ACTIONS OF FINITE GROUPS ON COMMUTATIVE RINGS AND INVARIANT IDEALS

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Abstract

Let R be a commutative ring with ١, and let $Aut(R)$ denote the group of ring automorphisms of R. We will usually consider a group $G\subseteq Aut(R)$.
In this paper we will study the relation between G- invariant ideals and their traces.
Also we study C.P modules and C.F modules.

افعال الزمر المنتهية على الحلقات الابدالية والمثاليات المتغايرة

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الخلاصة

تكن R حلقة إبدالية ذات عنصر محايد ١، ولتكن. $G\subseteq Aut(R)$ زمرة منتهية من التشاكلات المتقابلة الذاتية من R إلى R ذات رتبة n.
في هذا البحث سوف ندرس العلاقة بين المثالي اللامتغاير بفعل G وأثره. كذلك سوف ندرس المقاسات من النمط C.P ومن النمط C.F.

Introduction

Let R be a commutative ring with ١, and let $Aut(R)$ denote the group of ring automorphisms of R. We shall usually consider a group $G\subseteq Aut(R)$. The identity automorphism will be denoted by e. The fixed subring of G on R is $R^G=\{r\in R;\ g(r)=r, \text{ for all } g\in G\}$. An ideal I of the ring R is called a G-invariant ideal if $g(I)=I$, for all $g\in G$.
In this paper we study the relation between G-invariant ideals and their traces. On the other hand one can consider R as a module over R^G in the obvious way. In this paper we will study C.P modules and C.F modules. In fact we prove That if R is a Goldie ring that has no non-zero nilpotent elements and if R is a hereditary (flat) then R is a C.P (C.F) R^G -module.
We note finally that G in this paper is a finite group of order $|G|=n$.

١. The trace of R as R^G -module

Let R be a ring, and let G be a group of automorphisms of R. Let R^G denotes the fixed ring of R, we consider R as an R^G -module in the obvious way. Note that $R^G\neq$٠. Moreover since ١$\in R$, then R is a faithful R^G -module.

Proposition ١.١: Let R be any ring, and let G be a group of automorphisms of R. If R is a cyclic R^G -module, then $R= R^G$, and $G = \{e\}$. The converse is clear.

Proof: Since R is cyclic, then $\exists 0\neq x\in R$, such that $R= R^Gx$. But ١$\in R$ Then $\exists 0\neq y\in R^G$, such that ١$=yx$. Thus y is invertible in R, and hence is Invertible in R^G.
Thus $\exists 0\neq z\in R^G$, such that ١$=yz$. Then $x=xyz=z$, which implies that $R= R^Gz\subseteq R^G$, and hence $R= R^G$.
Next we give the following definitions:

Definition ١.٢: Let G be a group of automorphisms of R, and let $|G|=n$, be the order of G. Assume $|G|$ is invertible in R, hence is invertible in R^G. We define the trace of R to be the function $T(x)=\frac{1}{n}\sum_{i=1}^{n} g_i(x)$, for each $x\in R$, where $G=\{g_١=e_١, g_٢,\ldots g_n\}$.

Remark ١.٣:

١. Note that $T(x)\in R^G$. Moreover, if one considers R as an R^G -module, then T is an R^G-module, homomorphism. Observe that $1=\frac{1}{n}\sum_{i=1}^{n} g_i(1)$, and hence $T(R)=R^G$, and $T(١)=١$.

٢. $T(I)$ is an ideal in R^G; for each ideal I in R.

For any ring S, recall that for an S-module M, $\text{trace}(M)=T_S(M)=\sum_f f(M)$, where the sum is taken overall $f\in M^*=\underset{S}{Hom}(M,S)$, [٦]. And the S-module M is called a generator, if $T_S(M)=S$, [٩].

