} \frac{a_i}{m_i} \frac{b_i}{q_i} = \sum_{i=1}^{k} \frac{a_i b_i}{m_i q_i} \in (AB)R_M$$

(Since $a_i b_i \in AB$ and $m_i q_i \notin M$, for all i) and so that $AR_M BR_M \subseteq (AB)R_M$. Hence $(AB) R_M = AR_M BR_M$.

Now it is the time to give our first theorem

Theorem ٣.٣:

If R is an almost Noetherian domain, then the following conditions are equivalent:

١. R is an almost Dedekind domain.

٢. $A(B \cap C) = AB \cap AC$, for all ideals A, B and C of R.

٣. $(A+B)(A \cap B) = AB$, for all ideals A, B of R.

Proof:

First, we will prove (١↔٢).

Suppose that R is an almost Dedekind domain and A, B, C are ideals of R. Now, if M is any maximal ideal of R, then R_M is a Dedekind domain and AR_M, BR_M and CR_M are ideals of R_M. As R is an almost Noetherian domain, we get R_M is a Noetherian domain and hence by **Theorem ٢.٣**, we have

$AR_M (BR_M \cap CR_M) = AR_M BR_M \cap AR_M CR_M$.

Then, using **Theorem ٢.١(١)** and **Proposition ٣.٢** we get

$(A(B \cap C))R_M = (AB \cap AC)R_M$

and by **Theorem ٢.٢**, we get

$A(B \cap C) = AB \cap AC$.

Convesely, suppose that $A(B \cap C) = AB \cap AC$, for all ideals A, B and C of R and to show that R is an almost Dedekind domain. Let M be any maximal ideal of R, so that R_M is a Noetherian domain. If A', B' and C' are any ideals of R_M, then there exist ideals A, B and C of R such that $A' = AR_M$, $B' = BR_M$ and $C' = CR_M$, then by the given condition we have $A(B \cap C) = AB \cap AC$ and by making the use of **Theorem ٢.١(١)**, **Proposition ٣.٢** and **Theorem ٢.٢**, we get

$AR_M (BR_M \cap CR_M) = AR_M BR_M \cap AR_M CR_M$,

that is,

$A'(B' \cap C') = A'B' \cap A'C'$ and as R_M is a Noetherian domain,

so by **Theorem ٢.٣**, we get R_M is a Dedekind domain and hence R is an almost Dedekind domain.

To prove (١↔٣), we use exactly the same technique as in the above and getting the result. Combining **Theorem ٢.٤**, **Theorem ٢.٥** and **Theorem ٣.٣**, we give the following corollary:

Corollary ٣.٤:

If R is an almost Noetherian domain, then the following statements are equivalent:

١. R is an almost Dedekind domain.

٢. $A(B \cap C) = AB \cap AC$, for all ideals A, B and C of R.

٣. $(B+C)(B \cap C) = AB$, for all ideals A, B of R.

٤. R is an arithmetical domain.

٥. If A, B and C are ideals of R with A nonzero and contained in each maximal ideal of R such that any multiple of A is a prime ideal, then $AB = AC$ implies $B = C$.

٦. R is semihereditary.

Lemma ٣.٥:

If R is a valuation domain which is not a field, then the nonunits of R form a nonzero prime ideal of R.

Proof:

Let P be the set of all nonunits of R. If P=٠, this means that the zero element ٠ is the only nonunit element of R and thus R is a field which is a ontradiction. Hence P≠٠. Clearly, P is a nonempty subset of R. Let $a, b \in P$ and $r \in R$. If $ar \notin P$, then ar is a unit of R and hence a is a unit of R, so that $a \notin P$ which is a contradiction. Thus $a \notin P$. Similarly we can get that $ra \in P$. Also, if a-$b \notin P$, then a-b is a unit of R and thus $(a$-$b)\ x$ = ١, for some $x \in R$, then ax-bx=١. But since R is a valuation domain, so we have $\langle a\rangle \subseteq \langle b\rangle$ or $\langle b\rangle \subseteq \langle a\rangle$. If $\langle a\rangle \subseteq \langle b\rangle$, then $ax \in \langle a\rangle \subseteq \langle b\rangle$ and hence $ax = by$, for some $y \in R$, then we get $b(y-x)=1$, which means that b is a unit of R and thus $b \notin P$ which is a contradiction and if $\langle b\rangle \subseteq \langle a\rangle$, then by using the same technique again we get a contradiction and so $a-b \in P$. Hence P is a nonzero ideal of R and clearly $1 \notin P$, so that $P \neq R$ and finally, suppose that for $a, b \in R$, we have $ab \in P$. If $a \notin P$ and $b \notin P$, then both a and b are units of R and hence ab is a unit of R, so that $ab \notin P$, which is a contradiction and thus $a \in P$ or $b \in P$. Hence P is a nonzero prime ideal of R.

Lemma ٣.٦:

Let R be a valuation domain. If P is the set of all nonunits of R, then R_P is not a field.

