THE COMPOSITION OPERATOR C_{α_p} ON HARDY SPACE H^2

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Abstract
In this paper, we characterize the unitary composition operator C_{α_p} on the Hardy space H^2 where α_p is a special automorphism of a unit open disk U such that $p \in U$. In addition to we study the compactness and essential normality of C_{α_p} and give some other partial results.

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الخلاصة
في هذا البحث أعطينا وصف للمؤثر التركيبي الوحدوي C_{α_p} على فضاء هاردي H2 عندما يكون αp التحويل الخطي الخاص لكرة الوحدةU ,حيث إن $p \in U$. بالإضافة إلى ذلك درسنا تراص المؤثر التركيبي C_{α_p} و المؤثر التركيبي السوي الجوهري C_{α_p} مع بعض النتائج الأخرى.

Introduction

Let U denote the unite ball in the complex plane, the Hardy space H^2 is the collection of holomorphic (analytic) functions.

$f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ with $\hat{f}(n)$ denoting the n-th Taylor coefficient of f such that $\sum_{n=0}^{\infty} \left|\hat{f}(n)\right|^2 < \infty$.

More precisely,

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n \in H^2 \Leftrightarrow$$

$$\|f\|^2 = \sum_{n=0}^{\infty} \left|\hat{f}(n)\right|^2 < \infty .$$

The inner product inducing the H^2 norm is given by

$$\langle f, g \rangle = \sum_{n=0}^{\infty} \hat{f}(n)\overline{\hat{g}(n)} \qquad (f,g \in H^2).$$

The particular importance of H^2 is due to the fact that it is a Hilbert space. Let ψ be a homomorphic function that take the unit ball U into itself (which is called homomrphic self-map of U). To each holomorphic self-map ψ of U, we associate the composition operator C_ψ defined for all $f \in H^2$ by $C_\psi f = f \circ \psi$.

In this paper, we are going to discuss some links between the function theory and the operator theory. We investigate the relationship between the properties of the symbol α_p and the

operator C_{α_p}. Composition operators have been studied in many different contexts. A good source of references on the properties of composition operators on H^2 can found in [1] and [2].
We state very loosely some basic facts on composition operator on H^2.

Theorem 1* [1]*: Every composition operator C_ψ is bounded.
Theorem 2* [2]*: C_ψ is normal if and only if $\psi(z) = \lambda z$, $|\lambda| \le 1$.
Theorem 3* [2]*: $C_\sigma C_\psi = C_{\psi o \sigma}$
Theorem 4* [2]*: C_ψ is an identity operator if and only if ψ is the identity self-map.

For each $\alpha \in U$, the reproducing kernel at α, denoted by k_α is defined by

$$k_\alpha(z) = \frac{1}{1-\overline{\alpha}z} .$$

It is easily seen for each $\alpha \in U$ and $f \in H^2$, $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ that

$$\langle f, k_\alpha \rangle = \sum_{n=0}^{\infty} \hat{f}(n)\alpha^n = f(\alpha) .$$

The reproducing kernels for H^2 will play an important role in this paper. Shapiro gave the following formula for the adjoint C_ψ^* of a composition operator C_ψ on the family $\{k_\alpha\}_{\alpha \in U}$.

Theorem 5* [1]*: Let ψ be a homomorphic self map of U, then for all $\alpha \in U$

$$C_\psi^* k_\alpha = k_{\psi(\alpha)} .$$

For $p \in U$, Shapiro [1] defined $\alpha_p(z) = \dfrac{p-z}{1-\overline{p}z}$

(Where $\overline{p}$ is the complex conjugate of p). In fact α_p is called special automorphism of U. He proved that α_p maps U into itself, and ∂U into itself. Since $\alpha_p(\alpha_p(z)) = z$, then α_p is called have self-inverse property.
This paper consists of two sections. In section one, we are going to characterize the unitary composition operator C_{α_p} on H^2 see (1.1). In section tow, we characterize the compactness and essential normality of C_{α_p} see (2.1) and (2.11). These results are new to the best of our knowledge.

1. The necessary and sufficient condition for normality of C_{α_p}.

Recall that an operator T on a Hilbert space H is called unitary if $TT^* = T^*T = I$ where T^* is the adjoint of T and I is the identity operator on H [3] . We start this section by the following result.

Theorem 1.1: C_{α_p} is a unitary operator on H^2 if and only if $p=0$.