Since the map $T:R\to R^G$ is an R^G-homomorphism, then $T\in \underset{R^G}{Hom}(R,R^G)$

Moreover, by remark ١.٣, if $|G|^{-١}\in R$, then $T(R)=R^G$. Thus $\underset{R^G}{trace}\,R= T(R)=R^G$. Hence we have:

Proposition ١.٤: Let R be any ring, and let G be a finite group of automorphisms of R. If $|G|^{-١}\in R$, then R is a generator as an R^G -module.

An R -module M is called a self-generator, if for every cyclic submodule Ra; $Ra=\sum_{\varphi}\varphi(M)$, where the sum is taken over all $\varphi:M\to Ra$. This implies that for each submodule N of M; $N=\sum_{\varphi:M\to N}\varphi(M)$ where the sum is taken over all $\varphi:M\to N$, [٩]. And a special self-generator, if for each cyclic submodule N of M, there exists $\varphi:M\to N$ such that $\varphi(M)=N$. Thus every special self-generator is a self-generator, [٩].

Hence we have the following:

Proposition ١.٥: Let R be any ring, with $|G|^{-١}\in R$, then R is a special self-generator R^G -module.

Proof: Let $T: R \to R^G$ be defined by $T(x)=|G|^{-1}\sum_{i=1}^{n} g_i(x);\ \forall x\in R$, and let φ_a: $R^G\to R^G{}_\alpha$ be defined by $\varphi_a(r)=ra$, $\forall r\in R^G$. $\varphi_a oT$ is an R^G-homomorphism, and $(\varphi_a oT)(R^G)=R^G{}_\alpha$.

An R -module M is called a cancellation module, if whenever $AM=BM$ for some ideals A and B in R, then $A=B$, [١٢]. It is known that every faithful generator module is a cancellation module, [١٢], hence by proposition ١.٥, we have:

Corollary ١.٦: R is a cancellation R^G-module**,** provided $|G|^{-١}\in R$.

Recall that an R-module M is called a multiplication module, if for each submodule N of M, $N=IM$ for some ideal I of R, [٢].

Proposition ١.٧: Let R be any ring, and let G be a finite group of automorphisms of R. If R is a multi R plication R^G -module, With $|G|^{-١}\in R$ then R is a finitely generated R^G-module, and hence $\underset{R^G}{End}(R)=R^G$. Thus $\underset{R^G}{End}(R)$ is commutative.

Proof: R is a faithful R^G -module. Moreover, by corollary ١.٦, R is a cancellation R^G -module. Thus by [١٢], R is finitely generated. The last assertion follows from [٨].

٢. G-invariant ideals

In this section we study G-invariant ideals in R. In fact we study the relation between G-invariant ideals and their traces. First we start by the following definition.

Definition ٢.١: Let I be an ideal in a ring R, and let G be a group of automorphisms of R. I is said to be G-invariant ideal of R or G-ideal. if $g(I)\subseteq I$, $\forall g\in G$. Equivalently $g(I)=I$, for each $g\in G$, where $g(I)=\{g(a);\ a\in I\}$.

Examples and remarks ٢.٢:

١. If I is a G-invariant ideal, then $T(I)\subseteq I$. In fact, if $x\in T(I)$ then $x=\frac{1}{n}\sum_{k=1}^{n} g_k(a)$, for some $a\in I$, hence $x\in I$.

٢. For any set $I=\{a_i,\ a_i\in R^G,\ i\in\Lambda\}$, the ideal RI in R generated by the set I in a G-invariant ideal. In fact, if $x\in I$, then $x=\sum r_i a_i, r_i\in R, a_i\in I$.

Let $y\in T(RI)$, then $y=\frac{1}{n}\sum_{k=1}^{n} g_k(x)$, for some $x\in RI$. Hence

$y=\frac{1}{n}\sum_{k=1}^{n} g_k\left(\sum_i r_i a_i\right)=\frac{1}{n}\sum_i\sum_{k=1}^{n} g_k(r_i)a_i$, thus $y\in RI$. Therefore $T(RI)\subseteq RI$.

