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The viral infections disease in discrete epidemic model with treatment

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Abstract
In this research, the viral infections disease in discrete epidemic model with treatment is presented. The treatment function imposed inside SIV model to see their effects. The conditions for the positivity of solutions to endemic and epidemic points are found and also conditions that require to achieve local stability of all the equilibria are satisfied. Possibility of bifurcation and chaos are discussed. Furthermore, the numerical analysis validates the established theoretical results by using MATLAB program and the iterative method was employed. Also, the influence of the treatment on the model's behavior confirmed numerically.

Keywords: Discrete model, Stability theory, Infections disease, Epidemic model.

مرض العدوى الفيروسية في نموذج وبائي منفصل مع العلاج

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المستخلص:
في هذا البحث تم عرض مرض العدوى الفيروسية في نموذج وبائي منفصل مع العلاج . وتم ادراج دالة العلاج داخل نموذج SIV لمعرفة تأثيراتها . تم إيجاد شروط الحل الموجبة واللازمة للنقاط المتوطنة والوبائية وكذلك الشروط التي تتطلب تحقيق الاستقرار المحلي لجميع نقاط التوازن. وتم مناقشة إمكانية التشعب والفوضى وعلاوة على ذلك تم التحقق من صحة التحليل العددي للنتائج النظرية المعمول بها باستخدام برنامج ماتلاب وتم استخدام الطريقة التكرارية كما تم تأكيد تأثير العلاج على سلوك النموذج عدديًا.

1- Introduction

The study of illnesses and their spread through mathematics began is at most over three centuries old [1]. For a natural history of diseases, the interested readers can consult Burnet, Fenner, or Bailey and Anderson [2-4]. Fenner provides an account of smallpox and its eradication, while Bailey and Anderson for an outline of the development of mathematical theories for epidemic spread [2].

Among studies of diseases, numbers of continuous and discrete mathematical models have been proposed and analyzed on the viral disease, and most of these models are introduced into medicinal studies such as well-known COVID-19 [5] that involve three variables that are function of time, namely, uninfected cell population, infected cell population that are producing viruses, and free virus particles [6-11]. For instance, in [11] they analyzed both a discrete-time and continuous-time SIV model of the gonorrhea transmission among

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homosexuals, demonstrating that the discrete-time model given some important dynamics.

Since ancient times, viruses-intracellular parasites have coexisted with cell [12–14]. In order to facilitate effective viral protein production and replication, viruses take over living cells and redirect many cellular processes [15]. To start a de novo infection, some of these copies must spread to new target cells for the virus to be successful [16]. A hitherto unanticipated heterogeneity has been shown by the evolution of multiple viral spread mechanisms in many animal viruses [17–19].

Meanwhile, technologies used to identify dangerous variations of viruses are sometimes outpaced by viral changes [20]. Since dangerous virus variations emerge so quickly so it is difficult to produce vaccines and antivirals that continue to protect against them [21, 22]. Rapidly determining the mechanisms of viral entry and transmission during emergent virus epidemics is essential for the timely development of vaccines [23]. Furthermore, the treatment rate is assumed to be constant when the number of infectious agents exceeds capacity and proportionate when the number of infectious agents is less than capacity [24]. Due to that, the treatment rate embedded because it has already been used successfully to treat infectious diseases [25-28].

Most of epidemic models compute the basic reproduction number which commonly denoted by R_0 and also called epidemiological threshold because it determines the number of secondary infections caused by a single primary one. Recently, Ahmed and his coauthors [29, 30] derived the value of R_0 for models of bacterial meningitis and HIV\AIDS, respectively.

Wang [31] suggested treatment function to comprehend the impact of treatment capacity in a resource limited epidemic model. In this work, we will propose the dynamics of a discrete SIV epidemic model consistent of treatment function and study the new model by calculation all possible equilibrium point to find their stability conditions and confirm that with numerical results. Actually, the current model came from discretization continuous model for availability of discrete data in time increments and a discrete model behavior exhibits a rich dynamic as well.

This paper come up with follows sections: the main model SIV is introduced in Section 2 along with treatment effect. The existence of the equilibrium points is given in Section 3. Section 4 devoted to study the stability analysis at each equilibrium point. Section 5 includes the numerical analysis and dynamical behavior when the positive parameters are fixed. Finally, Section 6 is devoted to the conclusions.

