ON STRONGLY PRIME SUBMODULES

A. G. Naoum, I. M. A. Hadi*
department of mathematics, college of Science, university of Baghdad. Baghdad- Iraq
*department of mathematics, Ibn al-Haitham college of Education, university of Baghdad. Baghdad- Iraq

Abstract

Let R be an integral domain with quotient field K. A prime ideal P of R is called a strongly prime ideal of R if for each $x, y \in$ K, $x\,y \in$ P implies $x \in$ P or $y \in$ P. In this paper, we generalize this concept to submodules, thus we define strongly prime submodules and give some of their properties.

حول المقاسات الجزئية الأولية بقوة

عادل غسان نعوم، أنعام محمد علي هادي*
قسم الرياضيات، كلية العلوم، جامعة بغداد. بغداد-العراق.
*قسم الرياضيات، كلية التربية ابن الهيثم، جامعة بغداد. بغداد-العراق.

الخلاصة

لتكن R ساحة و K حقل القسمة لها. يقال ان المثالي الاولي P مثالي أولي بقوة اذا كان لكل $x, y \in$ K و$x\,y \in$ P يؤدي الى $x \in$ P أو $y \in$ P. في هذا البحث نعمم هذا المفهوم الى المقاسات الجزئية، ونعرف المقاسات الجزئية الاولية بقوة ونعطي بعض خواصها.

Introduction

Let R be an integral domain with quotient field K. Following ١, we say that an ideal P of R is strongly prime if P is a prime ideal and $x, y \in$P for $x, y \in$ K, implies $x \in$ P or $y \in$ P. Note that primeness follows from second condition.

In this paper, we give a generalization of this concept to modules, and we study the basic properties. Among other things we study strongly prime submodules in multiplication modules. We also use two known constructions IN ٢, ٣ to construct examples of strongly prime submodules. In ١, an integral domain is called a pseudo valuation domain (PVD) if every prime ideal of R is strongly prime. It is shown in ٤, that every valuation domain is a PVD.

In the last section of our paper we study the relation between strongly prime submodules and PVD.

Finally, we remark that R in this paper stands for an integral domain with quotient field. And M stands for a (left) unitary R module.

S.١ Strongly Prime Submodules (Basic Results)

Let R be an integral domain with quotient field K.

Recall that a prime ideal P of R called a strongly prime ideal (briefly S-prime ideal) if, whenever x, y $\in$ K and x, y $\in$ P, then x $\in$ P or y $\in$ P, ١.

Houston in ١ proved the following.

Proposition ١.١

Let R be an integral domain with quotient field K and P ia a prime ideal of R. Then P is an S-prime ideal iff r – ١ p $\subseteq$ p whenever r $\in$ K\R.

As an extension of the concept of an S-prime ideal to submodules. We proceed as following:

Let R be an integral domain with quotient field K and let M be an R-module. Let N an

R-submodule of M. Foe each $t=\frac{a}{b}\in$ K\R, we say that tN ⊆ N if for each x ∈ N, there exists y ∈ N such that a x = b y, ٥. In this case we write $y=\frac{a}{b}x$. Note that if N is torsion free, then y is unique.

Recall that a submodule P of an R-module M is prime if P is proper and r x ∈ P implies x ∈ P or r ∈ (P:M) when x ∈ M, r ∈ R, ٦, ٧, ٨.

Definition ١.٢

Let R be an integral domain with quotient field K. A submodule P of the R-module M is called strongly prime (briefly s-prime submodule) if whenever r∈K, x∈M implies x∈P or r∈(P:M).

The following is a characterization of s-prime submodules.

Proposition ١.٣

Let P be a prime submodule of the R-module M. Then the following are equivalent:

P is an s-prime submodule

r – ١ p ⊆ p, ∀ r ∈ K\R.

r x∈P implies x∈ P, when x∈M and r∈K\R.

proof: **١⇒٢:**Assume P is an s-prime submodule of M. Let y=r–١x where r∈K\R, x∈P. To prove y∈P. Now r y=r (r–١x)=x∈P. Hence y∈P by definition ١.٢.

٢⇒٣: Let y= r x∈P, where r∈K\R, x∈M. Then x=r–١y; that is x∈r–١P. But r – ١ p⊆p, by (٢). Thus x∈P.

٣⇒١: Let y = r x∈P, where r∈K, x∈M. Assume y∈P. If r∈P, then there is nothing to prove since P is prime. If r∉R, then r∈K\R, and hence x∈ P, by (٣). Thus p is an s-prime-submodule.