٣. If I is any subset of R^G, then $\underset{R}{ann}(I)$ is a G-invariant ideal. In particular for each

$x\in R^G$, $\underset{R}{ann}(a)$ in G-invariant ideal. In fact, $\forall r\in \underset{R}{ann}(I)$, and $\forall x\in I$, rx=٠. Thus ٠=$g(rx)$= $g(r)g(x)$= $g(r)x$, i.e. $g(r)\in \underset{R}{ann}(I)$.

٤. Note that every ideal of R^G is G-invariant ideal, but there are ideals not in as R^G, that are G-invariant as is seen by the following example. This example shows also that the trace of a G-invariant ideal may be {٠}.

٥. Let F be a field of characteristics ≠٢, and let $R=F[x_1, x_2, \dots]$ be the ring of polynomials in an infinite countable set of indetermines $x_1, x_2, \dots, x_m$, with the relation $x_i.x_j$=٠ $\forall i, j$, in particular x_i^2=٠, $\forall_i$. Define $g : R\to R$ by $g(x_i)=-x_i, \forall i$ and $g(a)=a, \forall a\in F$, and then extend this action to all the elements of R in the obvious way to make g an automorphism of g. It is clear that $g^2=e$, and if $G=\{e, g\}$ then $R^G=F$.

Now let $A=\{ax_1, a\in F\}$, then A is an ideal in R, A is a G-invariant ideal, since $g(ax_1)=g(a)g(x_1)=-ax_1$. However A is not contained in R^G. Note that $T(A)$=٠. Moreover, it is easily seen that every ideal in R is G-invariant, moreover, if $A_1=id(x_1)$, $A_2=id(x_2)$, then $T(A_1)$=٠, $T(A_2)$=٠ but $A_1\neq A_2$.

The following theorem shows that under certain conditions, $T(I)$ is not zero, but first we make the following simple remarks.

Remark ٢.٣: If I is a G-invariant ideal in R, then $T(I)=I\cap R^G$.

Remark ٢.٤: Let I be a subset of R^G. And let RI. R^GI be the ideals generated by I in R and R^G respectively, then $T(RI)=R^GI$.

We observed in ٢.٢ (example ٥) that $T(I)$ may be zero even if I is G-invariant. The following theorem shows that under certain conditions, $T(I)$ is not zero.

Theorem (Bergman-Issacs) ٢.٥. [٧]

Let R be any ring with no non zero nilpotent elements, and let I be a non zero G-invariant ideal of R, with $|G|^{-1}\in R$, then $T(I)\neq$ {٠}.

We need the following stronger result:

Proposition ٢.٦. [٣]: Let R be any ring which has no non-zero nilpotent elements, and $|G|^{-1}\in R$, then $T(I)$ does not vanish for any non-zero ideal I of R.

Proof: Suppose $T(I)$=٠, then $T(a)$=٠, for all $a\in I$. Let

$$J=\sum_{k=1}^{n} g_k(I)=\left\{\sum_{k=1}^{n} g_k(a), a\in I\right\}$$

Then it is easily checked that J is a G-invariant non-zero ideal of R. with $T(J)$={٠}. Thus by theorem ٢.٥, J={٠}, which is a contradiction.

Observe that example ٢.٢ (٥) shows that theorem ٢.٥ is false if the ring R has nilpotent elements.

Note: We will assume in what follows, that R is a ring with G a subgroup of $Aut(R)$, and $|G|^{-1}\in R$, unless otherwise stated.

Proposition ٢.٧: Let I be a G-invariant ideal of R. if I is a prime ideal in R. then $T(I)$ is a prime ideal in R^G.

Proof: Let $a, b\in R^G$, such that $a\cdot b\in T(I)$. Since I is G-invariant, then $a\cdot b\in I$, hence either $a\in I$ or $b\in I$. Assume $a\in I$ then $a\in I\cap R^G$. And by remark ٢.٣, $a\in T(I)$.

Corollary ٢.٨, [١١]: Let I be a subset of R^G, and let RI, R^GI be the ideals, generated by I in R and R^G respectively. if RI is a prime ideal in R then R^GI is a prime ideal in R^G.