2- Main model

Consider a population that consists of individuals dented by $N(t)$ at time t such that it is divided into three individuals viz. The individuals of susceptible, infected and virus that denoted by S_t, I_t and V_t , respectively. The model under investigation takes disease-related mortality into account for natural deaths. Additionally, recruitment in the population consists the logistic growth rate. Both susceptible individuals and recently entered members of the population are vaccinated. The vaccination is assumed to be perfect; thus, those who have received the vaccinated individuals do not become infected. All changes in population and individual transmissions between compartments are to happen once every unit of time (during time intervals) t, and to have a uniform distribution. $N(t)$ indicates the various population totals, where $N(t) = S(t) + I(t) + V(t)$. In actuality, when the treatment function is utilized, Wang [31], every community should have an appropriate capacity for treatment. The $T(I)$ used a constant therapy, simulating a treatment capacity that is finite. Note that, a constant treatment is appropriate when there are several infectious is large. The treatment's function can be expressed as follows:

$$T(I) = \begin{cases} \beta I, & if\ 0 \le I \le I_0; \\ \beta I_0, & if\ \ I > I_0. \end{cases} \quad (1)$$

This indicates that, in cases where treatment capacity is not reached, the rate of treatment is proportionate of the number to the number of the infective; in other cases, the maximum capacity is used. Straightforward, we imposed treatment function $T(I)$ inside the epidemic model to understand the effect of the treatment. Now, the system of difference equations that follows presents the discrete SIV epidemic model with treatment:

$$\begin{aligned} S_{t+1} &= S_t + h\left[rS_t\left(1 - \frac{S_t + I_t}{K}\right) - \gamma S_t V_t + \beta I_t \right], \\ I_{t+1} &= I_t + h(\gamma S_t V_t - cI_t - \beta I_t), \\ V_{t+1} &= V_t + h(acI_t - \mu V_t). \end{aligned} \quad (2)$$

Where S_t,I_t and V_t represent to the numbers of the susceptible, infected, and vaccinated individual, respectively. The positive parameters r is the intrinsic birth rate, γ is a positive infection rate, c is infected death rate before having a chance to reproduce, β is the treatment rate and μ is the virus death rate while the positive constant $ac$is the infected population that releases a virus into the environment, respectively.

3- The existence of the equilibrium points:

In order to get all possible equilibrium points of the model (2), we see that their equilibrium points satisfy the following equations:

$$\begin{aligned} &S + h\left[rS\left(1 - \frac{S+I}{K}\right) - \gamma SV + \beta I \right] = S, \\ &I + h(\gamma SV - cI - \beta I) = I, \\ &V + h(acI - \mu V) = V. \end{aligned} \quad (3)$$

The solution of Equations (3) yield to two equilibrium points are as follows:

1- The disease-free equilibrium point $E_{df} = (K, 0, 0)$, always exists, in which *I=0*.

2- The endemic equilibrium point $E_e = (S^*, I^*, V^*)$ that exists if and only if the condition $\mu(c+\beta) < a\gamma cK$ holds, where $S^* = \frac{\mu(c+\beta)}{a\gamma c}, V^* = \frac{ac}{\mu} I^*$, and $I^* = rS^* \frac{ac\gamma K - \mu(c+\beta)}{r\mu(c+\beta) + ac\gamma K(c+\beta) - ac\gamma K\beta}$.

4- Stability Analysis

First, the Jacobi matrix must be found before analyzing model (2) stability.

$$J(E_{(S,I,V)}) = \begin{pmatrix} 1 + h[r(1 - \frac{2S+I}{K}) - \gamma V] & h(\beta - \frac{r}{K}S) & -h\gamma S \\ h\gamma V & 1 - h(c+\beta) & h\gamma S \\ 0 & hac & 1 - h\mu \end{pmatrix}. \quad (4)$$

Let

$$F(\lambda) = \lambda^3 + A\lambda^2 + B\lambda + C = 0. \quad (5)$$

From the characteristic Equation (5) of the matrix (4) we get the following propositions:

Proposition 4.1: For all $K > 0$, let the endemic point $E_{df} = (K, 0, 0)$ has the following Jacobian matrix:

$$J(E_{df}) = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ 0 & S_{22} & S_{23} \\ 0 & S_{32} & S_{33} \end{bmatrix}. \quad (6)$$

Where

$S_{11} = 1 - hr$, $S_{12} = h(\beta - r)$, $S_{13} = -h\gamma k$,
$S_{22} = 1 - h(c+\beta)$, $S_{23} = h\gamma k$,
$S_{32} = hac$, $S_{33} = 1 - h\mu$.
Also, the Jacobian matrix (6) has the following characteristic equation:
$P(\lambda) = \lambda^3 + A_2\lambda^2 + A_1\lambda + A_0$,
(7)
where
$A_2 = -(S_{33} + S_{22} + S_{11})$,
$A_1 = S_{11}S_{22} + S_{11}S_{33} - S_{23}S_{32} + S_{33}S_{22}$,
$A_0 = -(S_{32}S_{23}S_{11} - S_{33}S_{22}S_{11})$.
And the eigenvalues of the Equation (7) at the point E_{df} are
$\lambda_1 = S_{11} = 1 - hr$ and $\lambda_{2,3} = \frac{1}{2}[(S_{22} + S_{33}) \mp \sqrt{(S_{22} - S_{33})^2 + 4S_{23}S_{32}}\ ]$.
Then, if:

i. $(S_{22} - S_{33})^2 + 4S_{23}S_{32} > 0$ so, the three eigenvalues are real,
ii. $(S_{22} - S_{33})^2 + 4S_{23}S_{32} < 0$ so, one eigenvalue is real and the others are complex pairs eigenvalues with stable real parts along with selected parameters sets.