Corollary ١.٤

Let P be a prime ideal in R, then p is an s-prime-ideal iff p is an s-prime R-submodule of R.

proof: It follows directly by proposition ١.١ and proposition ١.٣.

Note: Niether statement (٢) nor statement (٣) of proposition ١.٣ implies the primeness, as the following example shows.

Example: The submodule $P=(\bar{0})$ of the Z-module Z٦ satisfies statements (٢) and (٣) of proposition ١.٣. But P is not a prime Z-submodule of Z٦.

Recall that an R-module M is called prime if annRN = annRM for each non-zero submodule N of M, ٩. Equivalently, M is a prime R-module iff (٠) is a prime submodule in M, ٩.

Now the following consequences are immediate.

Remark ١.٥

١. The zero submodule of any prime R-module is an s-prime-submodule.
٢. Since K is the total quotient field of R, then K, as an R-module, has (٠) as the only prime submodule. Thus K has (٠) as the only s-prime-submodule.

Proposition ١.٦

Let M be an R-module. Then T(M) is an s-prime-submodule, where

T(M)={*x*:*x*∈M such that ∃ *r*∈R,*r*≠٠ and *rx*=٠}.

proof: T(M) is a prime submodule of M by ١٠, remark ١.٢ (d), chapter ١. To prove $r^{-١}$T(M) ⊆ T(M) for each *r* ∈ K\R. Assume *m* ∈ T(M), then there exists *a*∈R, *a* ≠ ٠ such that *a m* = ٠. Hence $r^{-١}$ *a m* = ٠ and so *a* ($r^{-١}$ *m*) = ٠. Thus $r^{-١}$ *m* ∈ T(M); that is $r^{-١}$T(M) ⊆ T(M).

Remark ١.٧

Any prime submodule *p*Z of the Z-module Z is not an s-prime submodule, where *p* is a prime number.

proof: for any prime number *p*, $\frac{p}{p+1}$ ∈ Q\Z, where Q is the total quotient field of Z. $\frac{p}{p+1}$(*p*+ ١) = *p* ∈ *p*Z and *p* + ١ ∉ *p*Z. Thus *p*Z is not an s-prime submodule, for any prime number *p*.

Now we shall give a condition under which a prime submodule is an s-prime submodule

Proposition ١.٨

If P is a prime divisible submodule of an R-module M, then P is an s-prime submodule.

proof: Let $\frac{a}{b}$ ∈ K\R, *x* ∈ P. To prove $\frac{a}{b}$ *x*∈P. Since *b*∈R, *b*≠٠ and P is divisible, *b*P=P. Then there exists *x*′∈P such that *x*=*b x*′. Hence $\frac{a}{b}$ *x* = $\frac{a}{b}$ *b x*′ ∈ P. Thus P is an s-prime submodule by proposition ١.٣.

Now, we can give the following example.

Example١.٩: Let M be the Z-module Q ⊕ Z. Let P = Q ⊕ (٠). It is clear that P is prime and divisible. Hence P is an s-prime submodule by proposition ١.٨.

Proposition ١.١٠

If N is an s-prime submodule of an R-module M, then (N:M) is an s-prime ideal of R, where (N:M) = $\{r \in R \mid rM \subseteq N\}$.

proof: Since N is an s-prime submodule, then N is a prime submodule of M. Hence (N:M) is a prime ideal of R by ١٠, proposition ٢.٨, chapter ١. Now for any $x \in$ (N:M), $xm \in$ N for each $m \in$ M. Hence for each $r \in$ K\R, $r^{-1}(xm) \in$ N by proposition ١.٣. Thus $(r^{-1}x)m \in$ N; that is $r^{-1}x \in$ (N:M) and so (N:M) is an s-prime ideal of R, by proposition ١.١.

Remark ١.١١

The converse of proposition ١.١٠ is not true in general as the following example shows.

Example: Let M be the Z-module Z ⊕ Z and N = O ⊕ ٢Z. (N:M) = (٠) which is an s-prime ideal of Z. However, N is not an S-prime submodule of M because it is not prime.

Recall that an R-module M is called a multiplication module if for each submodule N of M, there exists an ideal I of R such that N=IM, ١١.

The following is a characterization of s-prime submodules in multiplication modules.