Proof: By remark ٢.٤, $T(RI)=R^GI$, and hence by proposition ٢.٧ R^GI Is Prime.

Proposition ٢.٩: Let I be a maximal ideal in R, then $T(I)$ is a maximal ideal in R^G.

Proof: Let $y\in R^G$, and $y\notin T(I)$, then $y=T(y)$, and hence $y\notin I$. Since I is maximal in R, then $R=Ry+I$. Hence $1=ry+m$, where $r\in R$, $m\in I$.

Now $1=T(1)=T(ry)+T(m)=T(r)y+T(m)$, and thus $R^G=R^Gy+T(I)$ which implies that $T(I)$ is a maximal ideal in R^G.

Corollary ٢.١٠: Let I be a subset of R^G, and let RI, R^GI be the ideals generated by I in R and R^G respectively. If RI is maximal in R, then R^GI is maximal in R^G.

The converse of proposition ٢.٩ is not true even if I is G-invariant as example (٥) in remark ٢.٢ shows.

٣. small ideals and essential ideals

In this section we study the relation between small (essential) ideals in R and small (essential) ideals in R^G respectively.

Recall that an ideal I of a ring R is small, if whenever $I+J=R$, where J is ideal in R, then $J=R$, [١]. We have the following result.

Proposition ٣.١: Let I be a G-invariant ideal in R. if I is small in R, then $T(I)$ is small in R^G. The converse is true if every ideal in R Is G-invariant.

Proof: Let I be a small ideal in R, to prove $T(I)$ is small in R^G. Let $T(I)+J=R^G$, where J is an ideal in R^G. Hence $1=T(a)+b$, where $a\in I$ and $b\in J$. Since I is G-invariant, then $T(I)\subseteq I$, hence $\forall x\in R$, $x=x(T(a)+b)$, and thus $R=I+RJ$. But I is small in R, then $RJ=R$. Now, $J=R^GJ=T(RJ)=T(R)=R^G$. Thus $T(I)$ is small in R^G.

To prove the converse, let $I+K=R$, where K is an ideal in R. Then $T(I)+T(K)=T(R)=R^G$, thus $T(I)+K\cap R^G=R^G$. But $T(I)$ is small in R^G, then $K\cap R^G=R^G$, hence $K\supseteq R^G$. But $1\in R^G$, then $1\in K$, and $K=R$.

Corollary ٣.٢: Let I be a subset of R^G, and let RI. R^GI be the ideals generated by I in R and R^G respectively. If RI is small in R, then R^GI is small in R^G. The converse is true if every ideal in R is G-invariant.

Let us say that a ring R satisfies (A.C.C.) on small ideals if every ascending chain of small ideals is stationary.

Corollary ٣.٣: Let R be a ring that satisfies A.C.C. on small ideals. If ever ideal in R is G-invariant, then R satisfies A.C.C. on small ideals.

Proof: Let $I_1\subseteq I_2\subseteq\ldots\subseteq I_n\subseteq\ldots$ be an ascending chain of small ideals in R, and let $RI_1,RI_2,\ldots,RI_n,\ldots$ be the ideals in R generated by $I_1,I_2,\ldots,I_n,\ldots$ respectively. Hence $RI_1\subseteq RI_2\subseteq\ldots\subseteq RI_n\subseteq\ldots$ is an ascending chain of small ideals in R (by proposition ٣.٢). Then $\exists k\in N$, such that $RI_k=RI_{k+1}=\ldots$, hence $T(RI_K)=T(RI_{K+1})=\ldots$, and by remark ٢.٤, $R^GI_k=R^GI_{k+1}=\ldots$.

