# HOFFMAN INDEX OF MANIFOLDS

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**Abstract**

For a connected topological space M we define the homeomorphism and period noncoincidence indices of M, each of them is topological invariant reflecting the abundance of fixed point free self homeomorphisms and periodic point free self maps defined on M respectively.
We give some results for computing each of these indices and we give some examples and some results relating these indices with Hoffman index.

# دليل هوفمان للمطويات

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**الخلاصة**

لفضاء توبولوجي متصل M سنقدم في هذا البحث تعريفان الدليل التقابلي والدوري إلى M، كلّ منهما صفة توبولوجية يعاكسان عدد التقابلات والتطبيقات المعرفة على M بدون نقاط صامدة ودورية على التوالي.
أعطينا عدة نتائج نحسب فيها هذين الدليلين لفضاءات توبولوجية معروفة وبعض النتائج التي تبين علاقة هذين الدليلين بدليل هوفمان إضافة إلى أعطاء بعض الأمثلة.

## 1. Introduction

Hoffman in [1] gave the concept of noncoincidence index for a manifold M, #M as follows. If M admits k fixed point free self maps, no pair of which has a coincidence, he put $\#M \geq k + 1$. If $\#M \geq I$ for all I he put $\#M = \infty$. We could give this concept for any topological space and we could show also this is a topological invariant and it is no thomotopy type invariant since $S^{2n} \approx S^{2n} \times I$ and $\#(S^{2n} \times 1) = \infty$, $\#(S^{2n}) = 2$ see [2]. He also gave the concept of restricted noncoincidence index of a manifold M, RNI (M) which is reflecting the number of fixed point free self maps of M each of which has nonzero degree and no pair of which has a coincidence.
One can show that RNI $(S^n) = \#(S^n)$ for all n, since for each $f : S^n \to S^n$ of degree m the Lefschetz number of f, $Ł(f) = \sum_i (-1)^i \text{ tr } f_i = 1 + (-1)^n m \neq 0$ if $m \neq \pm 1$ and by Lefschetz fixed point point theorem [3], f has a fixed point.
Also RNI (R P(n)) = # (RP(n)) for all n (RP(n) the real projective space), since $H_k$ (RP(n); Q) = Q for k = 0, n and $H_k$ (RP(n), Q) = 0 other wise see [4], and hence for odd n, $Ł(f) = 1 + (-1)^n . m \neq 0$ for any map f with degree $f = m \neq 1$ and by Lefschetz fixed point theorem, f has a fixed point which implies that any fixed point free map defined on RP(n) (n odd), if it exists, must have degree 1, (note that RP(n) (n even) has the fixed point property see [5]).
Let M be a connected topological space, we say that M has the fixed point home-omporphisms property if each self homeomporphisms defined on M has a fixed point and we say M has the periodic point property if each self map defined

on M has a periodic point. We define the homeompophism noncoincidence index of M, $\#_h M$, as follows:

If M admits k fixed point-free self homeomporphisms no pair of which has a coincidence, set $\#_h M \geq k + 1$. If $\#_h M \geq i$ for all i, put $\#_h M = \infty$: otherwise #hM is the greatest number i with $\#_h M \geq i$. Similarly we define the period noncoincidence index of M, $\#_p M$ as follows:

If M admits k periodic-point-free self maps no pair of which has a coincidence, set $\#_p M \geq k + 1$. If $\#_p M \geq I$ for all I, put $\#_p M = \infty$: otherwise $\#_p M$ is the greatest number i with $\#_p M \geq i$.

Evidently $\#_h M = 1$ (where M as above) if and only if M has the fixed point homeomorphisms property and $\#_p M = 1$ if and only if M has the periodic-point property.

In §2 and §3 of this paper we study $\#_h M$, $\#_p M$ respectively, we show that these indices are also topological invariant and we give some results for computing each of them, we also give some examples and some results relating theses indices with Hoffman index.