Proof: The Jury conditions [32] state, in order to find the asymptotically stable region of E_{df}, that we need to find the region to satisfy the following requirements:
$P(1) = 1 + A_2 + A_1 + A_0 > 0$,
$P(-1) = -1 + A_2 - A_1 + A_0 < 0$,
(8)
$|A_0| < A_n$, $|B_0| > |B_{n-1}|$,
where $B_k = \begin{vmatrix} A_0 & A_{n-k} \\ A_n & A_k \end{vmatrix}$.
Finally, as in [33] and depends on the conditions (8), we get $|A_0| < 1$, $|A_0 + 1| < |A_1|$ and $|A_0 - 1||A_0 + A_1 + 1| > |A_0A_1 - A_2|$ as equivalent conditions that confirm stability of the endemic point E_{df}.
Now, before undergo to set the proposition of endemic point E_e stability we have the following lemmas.

Lemma 4.2: Let equation $x^3 + bx^2 + cx + d = 0$, where $b, c, d \in R$. Let $A = b^2 - 3c$, $B = bc - 9d$, $C = c^2 - 3bd$ and $\Delta = B^2 - 4AC$. Then
(1) If $\Delta \le 0$, there are three real roots exist for the equation.
(2) If $\Delta > 0$, there are two conjugate complex roots and one real root in the equation. Moreover, the conjugate complex root has
$\tau = \frac{-2b + Y_1^{\frac{1}{3}} + Y_2^{\frac{1}{3}}}{6} \pm i\frac{\sqrt{3}(Y_1^{\frac{1}{3}} - Y_2^{\frac{1}{3}})}{6}$ where $Y_{1,2} = bA + \frac{3(-B \pm \sqrt{B^2 - 4AC})}{2}$.

Proof: Lemma 4.2 and its proof are given in [34], but many readers find it difficult to locate it. Consequently, we shall present a proof of Lemma 4.2 here. The following two wells are relevant to the proof: Learn the Cardano formula for the formulation of the cubic equation we can see in [35-37].

$$x^3 + bx^2 + cx + d = 0, \tag{9}$$

where $b, c, d \in R$.

Lemma 4.3:(Cardano Lemma): The following forms correspond to the three roots of Equation (9)

$$\begin{aligned} x_1 &= -\frac{b}{3} + [-\frac{q}{2} + \sqrt{H}]^{\frac{1}{3}} + [-\frac{q}{2} - \sqrt{H}]^{\frac{1}{3}}, \\ x_2 &= -\frac{b}{3} + \tau_1\left[-\frac{q}{2} + \sqrt{H}\right]^{\frac{1}{3}} + \tau_2\left[-\frac{q}{2} - \sqrt{H}\right]^{\frac{1}{3}}, \\ x_3 &= -\frac{b}{3} + \tau_2\left[-\frac{q}{2} + \sqrt{H}\right]^{\frac{1}{3}} + \tau_1\left[-\frac{q}{2} - \sqrt{H}\right]^{\frac{1}{3}}, \end{aligned} \tag{10}$$

where $p=\frac{3c-b^2}{3}$, $q=\frac{27d-9bc+2b^3}{27}$, $H=(\frac{q}{2})^2+(\frac{p}{3})^3$ and $\tau_{1,2}=\frac{-1\pm i\sqrt{3}}{2}$.
In addition, when $H>0$, the roots x_2 and x_3 are two conjugate complex roots while x_1 is a real root, and when $H\le 0$, $x_j (j=1,2,3)$ are three real roots.
Since

$$H=(\frac{q}{2})^2+(\frac{p}{3})^3$$
(11)
$$=\frac{1}{324}(81d^2-3b^2c^2+12b^3d-54bcd+12c^3)$$
$$=\frac{1}{324}\left((bc-9d)^2-4(b^2-3c)(c^2-3bd)\right)=\frac{1}{324}\Delta.$$

We have to show that $\Delta>0$ if and only if $H>0$ and $\Delta\le 0$ if and only if $H\le 0$.
So, by Lemma 4.2, when $\Delta\le 0$, Equation (10) have three real roots.
When $\Delta>0$, since $\frac{q}{2}=\frac{27b-9bc+2b^3}{54}=\frac{2bA-3B}{54}$,
we have

$$-\frac{q}{2}-\sqrt{H}=-\frac{2bA-3B+3\sqrt{B^2-4AC}}{54}=-\frac{Y_1}{27},$$
and
$$-\frac{q}{2}+\sqrt{H}=-\frac{2bA-3B-3\sqrt{B^2-4AC}}{54}=-\frac{Y_2}{27}.$$