The converse of corollary ٣.٣ is false. In remark ٢.٢ example (٥) $\forall n\in N$, let $I_n=id\{x_1,x_2,\ldots,x_n\}$. Then $I_1\subseteq I_2\subseteq\ldots\subseteq I_n\subseteq\ldots$. Each of the ideals I_n is small. In fact, if $I_n+J=R$, where J is an ideal in R, then $1=a_1x_1+a_2x_2+\ldots+a_nx_n+b$ where $b\in J$, and $a_i\in F$, $I\le i\le n$. Thus $b=1-\sum_{i=1}^{n}a_ix_i$. It is easily seen that b is invertible in R, in fact $b^{-1}=1+\sum_{i=1}^{n}a_ix_i$, thus $J=R$. This shows that R does not satisfy A.C.C. on small ideals. However $R^G=F$ satisfies this condition.

Recall that an ideal I of a ring R is said to be essential, if whenever $I\cap J=٠$, where J is an ideal in R implies that $J=٠$. Equivalently, I is essential if and only if $\forall ٠\neq r\in R$, $\exists t\in R$, such that $٠\neq rt\in I$, [١٤].

We have the following result:

Proposition ٣.٤: Let R be any ring which has no non-zero nilpotent elements. If I is an essential ideal in R, then $T(I)$ is an essential ideal in R^G. The converse is true if I is G-invariant.

Proof: Let I be an essential ideal in R, and let $٠\neq x\in R^G$, then $x\in R$, hence $\exists y\in R$, such that $٠\neq xy\in I$. Let $A=id(xy)$ in R, then by proposition ٢.٦ $\exists a\in A$, such that $T(a)\neq 0$. Hence $\exists r\in R$, such that $a=rxy$, thus $0\neq T(rxy)=xT(ry)\in T(I)$.

Now let $T(I)$ be an essential ideal in R, and let $0\neq x=R$. Let $B=id(x)$ in R, then by proposition ٢.٦, $\exists b\in B$, such that $0\neq T(b)$ in R^G. But $T(I)$ is essential in R, thus $\exists 0\neq y\in R$ such that $0\neq yT(b)\in T(I)$. Hence $0\neq T(yrx)$, thus $yrx\in I$, since if $yrx\notin I$, then $T(yrx)\notin T(I)$, and $yrx\neq 0$, for if $yrx=0$, then $0=T(yrx)=yT(rx)$, a contradiction, hence I is essential in R.

Corollary ٣.٥: Let R be any ring which has no non-zero nilpotent elements. Let I be a subset of R^G, and let RI, R^GI be the ideals generated by I in R and R^G respectively, then R^GI is essential in R^G iff RI is essential in R.

٤. Annihilator ideals

In this section we study annihilator ideals. First we recall the following definition.

Definition ٤.l: An ideal I of a ring R is called an annihilator ideal, if I is the annihilator of some subset of R.

Note that if $I=\underset{R}{ann}(S)$, for some subset of R. then $I=\underset{R}{ann}(RS)$.

It is well known, and is easy to check, that I is an annihilator ideal in R if and only if $I=\underset{R}{ann}\,\underset{R}{ann}(I)$, [١٤].

We prove the following theorem:

Theorem ٤.٢: Let I be a subset of R^G. and let RI, $R^G I$ be the ideals generated by I in R and R^G respectively. If $R^G I$ is an annihilator ideal in R^G, then RI is an annihilator ideal in R. The converse is true if R has no non-zero nilpotent elements.

Proof: Let $R^G I = \underset{R^G}{ann}(S)$, where S is a subset of R^G. We claim that $RI = \underset{R}{ann}(RS)$, where RS is the ideal in R generated by S. Let $a \in I$, then $a \in R^G I$, thus $a(RS) = (aS)R = 0$, hence $RI \subseteq \underset{R}{ann}(RS)$.

Now let $x \in \underset{R}{ann}(RS)$, then $xRS = 0$, and $xS = 0$. Thus $T(x)S = 0$ and $T(x) \in \underset{R^G}{ann}(S)$. Therefore, $T(x) \in R^G I$, hence $x \in RI$, that is $ann(RS) \subseteq RI$. Thus $RI = \underset{R}{ann}(RS)$.