Proof:

Assume that C_{α_p} is unitary. Assume that $p \ne 0$. Then $\alpha_p(0) = p \ne 0$.
By assumption C_{α_p} is normal, then

$C_{\alpha_p} C_{\alpha_p}^* = C_{\alpha_p}^* C_{\alpha_p}$. It follows that

$C_{\alpha_p} C_{\alpha_p}^* k_0(z) = C_{\alpha_p}^* C_{\alpha_p} k_0(z)$.

But $C_{\alpha_p} k_0 = k_0$ and by theorem (5)

$C_{\alpha_p}^* k_0 = k_{\alpha_p(0)}$. Thus

$C_{\alpha_p} k_{\alpha_p(0)}(z) = C_{\alpha_p}^* k_0(z)$. Thus from definition of the composition operator and theorem (5) we have $k_{\alpha_p(0)}(\alpha_p(z)) = k_{\alpha_p(0)}(z)$. Hence,

$$\frac{1}{1-\overline{\alpha_p(0)}\ \alpha_p(z)} = \frac{1}{1-\overline{\alpha_p(0)}\ z} .$$

It follows that $\overline{\alpha_p(0)}\ \alpha_p(z) = \overline{\alpha_p(0)}\ z$. But $\alpha_p(0) \ne 0$, then $\alpha_p(z) = z$, which a contradiction (since α_p can not be identity map). Therefore $p=0$
Conversely, suppose that $p=0$, then clearly $\alpha_p(z) = -z$. To prove that C_{α_p} is unitary, it is enough to show that

$$C_{\alpha_p} C_{\alpha_p}^* = C_{\alpha_p}^* C_{\alpha_p} = I .$$

But it is well known that the span of the family $\{k_\alpha\}_{\alpha \in U}$ is dense subset in H^2, then we can prove this equation on this family. Let $\beta \in U$,

$$C^*_{\alpha_p} C_{\alpha_p} k_\beta(z) = C^*_{\alpha_p} k_\beta(\alpha_p(z))$$

$$= k_{\alpha_p(\beta)}(\alpha_p(\mathrm{z}))$$

$$= \frac{1}{1 - \overline{\alpha_p(\beta)}\ \alpha_p(z)}$$

$$= \frac{1}{1 - \left(-\overline{\beta}\right)\ (-z)}$$

$$= \frac{1}{1 - \overline{\beta} z}$$

$$= k_\beta(z)\ .$$

Hence $C^*_{\alpha_p} C_{\alpha_p} k_\beta(z) = k_\beta(z)$ for each $\beta \in U$. This implies that $C^*_{\alpha_p} C_{\alpha_p} = I$. On the other hand,

$$C_{\alpha_p} C^*_{\alpha_p} k_\beta(z) = C_{\alpha_p} k_{\alpha_p(\beta)}(z)$$

$$= k_{\alpha_p(\beta)}(\alpha_p(\mathrm{z}))$$

$$= \frac{1}{1 - \overline{\alpha_p(\beta)}\ \alpha_p(z)}$$

$$= \frac{1}{1 - \overline{\beta} z}$$

$$= k_\beta(z)\ .$$

$C_{\alpha_p} C^*_{\alpha_p} k_\beta(z) = k_\beta(z)$ for each $\beta \in U$. This implies that $C_{\alpha_p} C^*_{\alpha_p} = I$. So $C_{\alpha_p} C^*_{\alpha_p} = C^*_{\alpha_p} C_{\alpha_p} = I$ .Therefore C_{α_p} is a unitary operator on H^2 ■

Notation: Let φ be a holomorphic self-map of U and n is a non-negative integer, the nth-iterate of φ is

$$\varphi_n \overset{def}{=} \underbrace{\varphi \circ \varphi \circ \cdots \circ \varphi}_{n-times}.$$

Now we give the following result of nth-iterate of C_{α_p}.

Corollary (1.2):

1) If n is even, then $C^n_{\alpha_p}$ is a unitary operator on H^2 for each $p \in U$.

2) If n is odd, then $C^n_{\alpha_p}$ is a unitary operator on H^2 if and only if $p=0$.

Proof:

It is easily seen that by theorem (3) $C^n_{\alpha_p} = C_{\underbrace{\alpha_p \circ \alpha_p \circ \ldots\ldots \circ \alpha_p}_{n-times}}$.

1) If n is even, then it is clear by the self-inverse property of α_p that $C^n_{\alpha_p} = I$. Therefore it is easily seen that $C^n_{\alpha_p}$ is a unitary operator on H^2.