Thus, from Lemma 4.3 we can obtain that:

$$x_1=-\frac{b}{3}+[-\frac{q}{2}+\sqrt{H}]^{\frac{1}{3}}+[-\frac{q}{2}-\sqrt{H}]^{\frac{1}{3}}=-\frac{b+Y_1^{\frac{1}{3}}+Y_2^{\frac{1}{3}}}{3},$$
is a real root, while
$$x_2=-\frac{b}{3}+\tau_1\left[-\frac{q}{2}+\sqrt{H}\right]^{\frac{1}{3}}+\tau_2\left[-\frac{q}{2}-\sqrt{H}\right]^{\frac{1}{3}}=\frac{-2b+Y_1^{\frac{1}{3}}+Y_2^{\frac{1}{3}}}{6}+i\frac{\sqrt{3}(Y_1^{\frac{1}{3}}-Y_2^{\frac{1}{3}})}{6},$$
and
$$x_3=-\frac{b}{3}+\tau_2\left[-\frac{q}{2}+\sqrt{H}\right]^{\frac{1}{3}}+\tau_1\left[-\frac{q}{2}-\sqrt{H}\right]^{\frac{1}{3}}=\frac{-2b+Y_1^{\frac{1}{3}}+Y_2^{\frac{1}{3}}}{6}-i\frac{\sqrt{3}(Y_1^{\frac{1}{3}}-Y_2^{\frac{1}{3}})}{6},$$
are the pair of roots with conjugate complexity and this ended the proof.
Actually, the Jacobian matrix of the point E_e at the system (2) is

$$J(E_e)=\begin{bmatrix}1+h[r(1-\frac{2S^*+I^*}{K}-\gamma V^*) & h(\beta-\frac{r}{K}S^*) & -h\gamma S^* \\ h\gamma V^* & 1-h(c+\beta) & h\gamma S^* \\ 0 & hac & 1-h\mu\end{bmatrix}.$$
(12)

Now, we denote the following assumptions:

$$s_1=-[r\left(1-\frac{2S^*+I^*}{k}-\gamma V^*\right)-(c+\beta)-\mu],$$
$$s_2=-r(c+\beta)\left(1-\frac{2S^*+I^*}{k}-\gamma V^*\right)-\gamma V^*(\beta-\frac{r}{K}S^*)+\mu(c+\beta)-ac\gamma S^*-r\mu(1-\frac{2S^*+I^*}{k}-\gamma V^*),$$
$$s_3=-[\gamma^2acV^*S^*+rac\gamma S^*\left(1-\frac{2S^*+I^*}{k}-\gamma V^*\right)-\gamma V^*\mu(\beta-\frac{r}{k}S^*)-r\mu(c+\beta)\left(1-\frac{2S^*+I^*}{k}-\gamma V^*\right)].$$

Then the eigenvalues of the matrix (12) satisfied the following characteristic polynomial equation:

$$L(\tau)=\tau^3+b_1\tau^2+b_2\tau+b_3=0. \quad (13)$$

Where $b_1=-3-hs_1$, $b_2=3-2hs_1+h^2s_2$ and $b_3=-1+hs_1-h^2s_2+h^3s_3$.
From Lemma 4.2 we can calculate
A= $b_1{}^2-3b_2=h^2(s_1{}^2-3s_2)$,
B= $b_1b_2-9b_3=-2h^2(s_1{}^2-3s_2)-h^3(s_1s_2+9s_3)$,

C= $b_2{}^2 - 3b_1b_3 = h^2(s_1{}^2 - 3s_2) + h^3(s_1s_2 + 9s_3) + h^4(s_2{}^2 + 3s_1s_3)$,
and $\Delta = B^2 - 4AC = h^6\Delta_*$, Where $\Delta_* = 9(3s_3 + s_1s_2)^2 + 12[s_2{}^3 - s_1{}^2(s_2{}^2 + s_1s_3)]$.
The derivative of $L(\tau)$ is:
$L'(\tau) = 3\tau^2 + 2b_1\tau + b_2$.
(14)