To prove the converse, let $RI = \underset{R}{ann}(S)$, where S is an ideal in R. We claim that $R^G I = \underset{R^G}{ann}(T(S))$, where $T: R \to R^G$ is the trace map. Let $a \in R^G I$, then $a \in \underset{R^G}{ann}(S)$, and $aS = 0$. Therefore, $0 = aT(S)$, hence $a \in \underset{R^G}{ann}(T(S))$ and $R^G I \subseteq \underset{R^G}{ann}(T(S))$.

Now let $b \in \underset{R^G}{ann}(T(S))$, then $bT(S) = 0$, $\forall s \in S$. Hence $T(bs) = 0, \forall s \in S$, thus $T(bS) = 0$. But R has no non-zero nilpotent elements, then by proposition ٢.٦, $bS = 0$, and $b \in \underset{R^G}{ann}(S) \subseteq \underset{R}{ann}(S)$, hence $b \in RI \cap R^G = R^G I$.

A ring R is said to satisfy the ascending chain condition (A.C.C) on annihilator ideals, if every ascending chain of annihilator ideals $I_١ \subseteq I_٢ \subseteq \ldots$ terminates after a finite number of steps, that is there exists $k \in N$, such that: $I_k = I_{k-١}$, [٤].

A ring R is said to have no infinite direct sum of non-zero ideals if every direct sum of non-zero ideals in R has a finite number of terms, [٤].

Recall that a ring R is said to be Goldie ring, if R satisfies the (A.C.C.) on annihilator ideals and R has no infinite direct sum of ideals, [٤].

We prove the following:

Proposition ٤.٣: If R satisfies the (A.C.C.) on annihilator ideals, then R satisfies the (A.C.C.) on annihilator ideals.

Proof: Let $I_١ \subseteq I_٢ \subseteq \ldots \subseteq I_n \subseteq$ be an ascending chain of annihilator ideals in R^G. Let RI_j be the ideal generated by I_j, $j=١, ٢, \ldots$ Then by theorem ٤.٢, RI_j, $j = 1, \ldots, n, \ldots$, is an annihilator ideal in R. Moreover $RI_1 \subseteq RI_2 \subseteq \ldots \subseteq RI_n \subseteq \ldots$ is an ascending chain, thus $\exists k \in N$ such that: $RI_k = RI_{k+1} = \ldots$, hence $RI_k \cap R^G = RI_{k+1} \cap R^G = \ldots$. i.e. $R^G I_k = R^G I_{k+1} = \ldots$.

Proposition ٤.٤: Let R be any ring which has no non-zero nilpotent elements. If R has no infinite direct sum of non-zero ideals, then R^G has no infinite direct sum non-zero ideals.

Proof: Let $R^G I_1 \oplus R^G I_2 \oplus \ldots$ be a direct sum of ideals in R^G where $I_n \subseteq R^G$, $\forall n$. Let RI_n be the ideal in R generated by I_n, $\forall n$. Let $RJ_n = \oplus \sum_{k \neq n} RI_k$, $R^G J_n = \oplus \sum_{k \neq n} R^G I_k$, then $R^G J_n \cap R^G I_n = \{0\}$.

Now,

$T(RJ_n \cap RI_n) = T(RI_n) \cap T(RJ_n) = R^G J_n \cap R^G I_n = \{0\}$. Thus by proposition ٢.٦ $RJ_n \cap RI_n = 0$, hence $\sum_n RI_n$ is a direct sum in R, hence is finite. Thus $\oplus \sum_n R^G I$ is finite.

Propositions ٤.٣ and ٤.٤ give the following:

Theorem ٤.٥: If R is a Goldie ring which has no non-zero nilpotent elements, then R^G is a Goldie ring.

The converse of theorem ٤.٥ is true, [٧]

Proposition ٤.٦: Let R be any ring which has no non-zero nilpotent elements. If R^G is a Goldie ring then R is a Goldie ring.

٥. The R^G -module as a submodule of a free R^G -module

In this section we state a theorem of Montgomery, [٧], which shows that under certain condition on R, R may be considered as a submodule of a free R^G -module of finite rank, and we will use this theorem to prove that under extra conditions on RI, RI is a C.P(C.F) module.

