JORDAN LEFT DERIVATIONS AND GENERALIZED JORDAN LEFT DERIVATIONS ON RINGS

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Abstract

In this paper we have studied Jordan left derivation of a ٢-torsion free ring which has a commutator left non- zero divisor and we initiate the study of generalized Jordan left derivation of a ٢-torsion free ring, which has a commutator left non- zero divisor.

اشتقاقات جوردان اليسارية واشتقاقات جوردان اليسارية المعممة على الحلقات

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الخلاصة

في هذا البحث سنتطرق لدراسة أشتقاق جوردان اليساري على الحلقة طليقة الألتواء من النمط الثاني التي تمتلك مبادلاً لا يكون قاسم صفري يساري وكذلك سنقدم دراسة حول تعميم أشتقاق جوردان اليساري على الحلقة طليقة الألتواء من النمط الثاني التي تمتلك مبادلاً لا يكون قاسم صفري يساري.

١. Introduction

Throughout the present paper R will denote an associative ring with center Z. Recall that R is a ٢-torsion free if x in R, ٢x =٠ implies that x=٠. As usual [x, y] will denote the commutator xy-yx. An additive subgroup U of R is said to be a Lie ideal of R if [u, r]∈ U for all u ∈U, r∈R, and a ring R has a commutator left non-zero divisor means that there exist a, b∈R such that [a, b] c=٠ for every c∈R implies that c=٠. An additive mapping d: R→R is called a left derivation (resp., Jordan left derivation) if d(xy)=x d(y)+yd(x) (resp., $d(x^2)$=٢xd(x)) hold for all x, y ∈R. In [١] Brešar and Vukman have proved that the existence of a nonzero Jordan left derivation on a prime ring *R* of char *R* ≠ ٢, ٣ forces *R* to be commutative. It should be mentioned that the result obtained in [١] concerning Jordan left derivation has been improved by Deng [٢]. Some more related results can be seen in [٣, ٤, ٥, ٦]. It is easy to see that every left derivation on a ring *R* is a Jordan left derivation. However, in general, a Jordan left derivation need not be a left derivation, the following example justifies this statement.

Example:-[٧, Example ١.١]

Let R be a commutative ring and let a∈R such that x a x=٠ for all x ∈ R but xay≠٠, for some x and y, x≠y. Define a map d: R→R as follows

d (x)= x a + ax for all x ∈ R

Then d is a Jordan left derivation but not a left derivation.

Recently M. Ashraf and Nadeem–ur-Rehman in [٣] proved that, let R be a ٢- torsion free prime ring and let U be a Lie ideal of R such that u^2∈U for all u ∈U. If d:R→R is an additive map satisfying d(u^2)=٢ud(u) for all u ∈U. Then

d(u v)=u d(v)+v d(u) for all u ,v ∈U .

In this paper (**Section two**), we have studied the concept of Jordan left derivation on a Lie ideal of a ٢- torsion free ring which has a commutator left non zero divisor when we have proved the following theorem:

Theorem:- Let R be a ٢-torsion free ring which has a commutator left non- zero divisor and let U be a Lie ideal of R such that u^2 ∈U for all u∈U. If d is an additive map of R into itself satisfying d (u^2) =٢ud (u) for all u ∈U. Then

d (uv)=u d(v)+v d(u) for all u ,v ∈U.

In ***section three***, we shall introduce the definition of generalized Jordan left derivation and we have studied this concept on a ٢-torsion free ring which has a commutator left non-zero divisor by proving the following theorem:

Theorem:-Let R be a ٢-torsion free ring which has a commutator left non zero divisor and let U be a Lie ideal of R such that u^2∈U for all u∈U. Then every generalized Jordan left derivation on U is a generalized left derivation on U.

٢. Jordan Left Derivation

In this section, we shall study the concept of Jordan left derivation on a ٢-torsion free ring which has a commutator left non zero divisor.

