OSCILLATION OF LINEAR NEUTRAL DIFFERENTIAL EQUATION OF THIRD ORDER

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Abstract

In this paper sufficient conditions for oscillation of bounded and all solutions of linear third order neutral delay differential equation are studied. Examples are inserted to illustrate the obtained results

تذبذب حلول المعادلات التفاضلية المحايدة من الرتبة الثالثة

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الخلاصة

قمنا في هذا البحث بدراسة المعادلة التفاضلية المحايدة من الرتبة الثالثة من النوع $[x(t)+p(t)x(\tau(t))]'''+q(t)x(\sigma(t))=0$ حيث أن كل من p(t),q(t) دوال مستمرة وأن $\tau(t),\sigma(t)$ هي دوال مستمرة وغيرمتناقصه بحيث $\lim_{t\to\infty}\tau(t)=\infty, \lim_{t\to\infty}\sigma(t)=\infty$. الهدف من البحث هو إيجاد شروط كافية تضمن تذبذب الحلول المقيدة للمعادلة (1.1) , وهي أخف من الشروط المستخرجة في [5],[6] كذلك تم إيجاد شروط كافية تضمن تذبذب كل حلول المعادلة (1.1) , وقد أعطيت بعض الأمثلة لتوضيح النتائج المستخرجة.

Introduction

Consider the third order linear neutral delay differential equation

$$[x(t)+p(t)x(\tau(t))]'''+q(t)x(\sigma(t))=0 \quad (1.1)$$

Subject to the conditions:

$C1: p(t)\in C[[t_0,\infty),R], \quad \tau(t)\ and\ \sigma(t)$ are positive non decreasing continuous functions such that $\lim_{t\to\infty}\sigma(t)=\infty, \lim_{t\to\infty}\tau(t)=\infty$

$C2:\ q:[t_0,\infty)\to R$ is continuous function, and not equivalent to zero.

Our aim is to obtain new sufficient conditions for the oscillation of all solutions of equation (1.1). By a solution of equation (1.1) we mean a continuous function $x:[t_x,\infty)\to R$ Such that $x(t)+p(t)x(\tau(t))$ is three times continuously differentiable and $x(t)$ satisfies equation (1.1) for all sufficiently large $t\ge t_x$. A solution of (1.1) is said to be oscillatory if it has an infinite sequence of zeros, otherwise is said non oscillatory.

The problem of oscillation and non oscillation for neutral differential equations of higher order has received considerable attention by many authors in recent years , see e.g. [1-6] and the references cited therein ,however many of these papers discuss the cases when coefficients and arguments are constants and a few of them investigate the cases of variable coefficients and variable arguments . In this paper the conditions (3.2),(3.3) and (3.4) improve the conditions of [5],[6] rather than we give some new other results.

Some Basic Lemmas

In this section we give some lemmas which we need in proving our main result.

Lemma 1:- [1], [3]

Suppose that

$$p;\sigma : R^+ \to R^+, \sigma(t) < t, \quad \lim_{t\to\infty} \sigma(t) = \infty$$

For $t \ge t_0$ and

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} p(s)ds > \frac{1}{e}$$

Then the inequality $x'(t) + p(t)x(\sigma(t)) \le 0$ has no eventually positive solution , and the inequality $x'(t) + p(t)x(\sigma(t)) \ge 0$ has no eventually negative solution.

Lemma 2 :- [1],[3]

Suppose that

$p, \sigma : R^+ \to R^+, \quad \sigma(t) > t, \quad \lim_{t\to\infty} \sigma(t) = \infty,$ for $t \ge t_0$ and

$$\liminf_{t\to\infty} \int_{t}^{\sigma(t)} p(s)ds > \frac{1}{e}$$

Then the inequality $x'(t) - p(t)x(\sigma(t)) \ge 0$ Has no eventually positive solution, and the inequality $x'(t) - p(t)x(\sigma(t)) \le 0$ has no eventually negative solution.