Theorem ٥.١ [٧]: Let R be a Goldie ring which has no non-zero nilpotent elements. Then R can be embedded in $\overset{m}{\underset{i=1}{\oplus}} \sum R^G$ as R^G -module of finite rank.

It is known that if R is Noetherian (Artinian) ring, with $|G|^{-1} \in R$, then R^G is a Noetherian (an Artinian) ring. Moreover, it was proved by Farkas and Sinder in [٣], that if R^G is a

Noetherian ring which has no non-zero nilpotent elements, then R is a Noetherian ring. We give a similar result for Artinian rings.

Proposition ٥.٢: Let R^G be an Artinian ring which has no non-zero nilpotent elements, then R is a finitely generated R^G -module. In fact R is an Artinian R^G -module in particular, R is an Artinian ring.

Proof: Since R^G is Artinian, then R^G is Goldie, then by, [٧], R is a Goldie ring, and by theorem ٥.١, R is isomorphic to a submodule of the free R^G -module $\left(\overset{m}{\underset{i=1}{\oplus}}\sum R^G\right)$ since R^G is Artinian hence Noetherian, $\overset{m}{\underset{i=1}{\oplus}}\sum R^G$ is Artinian R^G -module, and $\overset{m}{\underset{i=1}{\oplus}}\sum R^G$ is Noetherian R^G -module. And thus R is finitely generated R^G -module. And since every finitely generated module over an Artinian ring is Artinian, [١], then R is an Artinian R^G -module.
Recall that an R -module M is called a C.P module, if every cyclic submodule of M is projective, and M is a C.F module, if every cyclic submodule of M is flat. The following results are well known, [١٠].

Proposition ٥.٣: If R is a P.P. ring, and M is a projective R -module, then M is a C.P module. And if R is a hereditary (semi-hereditary) ring, and M is a projective R -module, then every submodule (finitely generated submodule) of M is projective.

Proposition ٥.٤, [٥]: If R is a P.F. ring and M is a flat R -module, then M is a C.F module. And if R is an F.F. ring and M is a flat R -module, then every submodule of M is flat.
Hence we have:

Theorem ٥.٥, [٥]: Let R be a Goldie ring which has no non-nilpotent elements. If R^G is a hereditary ring, then R is projective R^G -module and every submodule of R is projective. In particular R is a C.P. R^G -module.

Theorem ٥.٦, [٥]: let R be a Noetherian ring which has no non-zero nilpotent elements. If R^G is a hereditary ring, then R is a finitely generated projective R^G -module, and every submodule of R is projective. In particular R is a C.P. R^G -module.
Next we give similar results for flat rings.

Theorem ٥.٧: Let R be a Goldie ring which has non-zero nilpotent elements. If R is a flat ring, then R is a flat R^G -module, and every R^G -submodule of R is flat, hence R is a C.F. R^G -module.

Proof: By theorem ٥.١ R is isomorphic to a submodule of the free R^G -module $\left(\overset{m}{\underset{i=1}{\oplus}}\sum R^G\right)$ of finite rank, hence flat, [٦]. And by proposition ٥.٤; every R^G -submodule of R is flat, and hence R is a C.F. R^G -module.
Recall that an R -module M is faithfully flat, if and only if M is a flat cancellation module, [١٢]. Thus we have:

Corollary ٥.٨: Let R be a flat Goldie ring, then R is a faithfully flat R^G -module.

Proof: By Corollary ١.٦, R is a cancellation R^G -module. Now since R is isomorphic to a submodule of the free R^G -module $\left(\overset{m}{\underset{i=1}{\oplus}}\sum R^G\right)$, and R is flat, then by [١٣], R^G is a flat ring. But $\overset{m}{\underset{i=1}{\oplus}}\sum R^G$ is a flat R^G -module, hence by proposition ٥.٤ R is a flat R^G -module, then R is a faithfully flat R^G -module.

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