2) If n is odd, then by the self-inverse property of α_p, we have $C^n_{\alpha_p} = C_{\alpha_p}$. Thus $C^n_{\alpha_p}$ is operator a unitary on H^2 if and only if C_{α_p} is a unitary operator on H^2. Hence we get the conclusion by (1.2)■

2. The characterization of compactness and essential normality of C_{α_p}.

Recall that an operator T on a Hilbert space H is said to be compact if it maps every bounded set into a relatively compact one (The set is called relatively compact if its closure in H is a compact set). Moreover, T is called essentially normal if $T^* T - T T^*$ is compact [4].

Proposition 2.1: C_{α_p} and $C^*_{\alpha_p}$ are not compact operators on H^2.

Proof:

By self-inverse property of α_p we have $\alpha_p^{-1} = \alpha_p$. It is easily seen that $C_{\alpha_p}^{-1} = C_{\alpha_p^{-1}}$ and $C_{\alpha_p} C_{\alpha_p} = I$. Since it is well known that every invertible operator is not compact, then C_{α_p} is not compact. Now to prove the compactness of $C_{\alpha_p}^*$. Since

$C_{\alpha_p} C_{\alpha_p} = I$ then $(C_{\alpha_p} C_{\alpha_p})^* = I^*$.

This implies that $C_{\alpha_p}^* C_{\alpha_p}^* = I$.

Therefore $\left(C_{\alpha_p}^*\right)^{-1} = C_{\alpha_p}^* = C_{\alpha_p^{-1}}^*$. Since $C_{\alpha_p}^*$ is invertible operator, then it is not compact■

Corollary 2.2: $C_{\alpha_p}^n$ is not compact operator for each non-negative integer *n*.

Proof:

If *n* is even, then $C_{\alpha_p}^n = I$. Thus $C_{\alpha_p}^n$ is not compact (since the identity operator is not compact). Moreover, if *n* is odd then $C_{\alpha_p}^n = C_{\alpha_p}$. Hence by (2.1) $C_{\alpha_p}^n$ is not compact■

To study of the essential normality of C_{α_p} we need some preliminaries.

Recall that if *T* is a bounded operator on a Hilbert space *H*. The norm of *T* is defined as follows $\|T\| = \sup\{ \|Tf\| \mid f \in H, \|f\| = 1\}$.

A holomorphic self-map ψ is called an inner function if $|\psi(z)| = 1$ a.e. on ∂U.

Calculating the exact value of the norm of a composition operator is difficult. Nordgren gave an exact value of the norm of a composition operators induced by inner functions in the next theorem.

Theorem 2.3 [5]: A holomorphic self-map ψ is an inner function if and only if $\|C_\psi\|^2 = \frac{1+|\psi(0)|}{1-|\psi(0)|}$.

Proposition 2.4: α_p is an inner function.

Proof:

Since α_p maps ∂U into itself, then by the definition of inner function we have α_p is inner function■

Therefore by (2.3) and (2.4) we can give an exact value of the composition operator induced by α_p.

Corollary 2.5: $\|C_{\alpha_p}\|^2 = \frac{1+|p|}{1-|p|}$.

Corollary 2.6:

1) $\|C_{\alpha_p}\| = 1$ if and only if *p=0*.
2) $\|C_{\alpha_p}\| > 1$ if and only if *p≠0*.

Proof:

1) Follows immediately from (2.5).

2) From (2.5) $\|C_{\alpha_p}\|^2 = \frac{1+|p|}{1-|p|}$.

Since *p≠0*, then $1-|p| < 1+|p|$. Therefore $\|C_{\alpha_p}\| > 1$■

Recall that [3] the spectrum of an operator T on a Hilbert space H, denoted by σ(T) is the set of all complex numbers λ for which T-λI is not invertible. The spectral radius of T, denoted by r(T) is defined as

$r(T) = \sup\{ |\lambda| : \lambda \in \sigma(T)\}$.

Cowen gave an easy estimate of the spectral radius of composition operator.

Theorem 2.7 [6]**:** Suppose that ψ is a holomorphic self-map of *U* and suppose that ψ has a fixed point *c*, then $r(C_\psi) = 1$ when $|c| < 1$ and

$r(C_\psi) = |\psi'(c)|^{-1/2}$ when $|c| = 1$.

By (2.7) we can compute the spectral radius of composition operator C_{α_p} by determining the position of fixed points of α_p.