Main Results

In this section we studied the oscillation of all solutions of equation (1.1) and obtained some new sufficient conditions for the bounded and all solutions of (1.1) to be oscillatory. Let $u(t) = x(t) + p(t)\,x(\tau(t))$, so equation (1.1) reduce to $u'''(t) = -q(t)\,x(\sigma(t))$ (3.1)

The next theorem concerns for bounded oscillatory solutions of equation (1.1).

<u>Theorem 1.</u> Suppose that $0 \le p(t) < 1,\ q(t) \ge 0,\ \tau(t) > t,\ \sigma(t) < t$, and there exist a continuous functions α, γ such that $\alpha(t) > t,\ \gamma(t) > t,\ \sigma(\alpha(\gamma(t))) < t$ and

$$\liminf_{t\to\infty} \int_{\sigma(\alpha(\gamma(t)))}^{t} \int_{s}^{\gamma(s)} \int_{r}^{\alpha(r)} q(\xi)(1 - p(\sigma(\xi)))\, d\xi\, dr\, ds > \frac{1}{e}$$

(3.2)

Then every bounded solution of (1.1) is oscillatory.

Proof:

For the sake of contradiction suppose that (1.1) has nonoscillatory solution $x(t)$, and without loss of generality let $x(t) > 0,\ x(\tau(t)) > 0,\ x(\sigma(t)) > 0$ for $t \ge t_0$, then from (3.1) we get $u'''(t) \le 0$, we have only two cases to investigate.

Case1:

$$u'''(t) \le 0, u''(t) > 0, u'(t) > 0, u(t) > 0,$$

Case2:

$$u'''(t) \le 0, u''(t) > 0, u'(t) < 0, u(t) > 0$$

Case1: This case is impossible since $\lim_{t\to\infty} u(t) = \infty$ and $u(t)$ is bounded.

Case2: we have $x(t) = u(t) - p(t)x(\tau(t))$

$$x(\sigma(t)) = u(\sigma(t)) - p(\sigma(t))x(\tau(\sigma(t)))$$

$$q(t)x(\sigma(t)) = q(t)[u(\sigma(t)) - p(\sigma(t))x(\tau(\sigma(t)))]$$

Then equation (1.1) leads to

$$u'''(t) + q(t)[u(\sigma(t)) - p(\sigma(t))x(\tau(\sigma(t)))] = 0$$

$$u'''(t) + q(t)u(\sigma(t))[1 - p(\sigma(t))] \le 0$$

(3.3)

Since $u(t)$ is positive decreasing and $\tau(t) > t$ then integrating the last inequality from t to $\alpha(t)$ we get

$$u''(\alpha(t)) - u''(t) + \int_{t}^{\alpha(t)} q(s)u(\sigma(s))[1 - p(\sigma(s))]ds \le 0$$

$$-u''(t) + \int_{t}^{\alpha(t)} q(s)u(\sigma(s))[1 - p(\sigma(s))]ds \le 0$$

Integrating the last inequality from t to $\gamma(t)$ we obtain

$$-u'(\gamma(t)) + u'(t) +$$

$$+ \int_{t}^{\gamma(t)} \int_{s}^{\alpha(s)} q(s)u(\sigma(s))[1 - p(\sigma(s))]\, dsdt \le 0$$

$$u'(t) + u(\sigma(\alpha(\gamma(t)))) \int_{t}^{\gamma(t)} \int_{s}^{\alpha(s)} q(s)[1 - p(\sigma(s))]dsdt \le 0$$

according to Lemma 1 and (3.2) we get a contradiction.