## 2. The Homeomorphism Noncoincidence Index

For a topological space M, evidently $\#_h M \leq \#M$ (where $\#_h M$, #M mentioned in §1), hence the homeomorphisms noncoincidence index of an even dimensional sphere $S^{2n}$, complex projective space CP(n), and quaternionic projective space QP(n), n ≥ 2 are finite since #CP(n) = 2 (n odd), $\#S^{2n} = 2$ and #QP(n) = 1 (n ≥ 2) see [2]. In fact $\#_h CP(n) = \#CP(n)$ and $\#_h S^{2n} = \#S^{2n}$, since the only fixed point-free map defined on CP(n) (n odd) takes z to $\bar{z}$ see [2] and [5], and the only fixed point-free map defined on $S^{2n}$ is the antipodal map [5], (of course CP(2n) = 1 and we will see later that $\#_h S^1 = \infty$).

The following proposition follows from the definition.

**Proposition 2.1.**

If a group G acts freely on a space M, then $\#_h M \geq$ card G if G is finite, and $\#_h M = \infty$ if G is infinite.

*Proof.* Since G acts freely on M, then the set of fixed points.

$F(g) = \{x \in M \mid gx = x\} = \phi$ for all $e \neq g \in G$, hence $\varphi_g : M \to M$ is fixed point free homeomorphisms where $\varphi_g(x) = gx$ for all $x \in M$ and we have card G – 1 fixed point free homeomorphisms.

Now let $\varphi_{g_1}, \varphi_{g_2}$ be any pair of these homeomporphisms such that $\varphi_{g_1}(x), \varphi_{g_2}(x)$ for some $x \in M$, hence $\varphi_{g_1}^{-1}\varphi_{g2}(x) = \varphi_{g1}^{-1}\varphi_{g_2}(x) = \varphi_{g_1 g_2}^{-1}(x) = x$, a contradiction.

It follows immediately from (2.1) that $\#_h S^n = \infty$ (n odd), and any connected nontrivial Lie group has homeompophism noncoincidence index ∞. In fact we compute the noncoincidence index of some familiar manifolds.

**Remark 2.2**

Let $F(1^k, n)$ denote the homogeneouse space U (n + k)/H, where H is conjugate to $U(1^k) \times U(n)$. if n = 0 then $F(1^k, 0)$ is written $F(1^k)$ which is called the flag manifold.

Hoffman in [6] shows that $\#(F(1^k) = k!$, and since the symmetric group $\sum_n$ acts freely on $F(1^k)$ see [6], thus $\#_h F(1^k) = k!$. similarly unless k = 2 and (n even), $\#_h F(1^k, n) = k! = \#F(1^k, n)$ see [1].

Note also $S^1 \times S^1$ acts freely on $S^n \times S^n$ (n odd) see e.g. [5], hence $\#_h(S^n \times S^n) = \infty$. For even n, since $z_2 \oplus z_2$, $z_4$ do actually act freely on $S^n \times S^n$ see [5], hence $\#_h(S^n \times S^n) \geq 4$ (n even).

The following proposition shows that the homeompophism noncoincidence index is a topological invariant.

**Proposition 2.3.**

Let M, M′ be any two homeomorphic topological spaces, then $\#_h M = \#M'$.

*Proof.* If $\#_h M < \infty$ say $\#_h M = k + 1$, then we have k fixed point free self homeomorphisms on M and no pair of which has a coincidence $\{f_1, \ldots, f_k\}$. Let h : M′→M be a homeompophism, the homeompophism.