Obviously, equation $L' = 0$ have the following two roots $\tau^*{}_{1,2} = \frac{1}{3}(-b_1 \pm \sqrt{b_1{}^2 - 3b_2}) = \frac{1}{3}\left[(3 + hs_1) \pm h\sqrt{s_1{}^2 - 3s_2}\right]$.
When $\Delta_* \leq 0$, namely, $\Delta \leq 0$, by Lemma 4.2, we have that the Equation (14) have three real roots τ_1, τ_2 and τ_3. From this, it is simple to demonstrate prove that two roots $\tau_{1,2}{}^*$ of equation $L' = 0$ also are real. When $\Delta_* > 0$, namely, $\Delta > 0$, by Lemma 4.2, we have that Equation (14) has one real root τ_1 and a pair of conjugate complex roots $\tau_{2,3}$ and the roots of the conjugate complex are:
$\tau_{2,3} = \frac{-2b_1 + Y_1^{\frac{1}{3}} + Y_2^{\frac{1}{3}}}{6} \pm i\frac{\sqrt{3}(Y_1^{\frac{1}{3}} - Y_2^{\frac{1}{3}})}{6}$,
(15)
where $Y_{1,2} = b_1A + \frac{3(-B \pm \sqrt{B^2 - 4AC})}{2} = \frac{h^3}{2}\left(-2s_1{}^3 + 9s_1s_2 + 27s_3 \pm 3\sqrt{\Delta_*}\right)$.
Further, we have $L(1) = 1 + b_1 + b_2 + b_3$ and $L(-1) = -1 + b_1 - b_2 + b_3$.

Proposition 4.4: Let the endemic point E_e exists, it possesses the subsequent topological types:

(1) $E_e = (S^*, I^*, V^*)$ is sink, if one of the following requirements is met:
(M) $\Delta_* \leq 0$, *L(1) > 0, L(−1) < 0* and $-1 < \tau_{1,2}^* < 1$.
(N) $\Delta_* > 0$, *L(1) > 0, L(−1) < 0* and $|\tau_{2,3}| < 1$.
(2) $E_e = (S^*, I^*, V^*)$ is source, if one of the following requirements is met
(Q) $\Delta_* \leq 0$ and if one of the following requirements is met:
(Q1) *L(1) > 0, L(−1) > 0* and $\tau_2^* < -1$ ($\tau_2^* > 1$).
(Q2) *L(1) < 0, L(−1) < 0* and $\tau_1^* < -1$ ($\tau_1^* > 1$).
(R) $\Delta_* > 0$ and if one of the following requirements is met:
(R1) *L(1) < 0* and the conjugate complex roots $\tau_{2,3}$satisfy $|\tau_{2,3}| > 1$.
(R2) *L(−1) > 0* and $|\tau_{2,3}| > 1$.
(3) $E_e = (S^*, I^*, V^*)$ is a saddle, if one of the following requirements is met:
(T) $\Delta_* \leq 0$ and If one of the following requirements is met:
(T1) *L(1) > 0, L(−1) < 0* and $\tau_1^* < -1$ or $\tau_2^* > 1$.
(T2) *L(1) < 0* and *L(−1) > 0.*
(T3) *L(1) < 0, L(−1) < 0* and $-1 < \tau_1^* < 1$.
(T4) *L(1) > 0, L(−1) > 0* and $-1 < \tau_2^* < 1$.
(U) $\Delta_* > 0$ and if one of the following requirements is met:
(U1) *L(1) > 0, L(−1) < 0* and $|\tau_{2,3}| > 1$.
(U2) *L(1) < 0* and $|\tau_{2,3}| < 1$.
(U3) *L(−1) > 0* and $|u_{2,3}| < 1$.
(4) $E_e = (S^*, I^*, V^*)$ is non-hyperbolic, if one of the following requirements is met:
(V) $\Delta_* \leq 0$, *L(1) = 0* or *L(−1) = 0.*
(W) $\Delta_* > 0$, *L(1) = 0* or *L(−1) = 0* or $|\tau_{2,3}| = 1$.

Proof. Let $\Delta_* \leq 0$. From Lemma 4.2, Equation (13) has three real roots τ_1, τ_2 and τ_3. We can assume that the order of the three real roots as $\tau_1 \leq \tau_2 \leq \tau_3$. Straightforward, the zeros of Equation (14) has also two real roots τ_1 and τ_2 with the order can be as $\tau_1 \leq \tau_2$. From the expression of $L'(\tau)$ we have that *L'(τ) > 0* for all $\tau \in (-\infty, \tau_1^*) \cup (\tau_2^*, +\infty)$ and $L'(\tau) < 0$ for all $\tau \in (\tau_1^*, u_2^*)$. Hence, *L(τ)* is increasing for all $\tau \in (-\infty, \tau_1^*) \cup (\tau_2^*, +\infty)$ and decreasing for all $\tau \in (\tau_1^*, \tau_2^*)$. Therefore, we finally obtain $L(\tau_1^*) \geq 0$, $L(\tau_2^*) \leq 0$, $\tau_1 \in (-\infty, \tau_1^*]$, $\tau_2 \in [\tau_1^*, \tau_2^*]$ and $\tau_3^* \in [\tau_2^*, +\infty)$.

Let requirement (M) hold. Then we obviously have $\tau_1^* \in (-1, \tau_1^*]$, $\tau_2^* \in [\tau_1^*, \tau_2^*]$ and $\tau_3 \in [\tau_2^*, 1)$. Therefore, E_e is sink.

