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Γ-(λ,δ)-Derivation on Semi-Group Ideals in Prime Γ-Near-Ring

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Abstract
The main purpose of this paper is to investigate some results. When h is Γ-(λ,δ) – Derivation on prime Γ-near-ring G and K is a nonzero semi-group ideal of G, then G is commutative .

Keywords: Prime Γ-near-ring, Semi-group ideal, Γ - (λ,δ) – derivation

اشتقاق كاما− (λ,δ) على مثالي شبه اولي في حلقه كاما المقتربه الاوليه

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الخلاصه
الهدف الرئيسي من هذا البحث هو دراسة بعض النتائج عندما تكون h هي اشتقاق كاما− (λ,δ) على الحلقه كاما المقتربه الاوليه G و K هي مثالي شبه اولي غير صفري في G فان G حلقه ابدالية.

1. Introduction

Throughout this paper, G denotes a zero – symmetric left Γ-near-ring with a multiplicative center Z(G). For a Γ-near-ring G, the set G_0 ={s∈G: $0\rho s=0$, $\forall\ \rho\in\Gamma$}is called a zero symmetric part of G. If $G=G_0$,then G is called a zero symmetric [1,2,3,4]. An additive mapping $h:G\rightarrow G$ is called a Γ-(λ, δ)-derivation on a Γ-near-ring G If there exist two automorphisms mapping $\lambda,\delta :G \rightarrow G$, such that $h(s\rho r)=h(s)\rho\lambda(r)+\delta(s)\rho h(r)$, for every $s,r\in G$ and $\rho\in\Gamma$ [4,5]. A Γ-near-ring G is said to be a prime Γ-near-ring if $s\Gamma G\Gamma r=0$ implies $s=0$ or $r=0$, for every $s,r\in G$, and it said to be a semiprime if $s\Gamma G\Gamma s=0$ implies $s=0$ for every $s\in G$ [5,6]. Further, an element s ∈ G is called constant if h(s)= 0 [4,7]. A non-empty subset K of G is called semi-group ideal if $K\Gamma G\subset K$ and $G\Gamma K\subset K$ [8]. For s,r∈ G and ρ∈Γ, the symbol $[s,r]^{\rho}_{\lambda,\delta}$ will denote δ(s)ρr – rρλ(s), as previously described[4,9]. The other commutators are $[s,r]_{\rho}$= sρr–rρs and (s,r) = s+r–s–r which denote the additive-group commutator [4,9].

The purpose of this paper is to study and generalize some results of previous authors [4,7,9,10] on the commutativity of the prime Γ-near-ring. Some recent results on rings deal with commutativity of prime and semiprime rings admitting suitably constrained derivations. For further details on prime near-ring, we refer to some previous articles [11-15].

As a generalization of near-ring, the Γ-near-ring was discussed by Satyanarayana [6], while Booth and Groenewald [5,13] surveyed various portions in the Γ-near-ring .In this paper, we investigate the condition for a Γ-(λ,δ)-derivation on a prime Γ-near-ring to be commutative.

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2. The Main Results

In this section, we investigate some results of a semi-group ideal of a Γ- near- ring admitting a Γ-(λ,δ)-derivation.

To prove the main theorems, we need the following lemmas.

Lemma 2.1 . Let h be a Γ-(λ ,δ) – derivation on a prime Γ-near-ring and K a semi-group ideal of G, if and only if $h(s\eta r) = \delta(s)\eta h(r) + h(s)\eta\lambda(r)$, for all $s,r \in K$ and $\eta \in \Gamma$.

Proof . $\forall s,r \in K$ and $\eta \in \Gamma$, we have $s\eta(r + r) = s\eta r + s\eta r$.