<u>Theorem 2.</u> Suppose that $0 \le p(t) < 1, q(t) \le 0, \tau(t) < t, \sigma(t) > t$, and there exist a continuous functions $\alpha, \gamma, \beta, \theta$ such that $\alpha(t) < t, \gamma(t) < t, \beta(t) > t,$ $\theta(t) > t, \sigma(\alpha(\gamma(t))) > t$ and

$$\liminf_{t\to\infty} \int_{t}^{\sigma(\alpha(\gamma(t)))} \int_{\gamma(s)}^{s} \int_{\alpha(r)}^{r} |q(\xi)|\,(1 - p(\sigma(\xi)))\, d\xi\, dr\, ds > \frac{1}{e}$$

(3.4)

$$\liminf_{t\to\infty} \int_t^{\sigma(t)} \int_s^{\theta(s)} \int_r^{\beta(r)} |q(\xi)|(1-p(\sigma(\xi)))\,d\xi\,dr\,ds > \frac{1}{e}$$
(3.5)

then all solutions of (1.1) are oscillatory.

Proof. Let $x(t)>0, x(\tau(t))>0, x(\sigma(t))>0$ for $t\ge t_0$, then from (3.1) we get $u'''(t)\ge 0,\ u(t)>0$, we have only two cases to consider,

Case1:

$$u'''(t)\ge 0, u''(t)>0, u'(t)>0, u(t)>0,$$

Case2:

$$u'''(t)\ge 0, u''(t)<0, u'(t)>0, u(t)>0$$

Case1: In this case inequality (3.3) will be

$$u'''(t)+q(t)u(\sigma(t))[1-p(\sigma(t))]\ge 0$$
(3.6)

Integrating (3.6) from $\alpha(t)$ to t we get

$$u''(t)-u''(\alpha(t))+ + \int_{\alpha(t)}^{t} q(s)u(\sigma(s))[1-p(\sigma(s))]\,ds \ge 0$$

$$u''(t)- \int_{\alpha(t)}^{t}|q(s)|u(\sigma(s))[1-p(\sigma(s))]ds\ge 0$$

integrating the last inequality from $\gamma(t)$ to t we get

$$u'(t)+u'(\gamma(t))- - \int_{\gamma(t)}^{t}\int_{\alpha(s)}^{s}|q(\xi)|u(\sigma(\xi))[1-p(\sigma(\xi))]\,d\xi\,ds\ge 0$$

$$u'(t)-\int_{\lambda(t)}^{t}\int_{\alpha(s)}^{s)}|q(\xi)|u(\sigma(\xi))[1-p(\sigma(\xi))]\,d\xi\,ds\ge 0$$

$$u'(t)-u(\sigma(\gamma(\alpha(t))))\int_{\lambda(t)}^{t}\int_{\alpha(s)}^{s)}|q(\xi)|[1-p(\sigma(\xi))]d\xi\,ds\ge 0$$

According to Lemma 2 and (3.4) we get a contradiction.

Case2: integrating (3.6) from t to $\beta(t)$ we get

$$-u''(t)+\int_t^{\beta(t)}|q(s)|u(\sigma(s))[1-p(\sigma(s))]ds\ge 0$$ in tegrating the last inequality from t to $\theta(t)$ we obtain

$$u'(t)-u(\sigma(t))\int_t^{\theta(t)}\int_s^{\beta(s)}|q(\xi)|[1-p(\sigma(\xi))]\,d\xi\,ds\ge 0$$ according to Lemma 1 and (3.5) we get a contradiction.

Theorem 3. Assume that $0\le p(t)<1,\ q(t)\ge 0,\ \tau(t)>t$ and

$$\int_t^{\infty} s\,q(s)\,[1-p(\sigma(s))]ds=\infty \qquad (3.7)$$

Then every bounded solution of (1.1) are either oscillatory or $\lim_{t\to\infty}x(t)=0$

Proof: Assume that $x(t)$ is non-oscillatory bounded solution of equation (1.1) and suppose that $x(t)>0, x(\tau(t))>0, x(\sigma(t))>0$ for $t\ge t_0$, then from (3.1) we get $u'''(t)\le 0$, we have two cases to consider for $t\ge t_1\ge t_0$,

Case1:

$$u'''(t)\le 0, u''(t)>0, u'(t)>0, u(t)>0,$$

Case2:

$$u'''(t)\le 0, u''(t)>0, u'(t)<0, u(t)>0$$

Case1: This case is impossible since $\lim_{t\to\infty}u(t)=\infty$ and $u(t)$ is bounded.