Let requirement (Q1) hold. When $\tau_2^* < -1$, we have $\tau_1 < -1$ and $\tau_2 < -1$. Since $L(\tau)$ is increasing for all $\tau \in [\tau_2^*, +\infty)$ and $L(-1) > 0$, we can obtain $\tau_3 < -1$. Therefore, E_e is source. When $\tau_2^* > -1$, then from $L(-1) > 0$ we have $\tau_1 < -1$. If there is a $\tau_0 \in (-1, 1)$ such that $L(\tau_0) \leq 0$. Then from $L(1) > 0$ we have τ_2, $\tau_3 \in (-1, 1)$ and hence, $\tau_2^* \in (-1, 1)$ which is a contradiction. Hence $L(\tau) > 0$ for all $\tau \in (-1, 1)$. Consequently, $\tau_2 > 1$ and $\tau_3 > 1$. Therefore, E_e is also source. Similarly, we prove that when condition (Q2) holds, E_e is also source.

Let requirement (T1) hold. Then, when $\tau_1^* < -1$ we have $\tau_1 < -1$ and when $\tau_2^* > 1$ we have $\tau_3 > 1$. From $L(1) > 0$ and $L(-1) < 0$ we have $\tau_2 \in (-1, 1)$. Therefore, E_e is a saddle. Let requirement (T2) hold. Then, we immediately have $\tau_1 < -1$, $\tau_2 \in (-1, 1)$ and $\tau_3 > 1$. Therefore, E_e is a saddle too.

Let requirement (T3) hold. From $L(1) < 0$ we have $\tau_3 > 1$. Since $\tau_1^* \in (-1, 1)$, $L(-1) < 0$ and $L(\tau)$ is increasing for all $\tau \in (-\infty, \tau_1^*]$, we obtain $\tau_1 \in (-1, 1)$. Therefore, E_e is a saddle. Similarly, we can prove that when requirement (T4) holds, E_e is also a saddle.

Let requirement (V) hold. Then, it is simple to demonstrate prove that E_e is non-hyperbolic. Now, we let $\Delta * > 0$. From Lemma 4.2, Equation (13) has one real root τ_1 and a pair of conjugate complex roots $\tau_{2,3}$. If requirement (N) holds, then from $L(1) > 0$ and $L(-1) < 0$ we have that real root $\tau_1 \in (-1, 1)$. Therefore, from $|\tau_{2,3}| < 1$ we obtain that E_e is sink.

If requirement (R1) holds, then from $L(-1) < 0$ we have real root $\tau_1 > 1$. Thus, from $|\tau_{2,3}| > 1$ we obtain that E_e is source. Similarly, we can prove that if requirement (R2) holds, then E_e is also source.

If requirement (U1) holds, then we have real root $\tau_1 \in (-1, 1)$. Thus, from $|\tau_{2,3}| > 1$ we have that E_e is a saddle. Similarly, we can prove that when requirements (U2) and (U3) hold, then E_e is also a saddle. Finally, it is simple to demonstrate that if requirement (W) holds, then E_e is non-hyperbolic and this complete the proof of Proposition 4.4.

Due to bifurcation analysis [38], the bifurcation occurs when the eigenvalue of the Jacobian matrix has module one which means when a real eigenvalue is either 1 or -1, the fold bifurcation or the flip bifurcation occurs, respectively. So, when E_e exists and one condition of fourth point in the proposition 4.4 holds one type of bifurcation occurs, see [14].

5- Numerical analysis and dynamic behavior

This section presents the dynamic behavior of system (2) about each equilibrium point under different sets of parameter values by iteration method in a MATLAB program. Since there are only two equilibrium points so in this numerical study we give different initial points to see the trajectory how to attracted to each equilibrium point. Also, one can see the influence of the treatment function on model (2) when the treatment parameter β is varied.

To illustrate the stability of endemic point E_{df}, we take $h = 0.9$, $r = 0.05$, $\gamma = 0.6$, $c = 0.005$, $a = 0.3$, $\mu = 0.6$, $K = 0.5$ and $\beta = 2$. The initial conditions is $(S_0, I_0, V_0) = (3, 2, 1)$, view the Figure 1 below:

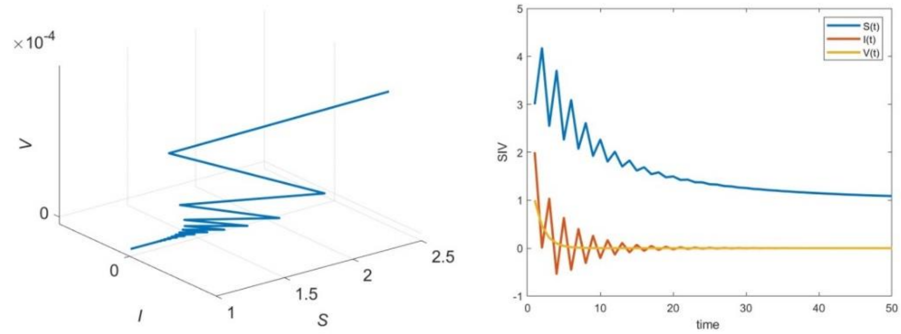

Figure 1: The phase portrait and time series of the point (S_t, I_t , V_t)=(1.08638,0,0).