Theorem ١.١٢

Let M be a multiplication R-module, and N is submodule of M. Then N is an s-prime submodule of M iff (N:M) is an s-prime ideal of R.

proof: The (if) part holds by proposition ١.١٠. To prove the (only if) part. (N:M) is an s-prime ideal of R, so (N;M) is a prime ideal of R. Hence N is a prime submodule of M by ١٠, corollary ٢.١٦, chapter ١. Let $\frac{a}{b} \in$ K\R, $x \in$ N, to prove $\frac{a}{b}x \in$ N. Since M is a multiplication R-module, N = (N:M) M, $x = \sum_{i=1}^{n} r_i x_i$; $r_i \in$ (N:M), $x_i \in$ M, i = ١, …,n. Thus $\frac{a}{b}x = \frac{a}{b}\sum_{i=1}^{n} r_i x_i = (\sum_{i=1}^{n} \frac{a}{b} r_i) x_i$. But (N:M) is an s-prime ideal of R, so $\frac{a}{b} r_i \in$(N:M) ∀ i = ١,…,n. Therefore $\frac{a}{b}x \in$ (N:M) = N; that is N is an s-prime submodule.

Now, we give some cases in which P is an s-prime ideal in R, implies that PM is

Remark ١.١٣

Let M be an R-module, let P an s-prime ideal in R. If PM is a prime submodule, then PM is an s-prime submodule of M.

proof: Let $\frac{a}{b} \in$ K\R, $x \in$ PM. To prove $\frac{a}{b}x \in$PM. Since $x \in$ PM, $x = \sum_{i=1}^{n} a_i m_i$ for some $a_i \in$ P, $m_i \in$ M, i = ١, …, n. Then $\frac{a}{b}x = \frac{a}{b}\sum_{i=1}^{n} a_i m_i = \sum_{i=1}^{n} (\frac{a}{b} a_i) m_i$. But $\frac{a}{b} a_i \in$P. ∀ i = ١, …,n by proposition ١.١, hence $\frac{a}{b}x \in$ PM and PM is an s-prime submodule.

Proposition ١.١٤

Let M be an R-module, and let P be an s-prime ideal of R. Then,

١. If M is a multiplication R-module, P⊇ann$_R$M and PM ≠ M, then PM is an s-prime submodule of M.
٢. If P is a maximal submodule of M, then PM is an s-prime submodule of M, provided PM ≠ M.
٣. If M is a flat R-module, then PM is an s-prime submodule of M, provided PM ≠M.

Proof:

١. Since P is an s-prime-ideal, P is a prime ideal of R. But M is a multiplication module, P ⊇ ann$_R$M and PM ≠ M implies PM is a prime submodule of M and (PM:M) = P by ١٠, proposition ٤.٦, chapter ١. Then PM is an s-prime submodule of M by remark ١.١٣.
٢. Since P is a maximal ideal of R and PM≠M, PM is a prime submodule of M and (PM:M)=P, by ١٠, corollary ٢.٦,

chapter ١. Then PM is an s-prime submodule of M by remark ١.١٣.

٣. Since P is an s-prime-ideal of R, P is a prime ideal of R. But M is flat and PM≠M, hence PM is a prime submodule of M by ٣, proposition ٤.٨ chapter ١. Then (PM:M) =P by remark ١.١٣.

Next we have the following:

Proposition ١.١٥

Let P be an s-prime R-submodule of an R-module M. Then P is an s-prime $\overline{R}$ -submodule of M, where $\overline{R}$ =R\ann$_R$M and ann$_R$M is a prime ideal of R.

Proof: P is an s-prime R-submodule of M, hence P is a prime R-submodule of M and then P is a prime $\overline{R}$ -submodule of M by ٧, Rresult ١.٢. Assume $\overline{K}$ is the total quotient field of $\overline{R}$. Let $\frac{\overline{b}}{\overline{a}} \in \overline{K} \setminus \overline{R}$, we prove $\frac{\overline{a}}{\overline{b}} p \subseteq$ P. Since $\frac{\overline{b}}{\overline{a}} \in \overline{K} \setminus \overline{R}$, then $\overline{a}, \overline{b} \in \overline{R}$; that is $\overline{a} = a$ +ann M, $\overline{b} = b$ + ann M for some $a, b \in$ R, $b \neq ٠$. Moreover $\frac{b}{a} \in$ K\R which implies $\frac{a}{b} x \in$ P, $\forall x \in$ P since P an s-prime R submodule of M. Hence $ax = by$ for some $y \in$P, which implies $\overline{a} x = \overline{b} y$ and hence $\frac{\overline{a}}{\overline{b}} x \in$ P. Thus the result follows from proposition ١.٣.

Remark ١.١٦

The converse of the previous proposition is not true in general, as the following example shows.

Let M be the Z-module Z٥ [$\sqrt{3}$], let N={$\overline{b}$ $\sqrt{3}$: $\overline{b}$ ∈ Z٥}. It follows that N is not an s-prime submodule of the z-module Z٥ [$\sqrt{3}$]. On the other hand, ann$_Z$M=٥z. Hence Z/ann$_Z$M= Z٥. We claim that N is an s-prime submodule of the Z٥ module M. We can see easily that N is prime and divisible Z٥-submodule. Hence N is an s-prime submodule of the Z٥-module M by proposition ١.٨.