$g_i = h^{-1} f_i h : M' \to M$,

for each $1 \leq i \leq k$ are fixed point free and no pair of which has a coincidence. Now suppose there exists a homeomorphism J : M′ → M′ which is fixed point free, then $hJh^{-1} : M \to M$ is fixed point free homeomorphism a contradiction since $\#_h M = k$, thus M′ admits exactly k fixed point free self homeomorphisms. Now if $g_\ell(x) = g_t(x)$, $\ell$, t ∈ {1, …, k}, then $h^{-1}f_\ell h(x) = h^{-1}f_t h(x)$ implies $f_\ell(h(x)) = f_t(h(x))$ which is obvious contradiction, hence $\#_h M' = k$. The proof when $\#_h M = \infty$, is similar.

It follows immediately from this proposition that $\#_h QP(1) = 2$, since QP(1) homeomorphic to $S^2$ see e.g. [3].
Recall that a manifold M is called L-rigid if $\chi(M) \neq 0$ and the set
$LZ(M) = \{f^* : H^*(M; Q) \to H^*(M; Q) \mid f$ is fixed point free$\}$,
consists of only endompophisms with deg $f \neq 0$, see [1].
The following two propositions give necessary conditions for homeomorphism noncoincidence index to be finite.

**Proposition 2.4.**

Let M be a manifold that has the same integral homology as $S^n \times S^n$ (n even) and M does not admit a self map f such that trace $f_n = -1$, then $\#_h M < \infty$.

*Proof.* Let $f : M \to M$ be a fixed point free map with deg f = 0, then

$$\begin{aligned} L(f) &= \sum_{q=0}^{2n} (-1)^q \operatorname{tr} f_q \\ &= 1 + \operatorname{tr} f_n + \deg f \\ &= 1 + \operatorname{tr} f_n \\ &\neq 0, \end{aligned}$$

hence by Lefschetz fixed point theorem, f has a fixed point. Thus LZ(M) consists of only endomorphisms with deg $f \neq 0$. Now since $\chi(M) \neq 0$, hence M is L-rigid and hence $\#M < \infty$, see [2] and we have done.

**Proposition 2.5.**

Let M be a closed orientable manifold that has the same integral homology as $S^n \times S^m$ (n+m even $n \neq m$) and M does not admit a map which reverse orientation then $\#_h M = 1$.

*Proof.* Let $f : M \to M$ be a homeomprphism, then deg $f = \pm 1$ and since deg $f \neq -1$, then by proposition 2.1 of [7], f has a fixed point, hence $\#_h M = 1$.
We saw earlier that $\#_h M = \#M$ for several examples of manifolds, the following example shows that $\#_h M < \#M$.

**Example 2.6.**

Let $M = S^1 = \{z \in C \mid |z| = 1\} = \exp(2\pi i\theta); 0 \leq \theta < 1$. For $x$ a real number, let $\{x\} = x - [x]$, where $[x]$ is the largest integer smaller than or equal to $x$, now let $h : S^1 \to S^1$ be the homeomprphism defined as follows:

$$h(\exp(\theta 2\pi i)) = \exp\left(\left\{\theta + \frac{1}{\sqrt{2}}\right\} 2\pi i\right).$$

Clearly

$$h^k(\exp(\theta 2\pi i)) = \exp\left(\left\{\theta + \frac{k}{\sqrt{2}}\right\} 2\pi i\right).$$

Hence
$h^k(\exp(\theta 2\pi i)) = \exp(\theta 2\pi i)$,
only if $k/\sqrt{2}$ where an integer for some integer k, so $h^k$ has no fixed point for all $k \geq 1$. Thus any rotation by an irrational number gives a fixed point free homeomprphism, and these homeomorphisms are noncoincident, hence $S^1$ actually has homeomorphism noncoincidence index $\infty$.
Now let $f : S^1 \vee S^2$ (where $\vee$ denotes the union of $S^1$ and $S^2$ in one point), be a map defined by:

$$f(x, y) = (h(x), \overline{y}),$$

Where $y \to \overline{y}$ is the antipodal map. Note that $\chi(S^1 \vee S^2) = 1$, and f is periodic point free. But by Halpern theorem [4], $S^1 \vee S^2$ has the fixed point property for homeomprphisms, hence $\#_h(S^1 \vee S^2) < \#(S^1 \vee S^2)$.