We can use the set of parameters in Figure 1 with four different initial point and we see that E_{df} still stable and these initial points are (2.5, 1.5, 0.5), (2.75, 1.75, 0.75), (3.25, 2.25,1.25), and (3.5, 2.5,1.5). See Figure 2:

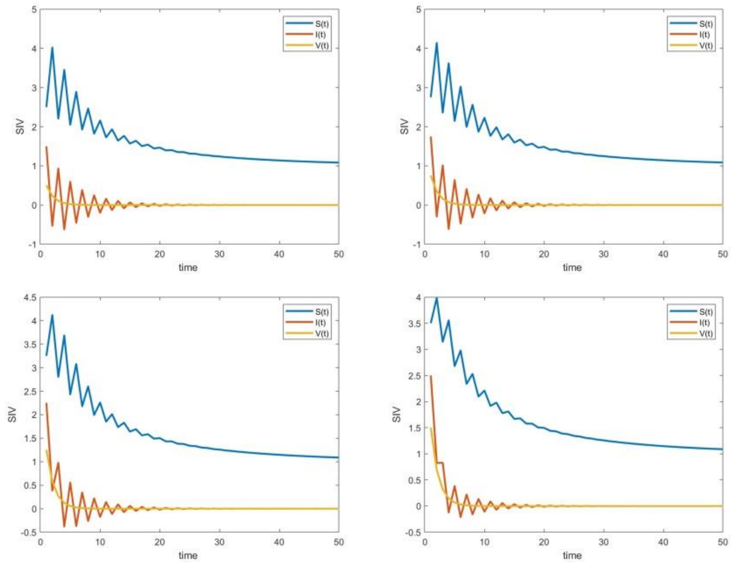

Figure 2: The time series of the point (S_t, I_t , V_t)=(1.08638,0,0) with different initial values.

One can see the influence of the treatment on the stability by fix the set of parameters in Figure 1 and varying the value of β around 2. The point E_{df} lost the stability and the dynamic behavior is changed when β reached 2.19. See the Figure 3 below:

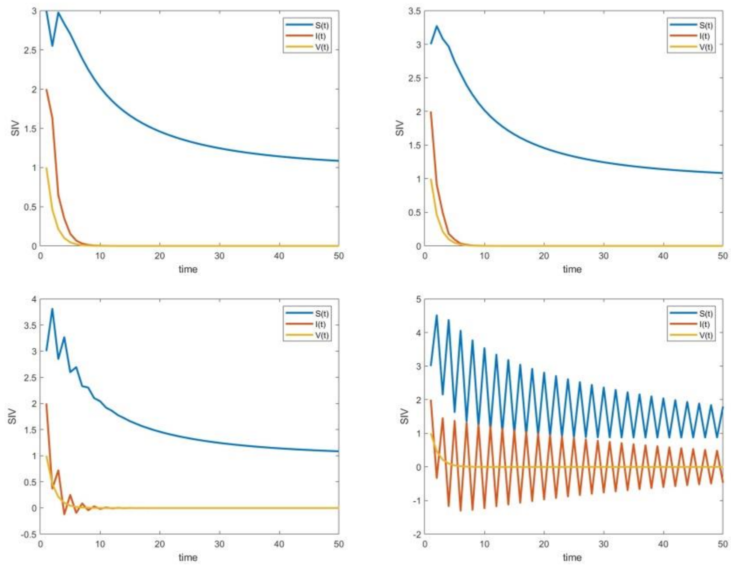

Figure 3: The time series of the point (S_t, I_t , V_t)=(1.08638,0,0) when $\beta = 1.1, 1.5, 1.7$ and 2.19.

Now, Choose the values $h = 0.04$ $r = 0.1$, $\gamma = 0.2$,$c = 40$, $a = 0.3$, $\mu = 0.5$, $K = 9$ and $\beta = 2$ with initial point (S_0, I_0 , V_0) = (*3, 2, 2*) to obtain the stability of the endemic point E_e, see the following Figure 4:

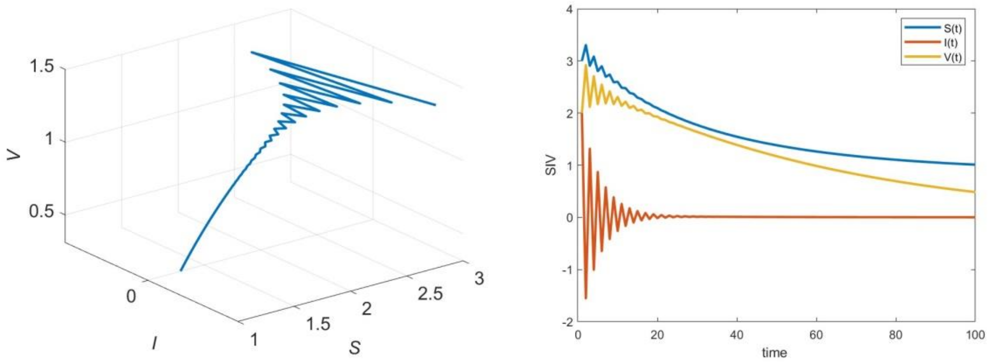

Figure 4: The phase portrait and time series of the point (S_t, I_t , V_t)=(1.01444,0.00225373,0.493938).

Similarly, we can use the set of parameters in Figure 4 with four different initial point and we see that E_e still stable and these initial points are (2.5, 1.5,1.5), (2.75, 1.75,1.75), (3.25, 2.25, 2.25), and (3.5, 2.5, 2.5). See Figure 5:

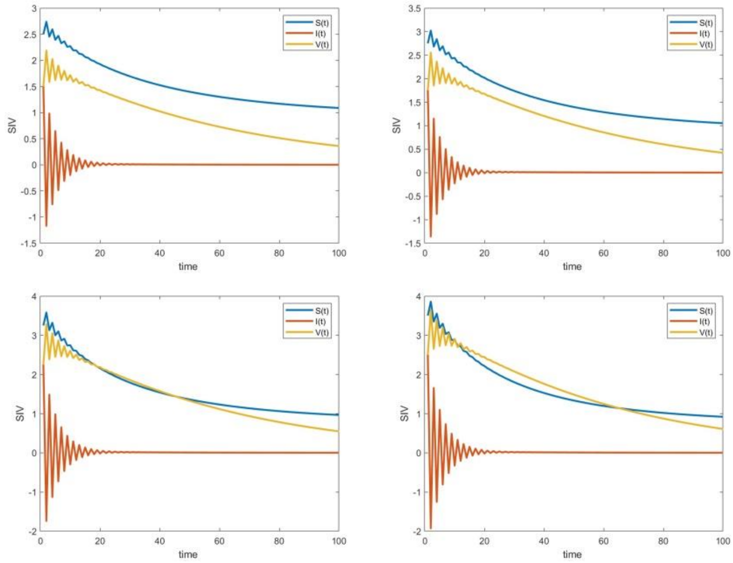

Figure 5: The time series of E_e=(1.01444,0.00225373,0.493938) with different initial values.

Finally, we can see the influence of the treatment on the stability by fix the set of parameters in Figure 4 and varying the value of β around 5. The point E_e lost the stability and the dynamic behavior is changed periodic 2 and this the sing of occurs flib bifurcation when β reached 9 and more above. See the Figure 6 below:

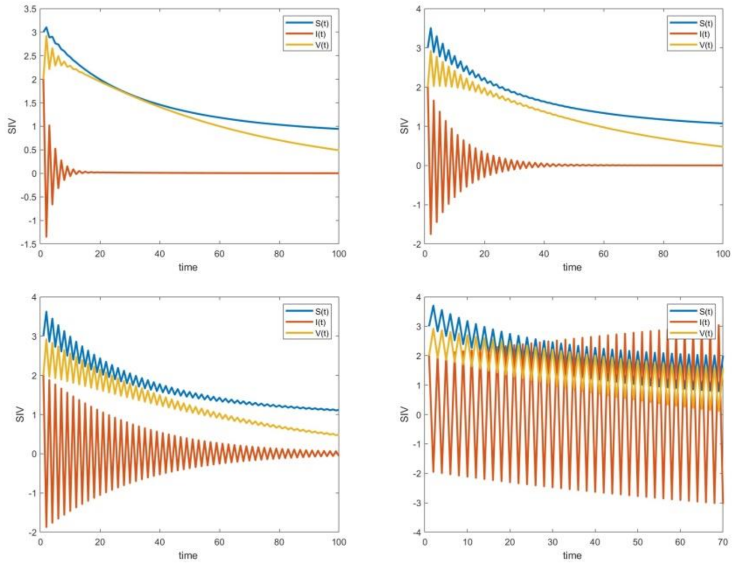

Figure 6: The time series of the point (S_t, I_t , V_t)=(1.08638,0,0) when $\beta = 2.5, 7, 9$ and 10.

6- Conclusions

The demographics of the population are changed as a result of the spread of infectious diseases. In this research, we presented an SIV model includes a viral infection with treatment. The equlibria of the model (2) were detected: The equilibrium point E_{df}, in which the infection will be extinct, and the equilibrium point E_e, in which the disease will continue to exist in the population. Numerical results present the treatment effect when each equilibrium point lost their stability such that the bifurcation and chaos appear.

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