S.٢ Construction of s-Prime Submodules

In this section we give constructions that lead to s-prime submodules.

Recall that if M is an R-module and P is a prime ideal of R, then set {m: $m \in$ M, A$m \subseteq$PM for some ideal A $\not\subset$ P} is denoted by M(P), ٢.

It is clear that M(P) is a submodule of M and PM $\subseteq$ M(P).

In the following remark, we consider R as R-module.

Remark ٢.١

Let P be an s-prime-ideal of R such that R(P) ≠ R. Then R(P) = P.

Proof: It is easy, so it is omitted.

Now, we rais the question: when is M(P) an s-prime submodule of M.

First we prove the following lemma:

Lemma ٢.٢

Let M be an R-module, and let P be a prime ideal of R. If N= M(P), then either N = M or N is a prime submodule of M and P=(N:M).

Proof: Suppose N≠M. $r \in$ R, $m \in$ M such that $rm \in$ N. If $r \in$ P, then $r \in$ (N:M) since otherwise, $\exists$ $m' \in$ M such that $r\ m' \notin$N. But $r\ m' \in$ PM $\subseteq$ M(P) = N, so we get a contradiction.

If $r \notin$ P, then $r\ m \in$ N = M(P) implies there exists an ideal A, A $\not\subset$ P and A(rm)$\subseteq$PM. Hence (Ar)$m \subseteq$ PM. If (Ar) $\subseteq$ P, then P is prime and $r \notin$ P,implies A $\subseteq$ P which is a contradiction. Thus (Ar) $\not\subset$ P and $m \in$ PM $\subseteq$ M(P)=N. Hence N is a prime submodule of M.

To prove P = (N:M). Let $r \in$ P, then rM $\subseteq$ PM $\subseteq$ M(P) = N, hence $r \in$(N:M) and P $\subseteq$ (N:M). Now assume that $\exists$ $r \in$ (N:M) and $r \in$ P. Then $r\ m \in$ N for every $m \in$ M, which implies $m \in$ N since N is prime. Thus M = N which is a contradiction. Therefore (N:M) $\subseteq$ P and (N:M) = P.

Theorem ٢.٣

Let M be an R-module satisfying tM$\subseteq$ M, $\forall t \in$ K. Let P be an s-prime ideal of R. If M(P) ≠ M, then M(P) is an s-prime submodule of M.

Proof: Since P is an s-prime ideal, P is a prime ideal and hence M(P) is a prime submodule of M and P = (M(P):M) by lemma ٢.٢.

Let $\frac{b}{a} \in$ K\R, $m \in$ M(P). To prove $\frac{a}{b} m \in$M(P). $m \in$ M(P) implies there exists an ideal A of R such that A $\not\subset$ P, A$m \subseteq$ PM. Hence there exists $r \in$ A, $r \notin$ P and r m $\in$ PM. It follows that $r\ m = \sum_{i=1}^{n} p_i m_i$, $p_i \in$ P, $m_i \in$M, i = ١, ٢, …,m. Thus $\frac{a}{b}$ r m= $\sum_{i=1}^{n}(\frac{a}{b}p_i)m_i$ $\in$PM since P is an s-prime-ideal of R. This implies (r) $\frac{a}{b}$ $m \in$ PM. But (r) $\not\subset$ P and $\frac{a}{b}$ $m \in$ M, so that $\frac{a}{b}$ $m \in$ M(P). Hence M(P) is an s-prime submodule of M.

The following example shows that there exists an R-module M such that $tm \subseteq$M, $\forall t \in$K.

Example ٢.٤

Let M be the Z-module Z٥. Then K = Q and QZ٥ $\subseteq$ Z٥.

Recall that if N a submodule of an R-module M, S is a multiplicative subset in R, then the set {x: $x \in$ M such that t $x \in$ M for some $t \in$ S} is denoted by N(S), ٣.

Note that N(S) is a submodule of M and N $\subseteq$ N(S).

If we consider R as R-module, we have the following:

Remark ٢.٥

Let P be an s-prime ideal of R. Then P(S)=P.

Proof: It is straightforward, hence is omitted.

Proposition ٢.٦

Let N be an s-prime submodule of the R-module M. If rM $\subseteq$ M, $\forall$ $r \in$ K and N(S) ≠M, then N(S) is an s-prime submodule of M, where S=R-{٠}.