**Note:** $S^1 \vee S^2$ in the preceding example is not manifold, in fact we don't know if the exists a manifold M with $\#_h M < \#M$.

## 3. The Period Noncoincidence Index

We introduced in section one the period noncoincidence index of a topological space M. It follows immediately from this definition that for even n, $\#_p CP(n) = \#_p QP(n) = \#_p RP(n) = 1$, (of course CP(n), QP(n) and RP(n), have the fixed point property), and since each periodic point free is fixed point free hence $\#_p(M) \leq \#(M)$.
The following example shows that the 1-sphere has infinite period noncoincidence index.

**Example 3.1.**

We saw in example (2.6), that each map $h_t : S^1 \to S^1$ such that:

$$h(\exp(2\pi i\theta)) = \exp\left(\left\{\theta + \frac{1}{t}\right\} 2\pi i\right); t \in Q',$$

is periodic point free ($Q'$ is the irrational numbers); suppose now we have a coincidence between a pair $h_{t_1}, h_{t_2}$ of these maps, hence there exists $x \in S^1$ such that $h_{t_1}(x) = h_{t_2}(x)$, implies.

$$\exp\left(\left\{\theta+\frac{1}{t_1}\right\}2\pi i\right)=\exp\left(\left\{\theta+\frac{1}{t_2}\right\}2\pi i\right),$$

which holds only if $t_1 = t_2$, this means $h_{t_1}, h_{t_2}$ have the same rotation a contradiction, hence we have infinitely many periodic point free maps defined on $S^1$ and no pair of which has a coincidence and hence $\#_p S^1 = \infty$.

The following proposition appears in [2] without proof, we give a proof.

**Proposition 3.2.**

If $\#M < \infty$, then any self map of M has a periodic point of period at most #M.

*Proof:* Let $\#M = k + 1$, and let $f : M \to M$ be a periodic point free map, hence f is fixed point free. If each of $f^2, f^3, f^4, \ldots, f^k : M \to M$ has no periodic point then $f, f^2, \ldots, f^k$ is a set of k fixed point free maps. If $f^{k+1} : M \to M$ is periodic point free then there exists $i, j \in \{1, \ldots, k\}$ such that $f^i(x) = f^j(x)$ for some $x \in M$,

let $i > j$ and let $f^j(x) = y$ then $f^{i-j}(y) = f^{i-j}(f^j(x)) = f^i(x) = f^j(x) = y$, a contradiction, and the conclusion follows.

It follows immediately from this proposition that $\#_p M = 1$, where M is one of the following manifolds see (section 2). CP(n) (n odd), $S^{2n}$ and QP(n) for all n, the flag manifold F(n), and the general flag manifold $F(1^k, n)$ ($k \neq 2$, n odd or n = 0).

Hoffman in [2] shows that for a compact manifold M, $\#M < \infty$ if and only if $\chi(M) \neq 0$ and M admits no fixed point free map of degree zero, thus we have the following remark.

**Remarks 3.3.**

A similar proof of (3.2) can be used to show, if $\chi(M) \neq 0$, then any self map of M of non-zero degree has a periodic point of period at most RNI (M). Thus if $\chi(M) \neq 0$, then any periodic point free map defined on M if it exist, has degree zero and hence we can not find a relation between $\#_h M$ an $\#_p M$.

**Note:** We call an action φ of a topological group G on a topological space M a periodic free action if the periodic point set of $g \in G$,

$F_p(g) = \{x \in M \mid \varphi^n(g, x) = x, n \in N\} = \phi$ for all $g \neq e$.

Now we have the following proposition.

**Proposition 3.4.**

If a group G acts periodic freely on a topological space M, then $\#_p M \geq$ card G if G is finite and $\#_p M = \infty$ if G is infinite.