Lemma ٢.١:-[٣, Lemma ٢.٢]
Let R be a ٢- torsion free ring and let U be a Lie ideal of R such that u^2□U for all u□U. If d:R→R is an additive map satisfying d(u^2)=٢ud (u) for all u□U. Then for all u, v, w□U

(i). d (u v +vu) =٢ u d(v)+٢ v d(u)
(ii). d (u vu) = u^2 d(v) +٣uvd(u)-v u d(u)
(iii). d(u v w + w v u)=(u w +w u)d(v)+٣uv d(w)+ ٣ w v d(u)-v u d(w)- v w d(u)
(iv). [u, v] u d(u)=u[u ,v]d(u)
(v). [u , v] (d(u v)-u d(v)-v d(u))=٠.

Remark ٢.٢:- Let R be a ٢-torsion free ring and let U be a Lie ideal of R such that u^2□U for all u□U. If d:R→R is an additive map satisfying d(u^2)=٢ud(u) for all u□U. Then for the purpose of this section, we shall write u^v =d(u v)-u d(v)-v d(u) for all u, v□U.
Remark that, it is easy to prove the following properties

(i).$u^v + v^u$ =٠
(ii).$u^{v+w} = u^v + u^w$
(iii).$(u+w)^v = u^v + w^v$

Now we can prove the main result of this section:

Theorem٢.٣:- Let R be a ٢-torsion free ring which has a commutator left non-zero divisor and let U be a Lie ideal of R such that u^2□U for all u□U. If d is an additive map satisfying d(u^2)=٢ud(u) for all u□U. Then
d(u v)=u d(v)+v d(u) for all u, v ∈U.

Proof:-

By [Lemma ٢.١,(v)]

[u, v](d (uv)-u d(v)-vd(u))=٠ for all u, v∈U

i.e. [u, v] u^v =٠ for all u, v∈U…………. (١)

By assumption of R has a commutator left non-zero divisor. Then there exist elements a and b of U such that

[a, b] c=٠ implies c=٠ for every c∈R….(٢)

By (١) we get

[a, b] a^b =٠ and so by (٢), we get a^b =٠..…(٣)

In (١) replace u by u +a

[u+ a, v]$(u+ a)^v$=٠ for all u, v ∈U . Then by [Remark ٢.٢, (iii)]

([u, v]+[a, v])(u^v+ a^v)=٠

[u, v] u^v+[u, v] a^v+[a, v] u^v+[a, v] a^v=٠

Then by using (١) we get

[u, v]a^v+[a, v]u^v=٠ for all u, v ∈U

Now replace v by v+ b and again using [Remark ٢.٢, (iii)]

$[u,v+b]\, a^{v+b}+[a,v+b]u^{v+b}=٠$

$[a, v]a^{v}+[u, v]a^{b}+[u, b]a^{v}+[u, b]a^{b}+[a ,v]u^{v}+[a ,v]u^{b}+[a,b]u^{b}=٠$

Then by using (٣) and (٤) we get

$[a,b]\, u^{v}=٠$ for all u ,v ∈U. Then by (٢) we get $u^{v}=٠$ for all u, v∈U .i.e

d(uv)=u d(v)+v d(u) for all u ,v ∈U .

Then we get the result

Corollary ٢.٤:- Let R be a ٢- torsion free ring which has a commutator left non-zero divisors and let d:R→R is a Jordan left derivation on R. Then d is a left derivation on R.

٣. Generalized Jordan Left Derivation

Throughout this section, R will be an associative ring, and U be a Lie ideal of R such that u²∈U for all u∈U. Now, we shall introduce the definition of generalized Jordan left derivation and the definition of generalized left derivation.

Definition ٣.١:- Let δ:R→R be an additive map, if there is a left derivation d:R → R such that

δ(a b) = a δ(b)+b d(a) for all a, b ∈R.

Then δ is called a ***generalized left derivation*** and d is called the relating left derivation.