Proof: N is an s-prime submodule, so N is prime. Then it is easy to check that N(S) is prime. Thus it is enough to show that $r^{-١}$N(S) $\subseteq$ N(S), $\forall$ $r \in$ K/R. Let r = $\frac{b}{a} \in$ K\R, $x \in$ N(S), to prove $\frac{a}{b}$ x $\in$N(S). But $x \in$ N(S) implies $tx \in$ N for some $t \in$ S. Hence $\frac{a}{b}$ (tx) $\in$N since N is an s-prime submodule of M. Thus t ($\frac{a}{b}x$) $\in$N. On the other hand, $\frac{a}{b}$ $x \in$ M, so that $\frac{a}{b}$ $x \in$ N(S) and N(S) is an s-prime submodule of M.

Theorem ٢.٧

Let M be an R-module such that rM$\subseteq$ M, $\forall$ r $\in$ K. Let P be an s-prime ideal of R. Then PM(S) is an s-prime submodule of M with (PM(S):M) = P or PM(S) = M.

Proof: Suppose PM(S)≠M. Since P is an s-prime ideal of R, P is prime and hence PM(S) is a prime submodule of M with (PM(S):M) = (PM:M) = P by ٣, proposition ٣.١١.

Let $\frac{b}{a} \in$ K\R, x $\in$ PM(S). To prove $\frac{a}{b}x \in$ PM(S).Since $x \in$ PM(S), t $x \in$ PM for some $t \in$ S. It follows that t $\frac{a}{b}$ $x \in$ PM, since P is an s-prime ideal of R. On the other hand, $\frac{a}{b}$ $x \in$ M, so $\frac{a}{b}$ $x \in$ PM(S). Thus PM(S) is an s-prime submodule of M.

S.٣ s-Prime Submodule and Pseudo Valuation Domains

Recall that an integral domain R is called a pseudo valuation domain (briefly PVD) if every prime of R is an s-prime ideal (see ١, ١٢).

Every valuation domain R is a pseudo valuation domain (١, proposition ١١), where an integral domain R is called a valuation domain if the ideals of are ordered by inclusion.

Recall that an R-module M is called a faithfully flat if M is a flat and PM $\subseteq$ M for each maximal ideal P of R, ٦, page ٢٦.

Proposition ٣.١

Let M be a faithfully flat R-module. If every prime submodule of M is an s-prime submodule, then R is a PVD.

Proof: Let I be a prime ideal of R. Then IM is a prime submodule of M and (IM:M) = I by (١٠, corollary ٤.٩, chapter ١). Hence IM is an s-prime submodule of M. It follows that (IM:M) is an s-prime ideal of R by proposition ١.١٠. Thus I is an s-prime ideal and R is a PVD.

Note that every free module is faithfully flat module and every faithful finitely generated multiplication module is faithfully flat module by (٨, page ٢٩) and (٨, theorem ٣.٨, chapter ٢). Hence we have the following consequence.

Corollary ٣.٢

Let M be a free R-module or faithful finitely generated multiplication R-module. If every prime submodule of M is an s-prime submodule, then R is a PVD.

Now we consider the converse of proposition ٣.١, we have the following:

Proposition ٣.٣

Let M be a multiplication R-module. If R is a PVD, then every prime submodule of M is a s-prime submodule.

Proof: Let N be a prime submodule of M. Then (N:M) is a prime ideal of R (٣, proposition ٢.٨, chapter ١). But R is a PVD, so (N:M) is an s-prime ideal of R. Hence N is an s-prime submodule of M by theorem ١.١٢.

Recall that an R-module M is called a chained module if the submodules of M are ordered by inclusion. A ring R is a chained ring if it is chained as R-module. For references see (٤, ١٣).

Note that when R is an integral domain, R is a chained ring iff R is a valuation ring (٤, proposition ٣.٨, chapter ١). It follows that every chained integral domain is a PVD by (٨, proposition ١.١); that is every prime ideal of a chained domain is an s-prime-ideal. However we notice that not every prime submodule of a chained module is an s-prime submodule, as the following example shows.

Example: The Z-module $Z_٨$ is a chained module. However the submodule N= <٢> of $Z_٨$ is a prime submodule but it is not an s-prime submodule.

However we have the following:

Proposition ٣.٤

Let M be a finitely generated faithful multiplication R-module. If M is a chained R-module, then every prime submodule of M is an s-prime submodule of M.

Proof: By (١١, theorem ٣.١), it follows that R is a chained domain. Hence R is a PVD, and so the result follows by proposition ٣.٣.

Remark ٣.٥

The condition M is faithful cannot be dropped from proposition ٣.٤, as is seen by the last example.

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