By applying h for both sides we obtain

$$\begin{aligned} h(s\eta(r+r)) &= h(s)\eta\lambda(r+r) + \delta(s)\eta h(r+r) \\ &= h(s)\eta\lambda(r) + h(s)\eta\lambda(r) + \delta(s)\eta h(r) + \delta(s)\eta h(r). \end{aligned}$$

and

$$\begin{aligned} h(s\eta r + s\eta r) &= h(s\eta r) + h(s\eta r) \\ &= h(s)\eta\lambda(r) + \delta(s)\eta h(r) + h(s)\eta\lambda(r) + \delta(s)\eta h(r). \end{aligned}$$

By comparing the two relations, we have

$$\begin{aligned} h(s)\eta\lambda(r) + \delta(s)\eta h(r) &= \delta(s)\eta\lambda(r) + h(s)\eta\lambda(r) \\ h(s\eta r) &= \delta(s)\eta\lambda(r) + h(s)\eta\lambda(r) \end{aligned}$$

$\forall s,r \in K$ and $\eta \in \Gamma$.

Conversely, assume for every $s,r \in K$ and $\eta \in \Gamma$, that

$$h(s\eta r) = \delta(s)\eta\lambda(r) + h(s)\eta\lambda(r) \ .$$

then,

$$\begin{aligned} h(s\eta(r+r)) &= \delta(s)\eta h(r+r) + h(s)\eta\lambda(r+r) \\ &= \delta(s)\eta h(r) + \delta(s)\eta h(r) + h(s)\eta\lambda(r) + h(s)\eta\lambda(r). \end{aligned}$$

and

$$\begin{aligned} h(s\eta r + s\eta r) &= h(s\eta r) + h(s\eta r) \\ &= \delta(s)\eta h(r) + h(s)\eta\lambda(r) + \delta(s)\eta h(r) + h(s)\eta\lambda(r). \end{aligned}$$

Comparing the two relation provides that:

$$\begin{aligned} \delta(s)\eta h(r) + h(s)\eta\lambda(r) &= h(s)\eta\lambda(r) + \delta(s)\eta h(r) \\ h(s\eta r) &= h(s)\eta\lambda(r) + \delta(s)\eta h(r) \end{aligned}$$

Lemma 2.2.If h be a Γ -(λ ,δ) – derivation on a Γ-near-ring G, K a semi-group ideal of G, and $\lambda(K)=K$, then

$(h(s)\eta\lambda(r) + \delta(s)\eta h(r))\rho v = h(s)\eta\lambda(r)\rho v + \delta(s)\eta h(r)\rho v$, for all $s,r,v \in K$ and $\eta, \rho \in \Gamma$.

Proof . Assume that $\forall s ,r,v \in K$ and $\eta,\rho \in \Gamma$.

$$\begin{aligned} h((s\eta r)\rho v) &= h(s\eta r)\rho\lambda(v) + \delta(s\eta r)\rho h(v) \\ &= (h(s)\eta\lambda(r) + \delta(s)\eta h(r))\rho\lambda(v) + \delta(s)\eta\delta(r)\rho h(v) \end{aligned}$$

and ,

$$\begin{aligned} h(s\eta(r\rho v)) &= h(s)\eta\lambda(r\rho v) + \delta(s)\eta h(r\rho v) \\ &= h(s)\eta\lambda(r)\rho\lambda(v) + \delta(s)\eta h(r)\rho\lambda(v) + \delta(s)\eta\delta(r)\rho h(v) \end{aligned}$$

Comparing the two relations above of $h(s\eta r\rho v)$, $\forall s,r,v \in K$ and $\eta,\rho \in \Gamma$.

, and since $\lambda(K)=K$, implies that

$$(h(s)\eta\lambda(r) + \delta(s)\eta h(r))\rho v = h(s)\eta\lambda(r)\rho v + \delta(s)\eta h(r)\rho v \ .$$

Lemma 2.3 . If h be a Γ -(λ ,δ) – derivation on a Γ-near-ring G and K a semi-group ideal of G such that $h([s, r]_\rho) = [s,r]_\rho$, $\lambda(K)=K$, and $\delta(K)=K$, then

(i) $h(v) = v$, for every commutator v in K.