Case2: integrating (3.3) two times from t to T, $t\in[t_1,T]$ we get

$$-u'(t)\ge\int_t^T\int_v^T q(\xi)u(\sigma(\xi))[1-p(\sigma(\xi))]d\xi\,dv$$

integrate the last inequality from t_1 to T we obtain

$$-u(T)+u(t_1)\ge \ge\int_{t_1}^T\int_s^T\int_v^T q(\xi)u(\sigma(\xi))[1-p(\sigma(\xi))]\,d\xi\,dv\,ds$$

Interchanging the order of integration of the last inequality and assuming that $\psi(\xi)=q(\xi)u(\sigma(\xi))[1-p(\sigma(\xi))]$ shows that

$$=\int_{t_1}^T\int_{t_1}^v\int_v^T\psi(\xi)\,d\xi\,ds\,dv=\int_{t_1}^T\int_v^T\int_{t_1}^{\xi}\psi(\xi)\,d\xi\,ds\,dv$$

$$=\int_{t_1}^T\int_v^T(\xi-t_1)\psi(\xi)d\xi\,dv=\int_{t_1}^T\int_{t_1}^{\xi}(\xi-t_1)\psi(\xi)dv\,d\xi$$

$$-u(T)+u(t_1)\ge\int_{t_1}^T(\xi-t_1)^2q(\xi)u(\sigma(\xi))[1-p(\sigma(\xi))]\,d\xi$$

$$\ge u(\sigma(T))\int_{t_1}^T(\xi-t_1)\,q(\xi)[1-p(\sigma(\xi))]\,d\xi$$

As $T\to\infty$ then the last inequality implies that $\lim_{t\to\infty}u(t)=0$ hence $\lim_{t\to\infty}x(t)=0$.

Example 1: Consider the neutral differential equation

$$[x(t)+(\frac{1}{2}+\frac{1}{4}\sin 4t)x(t+2\pi)]''' + (\frac{3}{2}+\frac{1}{4}\sin 4t)x(t-\frac{3\pi}{2})=0, \quad t\ge t_0 \qquad (E.1)$$

It easy to see that all the conditions of theorem 1 or theorem 3 are hold, if we compute condition (3.2) we get

$$\int_{t-\frac{\pi}{2}}^{t}\int_{t}^{t+\frac{\pi}{2}}\int_{t}^{t+\frac{\pi}{2}} q(\xi)(1-p(\xi-\frac{3\pi}{2}))\,d\xi\,dv\,ds$$

$$=\frac{23\pi^3}{128}>\frac{1}{e}$$

So according to theorem 1 or theorem 3 all bounded solution of (E.1) are oscillatory, for instance the solution $x(t)=\dfrac{\sin t}{\frac{3}{2}+\frac{1}{4}\sin 4t}$ is such oscillatory solution.

Theorem 4. Assume that $0\le p(t)<1,\ q(t)\le 0,\ \tau(t)<t$, and

$$\int_{t_1}^{\infty} s\,|q(s)|\,[1-p(\sigma(s))]ds=\infty \qquad (3.7)$$

Then every bounded solution of (1.1) oscillates.

Proof:

Assume that equation (1.1) have a non-oscillatory bounded solution $x(t)$, without loss of generality suppose that $x(t)>0, x(\tau(t))>0, x(\sigma(t))>0$ for $t\ge t_0$, then from (3.1) we get that $u'''(t)\ge 0, u(t)>0$ and $u(t)$ is bounded, we have two cases to consider for $t\ge t_1\ge t_0$,

Case1:

$$u'''(t)\ge 0, u''(t)>0, u'(t)>0, u(t)>0,$$

Case2:

$$u'''(t)\ge 0, u''(t)<0, u'(t)>0, u(t)>0$$

Case1: This case is impossible since $\lim_{t\to\infty}u(t)=\infty$ and $u(t)$ is bounded.