*Proof.* Since G acts periodic freely, then $F_p(g) = \phi$, for all $e \neq g \in G$, hence the homeoporphisms $\varphi_g : M \to M$ for all $e \neq g \in G$ are periodic point free. Now let $\varphi_{g_1}, \varphi_{g_2}$ be any pair of these homeomorphisms such that $\varphi_{g1}(x) = \varphi_{g2}(x)$ for some $x \in M$, hence.

$$\begin{aligned}\varphi_{g_2}^{-1}\varphi_{g_1}(x) &= \varphi_{g_2^{-1}}\varphi_{g_1}(x)\\ &= \varphi_{g_2^{-1}g_1}(x)\\ &= x,\end{aligned}$$

a contradiction, and hence we have card G-1 of periodic point free noncoincident maps, which completes the proof. □

**Example 3.5.**

Let $G = Z \times Z$ and $M = R \times R$, define an action φ of G on M as follows:

$\varphi((a,b), (t_1+t_2)) = (a+t_1, b+t_2)$; $(a,b) \in Z \times Z$, $(t_1+t_2) \in R \times R$.

This action is periodic free and hence by the (3.4) $\#_p(R \times R) = \infty$.

The following proposition shows that any homeomorphic topological spaces have the same period noncoincidence index.

**Proposition 3.6.**

Let M, M' be any two homeomorphic topological spaces then $\#_p M = \#_p M'$.

*Proof.* If $\#_p M < \infty$ say, $\#_p M = k+1$, then we have $\{f_1, \ldots f_k\}$ of periodic point free noncoincident maps.

Let $h: M' \to M$ be a homeomorphism, then the maps $g_i = h^{-1} f_i h : M' \to M'$, $1 \leq i \leq k$, are periodic point free maps. Now suppose there exists a map $J : M' \to M'$ which is periodic point free, then the map $hJh^{-1} : M \to M$ is periodic point free, but $\#_p M = k$, hence $J = h^{-1} f_t h = g_t$ for some $1 \leq t \leq k$, thus M′ admits exactly k periodic point free self maps. It is clear that no pair of $\{g_1, \ldots, g_k\}$ has a coincidence see the proof of (2.3) and this completes the proof. When $\#_p M = \infty$ the proof is similar.

Let $f : M \to M$ be a continuous map of a compact polyhedron M into itself. If $H_i(M; Q) \approx 0$ for odd i, then Halpern in [8] shows that at least one of the maps $f, f_2, \ldots, f^{\chi(M)}$ has a fixed point. Thus $\#_p(S^n \times S^n) = 1$ (n even) and $\#_p(S^n \times S^m) = 1$ (n, m even $n \neq m$). For the space $S^n$ (n odd) we have the following example.

**Example 3.7.**

Let $f : S^n \to S^n$ (n odd) be a transversal map with $\deg f \notin \{-1, 0, 1\}$, then the Lefschetz number of period m, $\ell(f^m)$ see [9] is:

$$\ell(f^m) = \sum_{d/m} \mu(d) L(f^{m/d}) = \sum_{d/m} \mu(d)[1 + (-1)^n \deg(f)^{m/d}]$$

Where $\mu(d)$ is the Möbius function see [9], then $\ell(f^m) \neq 0$ and by [10] the set $\{m \in N \mid m \text{ odd}\}$ is subset of period of f, hence the only periodic point free maps defined on $S^n$ (n odd) must be nontransversal with degree $\in \{-1, 0, 1\}$.
Finally we have the following proposition.

**Proposition 3.8.**

Let M be a compact polyhedron, if the Euler characteristic of M does not vanish and $\#_h M = \# M$ then $\#_p M = 1$.

*Proof.* Since $\#_h M = \# M$ then M admits ($\#M$ -1) fixed point free self homeomorphisms no pair of which has a coincidence and since $\chi(M) \neq 0$, then by [4] some iterate of each homeomorphism has a fixed point, hence $\#_p M = 1$.

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