(ii) $h(k)\gamma[s, r]_\rho = [s ,r]_\rho\gamma h(k)$, for every $s, k \in K$, $r \in G$ and $\rho,\gamma \in \Gamma$.

Proof .(i) Let $v = [s, r]_\rho$, where $s\in K$, $r\in G$ and $\rho\in\Gamma$.

$$h([s, r]_\rho) = [s,r]_\rho, \text{ for every } s\in K, r\in G \text{ and } \rho\in\Gamma.$$

Thus, $h(v) = v$, for each commutator v in K.

(ii) By the hypothesis that $h([s, r]_\rho) = [s ,r]_\rho$,we have

$$-[s,r]_\rho\gamma k + h([s,r]_\rho\gamma k) = -k\gamma[s,r]_\rho + h(k\gamma[s,r]_\rho), \ \forall\ s,k\in K, r\in G \text{ and } \rho,\gamma\in\Gamma$$

By using Lemma 2.1, we have

$$-[s,r]_\rho\gamma k + h([s,r]_\rho)\gamma\lambda(k) + \delta([s,r]_\rho)\gamma h(k) = -k\gamma[s,r]_\rho+ h(k)\gamma\lambda([s,r]_\rho) + \delta(k)\gamma h([s,r]_\rho)$$

By applaying of (i) and as $\lambda(K) = K$,$\delta(K) = K$,λ is an automorphism, we obtain:

$$[s, r]_\rho\gamma h(k) = h(k)\gamma[s, r]_\rho ,\forall\ s,k\in K, r\in G \text{ and } \rho,\gamma\in \Gamma.$$

Lemma 2.4 . If h is a Γ -(λ ,δ) – derivation on a Γ-near-ring G, K is a nonzero semi-group ideal of G, and $h([s, r]_\rho) = [s,r]_\rho$, $\lambda(K)=K$,$\delta(K)=K$, then

(i) If v is a commutator in K and $w\mu v = z\mu v$, where $w,z\in K$ and $\mu\in \Gamma$, then $v\mu h(w\text{-}z) = 0$.

(ii) If v_1 and v_2 are commutators in K with $v_1\mu v_2 = 0$, then $v_1 = 0$ or $v_2= 0$.

Proof. (i) Let $v = [s, r]_\rho$, $\forall s\in K$, $r\in G$ and $\rho\in \Gamma$.

Then, the hypothesis provides that $w\mu[s,r]_\rho = z\mu[s,r]_\rho,\forall\ s,w,z\in K,r\in G$ and $\rho,\mu\in \Gamma$.

Applying h for both sides, implies that

$$h(w\mu[s,r]_\rho) = h(z\mu[s,r]_\rho),\forall\ s,w,z\in K,\ r\in G \text{ and } \rho,\mu\in \Gamma.$$

Thus, $$h(w)\mu\lambda([s,r]_\rho) + \delta(w)\mu h([s,r]_\rho) = h(z)\mu\lambda([s,r]_\rho) + \delta(z)\mu h([s,r]_\rho).$$

Using Lemma 2.3 (i,ii) provides that:

$$h(w)\mu\lambda([s,r]_\rho) = h(z)\mu\lambda([s,r]_\rho),\forall\ s,w,z\in K , r\in G \text{ and } \rho,\mu\in\Gamma.$$

So , $[s ,r]_\rho\mu h(w\text{-}z) = 0$. Thus, $v\mu h(w\text{-}z) = 0$, for every commutator v in K, $w,z\in K$, and $\mu\in \Gamma$.

(ii) If $v_1\mu v_2 = 0 = 0\mu v_2$, since v_2 is a commutator in K, (i) yields

$$v_2\mu h(v_1) = 0 \qquad \text{………..…. (1)}$$

By using Lemma 2.3 (i), since v_1 is a commutator in K, we obtain

$$v_2\mu v_1 = 0 \qquad \text{…………… (2)}$$

B substituting $r\gamma v_1$ for v_1, where $r\in K$,$\gamma\in\Gamma$ in equation (1), we obtain:

$$v_2\mu h(r\gamma v_1) = 0 = v_2\mu h(r)\gamma\lambda(v_1) + v_2\mu\delta(r)\gamma h(v_1) \qquad \text{…. …………(3)}$$

Using Lemma 2.3 (ii) and equation (2) in equation (3) provides that:

$v_2\,\mu\delta(r)\gamma h(v_1) = 0$, for every commutator v_1, v_2 in K, $r\in K$, and $\mu,\gamma\in\Gamma$.