Proof:

By **Lemma ٣.٥**, P is a nonzero prime ideal of R and thus PR_P is an ideal of R_P. We will show that PR_P is non trivial. Since $P \neq 0$, so there exists $0 \neq x \in P$ and as $1 \notin P$, we get $\frac{x}{1} \in PR_P$. Now, if $\frac{x}{1} = \frac{0}{1}$, then there exists $u \notin P$ such that $ux = 0$. But $u \notin P$ gives $u \neq 0$ and as R is integral domain we get $x = 0$ which is a contradiction and thus $\frac{x}{1} \neq \frac{0}{1}$. Hence $PR_P \neq$ ٠. If $PR_P = R_P$, then $\frac{1}{1} \in PR_P$, so that $\frac{1}{1} = \frac{p}{m}$, for some $p \in P$ and $m \notin P$ and then there exists $v \notin P$ such that $vm = vp \in P$ and as P is a prime ideal we get $v \in P$ or $m \in P$, which is a contradiction and so, $PR_P \neq R_P$, that means PR_P is non trivial ideal of R_P. Hence R_P is not a field.

Lemma ٣.٧:

Let R be a valuation domain. If P is the set of all nonunits of R, then PR_P is the set of all non units of R_P.

Proof:

By **Lemma ٣.٥**, P is a nonzero prim ideal of R and thus PR_P is a local ring with PR_P as its unique maximal ideal and thus every element of PR_P is a nonunit of R_P and if $\frac{a}{m}$ is any nonunit of R_P, then it must contained in some maximl ideal of R_P and since PR_P is the unique maximal ideal of R_P, so $\frac{a}{m} \in PR_P$. Hence PR_P is the set of all non units of R_P.

Lemma ٣.٨:

Let R be a valuation ring. If P is a prime ideal of R, then R_P is also a valuation ring.

Proof:

Let A' and B' are any ideals of R_P, then there exist ideals A and B of R such that $A' = AR_P$ and $B' = BR_P$. As R is a valuation ring we have $A \subseteq B$ or $B \subseteq A$, which in consequence give $AR_P \subseteq BR_P$ or $BR_P \subseteq AR_P$, that is $A' \subseteq B'$ or $B' \subseteq A'$. Hence R_P is a valuation ring.

Lemma ٣.٩:

Let R be an almost Noetherian domain which is not a field. If R is a valuation domain and P is the set of all nonunits of R, then P is the only maximal ideal of R and R_P is Noetherian.

Proof:

Let M be any maximal ideal of R, then if $x \in M$, so x is a nonunit of R and hence

$x \in P$, so that $M \subseteq P$ and as M is maximal, we get $P = R$ or $M = P$, but by **Lemma ٣.٥**, we have P is prime, so $P \neq R$ and thus we get $M = P$, that means P is the only maximal ideal of R and as R is almost Noetherian, we get R_P is Noetherian.

Lemma ٣.١٠:

Let R be an almost Noetherian domain. If R is a valuation domain and P is the set of all nonunits of R, then PR_P is a principal ideal of R_P.

Proof:

By **Lemma ٣.٦**, R_P is not a field and thus by **Lemma ٣.٩**, we have P is a maximal ideal and hence a prime ideal of R and R_P is Noetherian and so by **Lemma ٣.٨**, R_P is a valuation domain and also by **Lemma ٣.٧**, PR_P is the set of all nonunits of R_P and as R_P is both Noetherian and valuation, by **Theorem ٢.٦**, we get PR_P is a principal ideal of R_P.

The last theorem of this paper is a generalization of **Theorem ٢.٦**, to almost Noetherian domains.

Theorem ٣.١١:

If R is an almost Noetherian domain which is not a field, then the following statements are equivalent:

١. R is a valuation domain.
٢. The non units of R form a nonzero principal ideal of R.
٣. R is integrally closed and has exactly one nonzero proper prime ideal.

Proof:

(١↔٢)

Let R be a valuation domain and let P be the set of all nonunits of R. By **Lemma ٣.٥**, P is a nonzero prime ideal of R. Then, by **Lemma ٣.١٠**, PR_P is a principal ideal ideal of R_P, so let $PR_P = \langle \frac{x}{y} \rangle$, for some $\frac{x}{y} \in R_P$. To show $P = \langle x \rangle$. Now, $\frac{x}{y} = \frac{a}{b}$ for some $a \in P$ and $b \notin P$, so that there exists $q \notin P$ such that $qxb = qya \in P$ and as P is a prime ideal and $qb \notin P$, so $x \in P$ and hence $\langle x \rangle \subseteq P$. Next, let $p \in P$, then $\frac{p}{1} \in PR_P = \langle \frac{x}{y} \rangle$ and hence $\frac{p}{1} = \frac{r}{s}\frac{x}{y}$, for some $\frac{r}{s} \in R_P$, then there exists $u \notin P$ such that $urx = upsy = usyp$. As $u, s, y \notin P$, we get $usy \notin P$ and hence usy is a unit of R, that means $(usy)^{-1} \in R$, then we get $p = (usy)^{-1}usyp = (usy)^{-1}urx \in \langle x \rangle$. Hence $P \subseteq \langle x \rangle$ and thus $P = \langle x \rangle$ as required.

To prove (٢↔٣)

Suppose that the nonunits of R form a nonzero principal ideal of R and let us denote it by P. By **Theorem ٢.٨**, R is a local ring and since R is an almost Noetherian domain, so by **Theorem ٢.٩**, R is a Noetherian domain. As R is not a field and the nonunits of R form a nonzero principal ideal of R, so by **Theorem ٢.٦**, we have R is integrally closed and has exactly one nonzero proper prime ideal.

To prove (٣↔١)

Suppose that R is integrally closed and has exactly one nonzero proper prime ideal and thus R must be a local ring and since it is an almost Noetherian domin, so it is a Noetherian domain which is also not a field, thus by **Theorem ٢.٦**, we get that R is a valuation domain.

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