Definition ٣.٢:- Let δ:R→R be an additive map, if there is a Jordan left derivation d:R→ R such that

δ(a²) = a δ(a)+a d(a) for all a ∈R.

Then δ is called a ***generalized Jordan left derivation*** and d is called the relating Jordan left derivation .

Now we shall study the concept of generalized Jordan left derivation on a ٢-torsion free ring which has a commutator left non zero divisor as follows:

Lemma ٣.٣:- Let R be a ٢-torsion free ring and let U be a Lie ideal of R such that u²∈U for all u∈U. Let δ:R→R be a generalized Jordan left derivation and d:R→R the relating Jordan left derivation. Then for all u, v∈U, δ satisfying the following:

(i).δ(uv+vu)=u δ(v)+v d(u)+v δ(u)+u d(v)

(ii).δ(uvu)= uv δ(u)+٢ uv d(u)-vu d(u)+ u²d(v)

(iii).δ(uvw+wvu)=uvδ(w)+wvδ(u)+٢uvd(w) -vud(w)+٢wvd(u)-vwd(u)+uwd(v)+wud(v)

(iv).[u , v](δ(u v)-u δ(v)-v d(u)) =٠.

Proof:-

(i) Since

uv+vu= (u +v) ²- u²- v²

Then

u v+ vu ∈ U ,and so

δ(uv+vu) = δ((u +v)²) – δ(u²)- δ(v²)

= (u +v)δ (u +v) +(u +v) d(u +v)- uδ(u)-u d(u)- v δ(v)-v d(v)

= uδ(u)+uδ(v)+vδ(u)+vδ(v)+ ud(u)+ud(v)+vd(u)+vd(v)- uδ(u)-ud(u) -vδ(v)-vd(v)

Then δ(uv+vu)=u δ(v)+v d(u)+v δ(u)+u d(v) for all u ,v ∈ U.

(ii) BY replacing v by u v+ vu in (i), we get

W= δ (u(uv+vu)+(uv+vu)u)

= u δ(uv+vu)+ (uv+vu)d(u)+ (uv+vu) δ (u)+u d(uv+vu)

Then

W=u(uδ(v)+vd(u)+vδ(u)+ud(v))+ uvd(u) +vud(u)+uvδ(u)+vuδ(u)+٢u²d(v)+٢u v d(u)

=u²δ(v)+٤uvd(u)+٢uvδ(u)+٣u²d(v)+ vu d(u)+ vu δ(u).

On the other hand

W = δ (u(u v + vu)+(u v + vu) u)

= δ (u²v+٢ u v u +v u²)

= δ (u²v+ v u²)+٢ δ(uvu)

= u² δ(v)+ v d(u²)+v δ(u²)+ u² d(v)+ ٢ δ(uvu)

= u²δ(v)+٢vud(u)+ u² d(v)+ ٢ δ(uvu).

By comparing these two expressions of W. We get

٢δ(uvu)+٢vud(u)-٢u²d(v)-٤uvd(u)-٢uvδ(u)=٠.

Since R is ٢-torsion free ring. Then we get

δ(uvu)=uvδ(u)+٢uvd(u)+u²d(v)-vud(u)

for all u, v ∈ U.

(iii) By replacing u by u+w in (ii)

$W=\delta((u+w)v(u+w))$
$=(u+w)v\delta(u+w)+((u+w)v-v(u+w))d(u+w)+(u+w)vd(u+w)+(u+w)^2d(v)$

$W=uv\delta(u)+uv\delta(w)+wv\delta(u)+wv\delta(w)+(uv-vu)d(u)+(uv-vu)d(w)+(wv-vw)d(u)+(wv-vw)d(w)+uvd(u)+uvd(w)+wvd(u)+wvd(u)+u^2d(v)+w^2d(v)+uw\,d(v)+wud(v)$