Hence, $v_2\Gamma K\Gamma h(v_1) = 0$.

By using Lemma 2.3 (i), since v_1 is commutator, we obtain $v_2\Gamma K\Gamma v_1 = 0$.

Since K is a nonzero semi-group ideal of G and G is a prime Γ-near-ring, we obtain $v_1 = 0$ or $v_2 = 0$.

Lemma 2.5 . If G be a prime Γ-near-ring and K is a nonzero semi-group ideal of G, then $Z(K)\subseteq Z(G)$.

Proof. Suppose that $t\in Z(K)$, this means that, $[t,s]_\rho = 0$, $\forall s\in K$ and $\rho \in \Gamma$.

Replacing s by $s\mu r$, so $r\in G$ in the above equation, we obtain

$[t, s\mu r]_\rho = 0 = s\mu[t,r]_\rho + [t,s]_\rho \mu r$, $\forall$ t,s$\in$ K, r$\in$ G and $\rho,\mu \in \Gamma$.

Thus, $K\mu[t,r]_\rho = 0$. Since K is a nonzero semi-group ideal of G and G is a prime Γ-near-ring, we get $[t,r]_\rho = 0$, ∀t∈ K, r∈ G and ρ∈ Γ. Hence, t∈ Z(G).

Lemma 2.6 . If h is a Γ -(λ ,δ) – derivation on a prime Γ-near-ring G and K be a semi-group ideal of G.

(i) If u is a nonzero element in Z(G), then u is not a zero divisor.

(ii) If there exists a nonzero element u of Z(G) such that u + u ∈ Z(G), then (K,+) is an abelian.

Proof . (i) If u∈ Z(G)\{0} and $u\eta s = 0$,∀s∈ K and η∈ Γ.

Then ,left multiplication of this equation by tγ, where t∈ G and γ∈ Γ, provides that

$t\gamma u\eta s = 0$. Since G is a multiplicative with the center Z(G), it implies that

$u\gamma t\eta s = 0$, ∀t∈ G and s∈ K ,thus , $u\Gamma G\Gamma s = 0$.

Since G is a prime Γ-near–ring and u is a nonzero element, it shows that s = 0.

(ii) Let u∈ Z(G)\{0} be an element, such that u+ u ∈ Z(G),

Let $s, r \in K$ and $\rho\in \Gamma$ so,

$$(s+r)\rho(u+u) = (u+u)\rho(s+r)$$

$$s\rho u + s\rho u + r\rho u + r\rho u = u\rho s + u\rho r + u\rho s + u\rho r$$

Since u ∈ Z(G),we get $u\rho s + u\rho r = u\rho r + u\rho s$

Thus, $u\rho(s+r-s-r) = 0$,∀ s, r∈ K and ρ ∈ Γ

Left multiplication this equation by aγ , where a∈G, γ∈ Γ, provides that:

$a\gamma u\rho(s,r) = 0$, ∀ s,r ∈ K , a∈G and γ,ρ ∈ Γ.

Because G is a multiplicative with the center Z(G) , this provides that:

$u\gamma a\rho(s,r) = 0$. Hence, uΓGΓ(s,r) = 0

Because G is a prime Γ-near-ring and u is a nonzero element, it implies that

(s, r) = 0, ∀ s,r ∈ K. Thus, (K, +) is an abelian. □

Lemma 2.7 . If h be a nonzero Γ-(λ,δ)-derivation on a prime Γ-near-ring G and K be a nonzero semi-group ideal of G. Then sΓh(K) = 0, which implies that s = 0 and h(K)Γs= 0, which means that s = 0, where s∈ G.