Proposition 2.8: If $p \neq 0$, then $\frac{1-\sqrt{1-|p|^2}}{\bar{p}}$ is an interior fixed point and $\frac{1+\sqrt{1-|p|^2}}{\bar{p}}$ is an exterior fixed point of α_p.

Proof:

Put $\alpha_p(z) = z$ then $\frac{p-z}{1-\bar{p}z} = z$. Therefore $\bar{p}z^2 - 2z + p = 0$. Hence α_p has two fixed points

$$z_1 = \frac{1+\sqrt{1-|p|^2}}{\bar{p}} \quad and \quad z_2 = \frac{1-\sqrt{1-|p|^2}}{\bar{p}}.$$

Since p is an interior point, then $|\bar{p}| < 1 < \left|1+\sqrt{1-|p|^2}\right|$, therefore $|z_1| = \left|\frac{1+\sqrt{1-|p|^2}}{\bar{p}}\right| > 1$. This implies that z_1 is an exterior fixed point of α_p. Now we must prove that $|z_2| = \left|\frac{1-\sqrt{1-|p|^2}}{\bar{p}}\right| < 1$.

Let r = |p|, then 0 < r < 1 (p ≠ 0). Suppose that $|z_2| \geq 1$ so that $\left|1-\sqrt{1-r^2}\right| \geq r$. This implies that $\sqrt{1-r^2} < 1$.

Therefore $1-\sqrt{1-r^2} = \left|1-\sqrt{1-r^2}\right|$. Hence $1 - r - \sqrt{1-r^2} \geq 0$. This inequality implies that $1 \leq r \leq 0$, this contradicts that $0 < r < 1$. Thus z_2 is an interior fixed point of α_p.

Now we are ready to compute the spectral radius of composition operator C_{α_p} on Hardy space H^2.

Corollary 2.9: $r\left(C_{\alpha_p}\right) = 1$.

Proof:

The proof follows directly by (2.8) and (2.7) .

Definition 2.10 [7]: Let *B(H)* be a Banach space of all bounded operators on a Hilbert space *H*, and *B(H)* be the ideal of all compact operators on *H*, then the Calkin algebra is the quotient space *B(H)/B(H)*. If $T \in B(H)$, then the canonical projection $\Pi(T)$ onto *B(H)/B(H)* will be denoted by $\tilde{T}$. The essential norm of T is $\|T\|_e = \|\tilde{T}\|$. The essential spectral radius of T is $r_e(T) = r(\tilde{T})$. The essential spectrum of T is $\sigma_e(T) = \sigma(\tilde{T})$.

So, one can show that if *T* is an essential normal operator on a Hilbert space *H* then $T^*T - TT^* = 0$ in Calkin algebra. It follows easily from the definition of the essentially normal operator that every normal operator and compact operator is essentially normal.
Shapiro proved the following result.

Theorem 2.11 [5]: A holomorphic self-map ψ is inner if and only if $\|C_\psi\|_e = \|C_\psi\|$ and $r_e(C_\psi) = r(C_\psi)$.

Now we give the following result.

Theorem 2.12: C_{α_p} is an essential normal operator on H^2 if and only if $p=0$.

Proof:

Assume that C_{α_p} is an essential normal operator on H^2. Thus by [8] we have that

$$\left\|C_{\alpha_p}\right\|_e = r_e\left(C_{\alpha_p}\right) \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Suppose that $p \neq 0$, this follows by (2.6)(2) that $\left\|C_{\alpha_p}\right\| > 1$ and by (2.9) $r\left(C_{\alpha_p}\right) = 1$.
Thus by (2.10) we have that $\left\|C_{\alpha_p}\right\|_e = \left\|C_{\alpha_p}\right\| > 1$. But by (2.4) α_p is inner. Therefore by (2.11)

$$r_e\left(C_{\alpha_p}\right) = r\left(C_{\alpha_p}\right) \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Hence from (1) and (2) we get

$$1 < \left\| C_{\alpha_p} \right\|_e = r_e\left(C_{\alpha_p}\right) = r\left(C_{\alpha_p}\right) = 1,$$

which is a contradiction. Thus *p=0*. Conversely, if *p=0,* then by (1.1) we have C_{α_p} is a normal operator on *H.* Thus C_{α_p} is essentially normal■

Thus by (1.1) and (2.12) we get the following consequence.

Corollary 1.13: The following statements are equivalent:

1) C_{α_p} is a unitary operator on H^2.
2) C_{α_p} is an essentially normal operator on H^2.

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