$W=uv\delta(u)+uv\delta(w)+wv\delta(u)+wv\delta(w)+$ ٢$uvd(u)-vud(u)+$٢$uvd(w)-vud(w)+$ ٢$wvd(u)-vwd(u)+$٢$wvd(w)-vw\ d(w)+u^2d(v)+w^2d(v)+u\,w\,d(v)+w\,u\,d(v).$

On the other hand

$W=\delta((u+w)v(u+w))$
$=\delta(u\,v\,u+w\,v\,w+u\,v\,w+w\,v\,u)$
$=\delta(uvu)+\delta(w\,v\,w)+\delta(uvw+wvu)$
$=uv\delta(u)+(uv-vu)d(u)+uvd(u)+u^2d(v)+wv\delta(w)+(wv-vw)d(w)+wvd(w)+w^2d(v)+\delta(uvw+wvu).$

Then by comparing these two expression of W. We get

٢$\delta(uvu)+$٢$vud(u)-$٤$uvd(u)-$٢$uv\delta(u)-$٢$u^2d(v)=$٠

Since R is ٢-torsion free ring .Then we get

$\delta(uvu)=uv\delta(u)-$٢$uvd(u)+u^2d(v)-vud(u)$ for all $u,v\in U$.

(iv) By replace u by u +w in (ii)

$W=\delta((u+w)v(u+w))$
$=(u+w)v\delta(u+w)+((u+w)v-v(u+w))d(u+w)+(u+w)vd(u+w)+(u+w)^2d(v)$
$=uv\delta(u)+uv\delta(w)+wv\delta(u)+wv\delta(w)+(uv-vu)d(u)+(uv-vu)d(w)+(wv-vw)d(u)+(wv-vw)d(w)+uvd(u)+uvd(w)+wvd(u)+wvd(w)+u^2d(v)+w^2d(v)+uw\,d(v)+wud(v)$
$=uv\delta(u)+uv\delta(w)+wv\delta(u)+wv\delta(w)+$ ٢$uvd(u)-vud(u)+$٢$uvd(w)-vud(w)+$ ٢$wvd(u)-vwd(u)+$٢$wvd(w)-vwd(w)+u^2d(v)+w^2d(v)+uwd(v)+wud(v).$

On the other hand

$W=\delta((u+w)v(u+w))$
$=\delta(u\,v\,u+w\,v\,w+uvw+wvu)$
$=\delta(uvu)+\delta(wvw)+\delta(uvw+wvu)$
$=uv\delta(u)+(uv-vu)d(u)+uvd(u)+u^2d(v)+wv\delta(w)+(w\,v-v\,w)d(w)+w\,vd(w)+w^2d(v)+\delta(uvw+wvu).$

Then by comparing these two expressions of W. We get

$\delta(uvw+wvu)=uv\delta(w)+wv\ \delta(u)+٢uvd(w)-vud(w)+٢wvd(u)-vwd(u)+uwd(v)+wud(v).$

(iv) In (iii) replace w by [u, v] .Then

$Y=\delta(uv[u,v]+[u,v]vu)$
$Y=uv\delta([u,v])+[u,v]v\delta(u)+٢uvd([u,v])-vud([u,v])+٢[u,v]vd(u)-v[u,v]d(u)+u[u,v]d(v)+[u,v]ud(v)$
$Y=uv\delta(uv)-uv\delta(vu)+[u,v]v\delta(u)+vd([u,v])+٢[u,v]v\,d(u)-v[u,v]d(u)+u[u,v]d(v)+[u,v]ud(v).$

On the other hand.