Proof. Assume that sΓh(K) = 0, ∀r∈ G , t∈ K and β∈Γ

Then, sηh(tβr) = 0, showing that:

sηh(t)βλ(r) + sηδ(t)βh(r) = 0

Therefore , ∀ s, r∈ G , t∈K and η,β∈Γ , we have sηδ(t)βh(r) = 0.

Since δ(K) = K , then sΓKΓh(r) = 0.

Since K is a nonzero semi-group ideal and G is a prime Γ-near–ring, h ≠0,

it implies that s = 0.

Similarly , we can show that if h(K)Γs = 0, ∀ s∈ G, it implies that s = 0. □

Lemma 2.8 . If G is a 2 – torsion free prime Γ-near-ring, h be a nonzero Γ-(λ,δ)-derivation of G, and K be a nonzero semi-group ideal of G. If $h^2(K) = 0$ and λ, δ commute with h , then h(K) = 0.

Proof . ∀s,r∈ K and ρ∈ Γ.

$$\begin{aligned} 0 = h^2(s\rho r) = h(h(s\rho r)) &= h(h(s)\rho\lambda(r) + \delta(s)\rho h(r)) \\ &= h(h(s)\rho\lambda(r)) + h(\delta(s)\rho h(r)) \\ &= h^2(s)\rho\lambda^2(r) + \delta(h(s))\rho h(\lambda(r)) + h(\delta(s))\rho\lambda(h(r)) + \delta^2(s)\rho h^2(r) \end{aligned}$$

By the hypothesis, we obtain that 2h(δ(s))ρh(λ(r))= 0, ∀ s,r ∈ K and ρ∈Γ.

Because G is a 2 – torsion free and λ(K) = K, this provides h(δ(s))ρh(K) = 0

By using Lemma 2.7, we obtain tthat h = 0. □

Lemma 2.9. Let G be a prime Γ-near–ring and K be a nonzero semi-group ideal of G. If K is a commutative then G is a commutative ring.

Proof. ∀ s,r ∈ K, $[s,r]_\rho = 0$.

By taking sγa instead of s and rγb instead of r , where a,b ∈ G and γ∈Γ , we obtain that $[s\gamma a, r\gamma b]_\rho = 0$. Since K is a commutative and semi-group ideal of G, this provides

$$0 = s\gamma a\rho r\gamma b - r\gamma b\rho s\gamma a \quad = s\gamma r\gamma a\rho b - s\gamma r\gamma b\rho a$$
$$= s\gamma r\gamma [a,b]_\rho$$

$\forall$a,b$\in$ G, s,r $\in$ K and $\gamma,\rho\in\Gamma$, this implies that $s\Gamma K\Gamma[a,b]_\rho =0$.
Because K is a a nonzero semi-group ideal of G, G is a prime Γ-near–ring , thus $[a,b]_\rho = 0$, $\forall$ a,b $\in$ G. Thus, G is a commutative ring.

Lemma 2.10 . If *G* is a prime Γ-near-ring and K is a nonzero semi-group ideal of G. If (K,+) is an abelian, then (G,+) is an abelian.

Proof . Since (K ,+) is an abelian, we obtain that z + c = c + z ,$\forall$ z,c$\in$ K.

By substituting sηz for z and rηz for c, for s,r$\in$ G and η$\in$ Γ, we have

$$s\eta z + r\eta z = r\eta z + s\eta z , \forall z \in K , s,r \in G \text{ and } \eta \in \Gamma.$$

Which gives (s+r-s-r)ηz= 0.

Thus, (s,r) Γ K=0. Since K ≠0 is a semi-group ideal and G is a prime , then

(s,r) = 0,$\forall$ s,r$\in$ G , Then (G,+) is abelian.