Case2: integrating (3.6) two times from t to T, $t\in[t_1,T]$ we get

$$-u'(T)+u'(t)\ge\int_t^T\int_v^T|q(\xi)|u(\sigma(\xi))[1-p(\sigma(\xi))]d\xi\,dv$$

$$u'(t)\ge\int_t^T\int_v^T|q(\xi)|u(\sigma(\xi))[1-p(\sigma(\xi))]d\xi\,dv$$

Integrate the last inequality from t_1 to T we obtain

$$u(T)-u(t_1)\ge\int_{t_1}^T\int_s^T\int_v^T|q(\xi)|u(\sigma(\xi))[1-p(\sigma(\xi))]d\xi\,dv\,ds$$

Interchanging the order of integration to the last inequality shows that

$$=\int_{t_1}^T\int_{t_1}^v\int_v^T\psi(\xi)\,d\xi\,ds\,dv=\int_{t_1}^T\int_{v}^T\int_{t_1}^{\xi}\psi(\xi)\,d\xi\,ds\,dv$$

$$=\int_{t_1}^T\int_v^T(\xi-t_1)\psi(\xi)\,d\xi\,dv$$

$$=\int_{t_1}^T\int_{t_1}^{\xi}(\xi-t_1)\psi(\xi)\,dv\,d\xi=\int_{t_1}^T(\xi-t_1)^2\psi(\xi)\,d\xi$$

Hence

$$u(T)-u(t_1)\ge\int_{t_1}^T(\xi-t_1)^2|q(\xi)|u(\sigma(\xi))[1-p(\sigma(\xi))]\,d\xi$$

$$\ge u(\sigma(t_1))\int_{t_1}^T(\xi-t_1)|q(\xi)|[1-p(\sigma(\xi))]\,d\xi$$

As $T\to\infty$ then the last inequality implies that $\lim_{t\to\infty}u(t)=\infty$ which is a contradiction.

Example 2: Consider the neutral differential equation

$$[x(t)+(\frac{1}{2}+\frac{1}{4}\cos 4t)x(t-2\pi)]''' - (\frac{3}{2}+\frac{1}{4}\cos 4t)x(t+\frac{3\pi}{2})=0, \quad t\ge t_0 \qquad (E.2)$$

We can see that all the conditions of theorem 2 or 4 are hold, if we compute condition (3.4) and (3.5) we get

$$\int_t^{t+\frac{\pi}{2}}\int_{s-\frac{\pi}{2}}^{s}\int_{r-\frac{\pi}{2}}^{r}(\frac{3}{2}+\frac{1}{4}\cos 4\xi)(\frac{1}{2}-\frac{1}{4}\cos 4\xi)\,d\xi\,dr\,ds$$

$$=\frac{23\pi^3}{256}>\frac{1}{e}$$

$$\int_t^{t+\frac{\pi}{2}}\int_{s}^{s+\frac{\pi}{2}}\int_{r}^{r+\frac{\pi}{2}}(\frac{3}{2}+\frac{1}{4}\cos 4\xi)(\frac{1}{2}-\frac{1}{4}\cos 4\xi)\,d\xi\,dr\,ds$$

$$=\frac{23\pi^3}{256}>\frac{1}{e}$$

According to theorem 2 or 4 all solutions of ($E.2$) are oscillatory, for instance the solution

$$x(t)=\frac{\cos t}{\frac{3}{2}+\frac{1}{4}\cos 4t}$$

is such oscillatory solution.

Remark: In similar way one can establish new conditions when $p(t) > 1$.

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