$Y=\delta(u\,v(uv-vu)+(uv-vu)vu)$
$=\delta((u\,v)^2-uv^2u+uv^2u-(vu)^2)$
$=\delta((uv)^2-(vu)^2)$
$=uv\delta(uv)+uvd(uv)-vu\delta(vu)-vud(vu).$

Then by comparing these two expressions of Y. We get

$-[u,v](\delta(vu)-v\delta(u)-ud(v))+uvd([u,v])+٢[u,v]vd(u)-v[u,v]d(u)+[u,v]d(uv)-v[u,v]d(u)-uvd(uv)+vud(vu)=$٠

Then

$-[u,v](\delta(vu)-v\delta(u)-ud(v))+uvd([u,v])+٢[u,v]vd(u)-٢v[u,v]d(u)+uvd(uv)-vu\,d(uv)-uvd(uv)+vud(vu)=$٠
$-[u,v](\delta(vu)-v\delta(u)-d(v))+[u,v]d([u,v])+٢[u,v]v\,d(u)-٢v[u,v]d(u)=$٠.

Then by [٣,Lemma ٢.٣].

We have $[u,v]d([u,v])=$٠ for all $u,v\in U$, then

$-[u,v](\delta(vu)-v\delta(u)-ud(v))$
$=$٢$(v[u,v]-[u,v]v)d(u)$
$=-$٢$(uv^2-$٢$vuv-v^2u)d(u)$

Then by proof of [١, lemma ٢.٣, (ii)]. We have

(u^2v-٢uvu+vu^2)d(v)-(v^2u-٢vuv+uv^2)d(u)=٠ for all u ,v ∈U.

Then we get

-[u, v](δ(vu)-vδ(u)-ud(v))
= - ٢(u^2v-٢uvu+vu^2)d(v)

and by [١,lemma٢.٣,ii] .We have

(u^2v-٢uvu+vu^2)d(v)=٠ for all u ,v ∈U.

Then we get

-[u ,v](δ(vu)-vδ(u)-ud(v))=٠, and so
[u ,v](δ(uv)-vδ(v)-v d(u))=٠

This completes the proof of the above lemma.

Remark٣.٤:- Let R be a ٢-torsion free ring and let U be a Lie ideal of R such that u^2∈U for all u∈U. Let δ:R→R be a generalized Jordan left derivation and d:R→R the relating Jordan left derivation .Then for the purpose of this section we shall write

u^v= δ(u v)-u δ(v)-v d(u) for all u ,v ∈U.

Remark that, it is easy to prove the following properties

(i). u^v +v^u=٠
(ii). u^{v+w}= u^v +u^w
(iii). $(u+w)^v$= u^v + w^v

Now we can prove the main result of this section.

Theorem ٣.٥:- Let R be a ٢-torsion free ring which has a commutator left non-zero divisor, and let U be a Lie ideal of R such that u^2∈U for all u∈U. Then every generalized Jordan left derivation on U is a generalized left derivation on U.

Proof:-Suppose that δ:R→R is a generalized Jordan left derivation on U and d:R→R is the relating Jordan left derivation on U. By [Lemma ٣.٣, (iv)], we have

[u, v](δ(uv)-uδ(v)-vd(u))=٠
for all u, v ∈U

i. e. [u, v] u^v=٠ for all u, v∈U……...(١)

By assumption of R has a commutator left non zero divisor, then there exist element a and b of U such that

[a, b]c=٠ implies c=٠ for every c∈R....(٢)

From (١) we get

[a,b]a^b=٠ and so by (٢), we get
a^b=٠……………………………...…….(٣)

In (١) replace u by u+a and using [Remark ٣.٤, (iii)] and (١).We get

[u, v]a^v+[a, v]u^v=٠ for all u, v∈U……(٤)

Now replace v by v+b and using [Remark ٣.٤, (iii)], we get

[u, v]a^v+[u, v]a^b+[u, b]a^v+[u, b]a^b+[a, v]u^v+[a, v]u^b+[a, b]u^v+[a, b]u^b=٠

Then by using (٣) and (٤).We get

[a,b] u^v=٠ for all u ,v ∈U.

Then by (٢) we get u^v=٠ for all u, v ∈U

δ(uv)=uδ(v)+vd(u) for all u, v ∈U.

Corollary ٣.٦:-Let R be a ٢- torsion free ring which has a commutator left non-zero divisor. Then every generalized Jordan left derivation is a generalized left derivation.

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