Lemma 2.11. If h be a Γ -(λ ,δ) – derivation on a prime Γ-near-ring G and K is a semi-group ideal of G. Suppose that t$\in$ K is not a left zero divisor. If $[t,h(t)]^{\beta}_{(\lambda,\delta)} = 0$, then (s,t) is a constant for every s$\in$ K and β$\in$Γ.
Proof. From tβ(s+t) = tβs + tβt, $\forall$t $\in$K and β $\in$Γ.
By applying h for both sides, we have

$$h(t\beta(s+t)) \quad = h(t)\beta\lambda(s+t) + \delta(t)\beta h(s+t)$$
$$= h(t)\beta\lambda(s) + h(t)\beta\lambda(t) + \delta(t)\beta h(s) + \delta(t)\beta h(t)$$

and

$$h(t\beta s + t\beta t)) \quad = h(t\beta s) + h(t\beta t)$$
$$= h(t)\beta\lambda(s) + \delta(t)\beta h(s) + h(t)\beta\lambda(t) + \delta(t)\beta h(t)$$

Which gives that h(t)βλ(t) + δ(t)βh(s) = δ(t)βh(s) + h(t)βλ(t), $\forall$t,s$\in$ K and β$\in$ Γ.
By using the hypothesis, we have δ(t)βh((s,t)) = 0.
By substituting δ(t) by δ(t)γm, where m$\in$ K and γ$\in$Γ, we get δ(t)γmβh((s,t)) = 0.
Hence, δ(t)ΓKΓh((s,t)) = 0, $\forall$t,s$\in$ K .
Because t is not a left zero divisor and δ(K) = K , K is a semi-group ideal and G is a prime Γ-near-ring, we obtain that
h((s,t)) = 0. Thus, (s,t) is a constant for every s$\in$ K.
Now we can prove the main theorems.
Theorem 2.12. Let h be a Γ -(λ ,δ) – derivation of a prime Γ-near-ring G and K is a semi-group ideal of G which has no nonzero divisors of zero, where h is commuting on K, λ(K)= K , then (G, +) is an abelian.
Proof. Let v be any additive commutator in K.
So, the application of Lemma 2.11 yields that v is a constant.
For any s$\in$ K, sγv is also an additive commutator in K. Then , sγv is also a constant.
Therefore , 0 = h(sγv) = h(s)γλ(v) + δ(s)γh(v) = h(s)γλ(v), $\forall$ s$\in$ K and γ$\in$ Γ.
Because h(s) ≠ 0, for some s$\in$ K, and K has no nonzero divisors of zero,
Which gives λ(v) = 0, thus v = 0 ,for every additive commutator v in K.
Hence, (K, +) is an abelian. By using Lemma 2.10, we obtain that (G, +) is an abelian. □

We need the following lemma to prove the main theorem.

Lemma 2.13. Let h be a nonzero Γ -(λ ,δ) – derivation on a prime Γ-near-ring G, and K is a semi-group ideal of G, so $\lambda\gamma h = h\gamma\lambda$,$\delta\gamma h = h\gamma\delta$ for every $\gamma\in\Gamma$, $\lambda(K)=K$, where $h(K)\subseteq Z(G)$, then (K, +) is an abelian. If G is a 2 – torsion free and $h(K)\subseteq K$, then K is a central ideal.

Proof. Since $h(K)\subseteq Z(G)$ and h is a nonzero Γ-(λ,δ)-derivation.
There exists a nonzero element t in K, such that $u = h(t)\in Z(G)\backslash\{0\}$.
And, $u+u = h(t)+ h(t) = h(t +t)\in Z(G)$.
Therefore , (K,+) is an abelian by Lemma 2.6 (ii).
Using the hypothesis, $\forall s, r \in K, c\in G$ and $\beta, \gamma\in\Gamma$ gives $\lambda(c)\gamma h(s\beta r) = h(s\beta r)\gamma\lambda(c)$.
Uusing Lemma 2.2, it provides

$$\lambda(c)\gamma h(s)\beta\lambda(r) + \lambda(c)\gamma\delta(s)\beta h(r) = h(s)\beta\lambda(r)\gamma\lambda(c) + \delta(s)\beta h(r)\gamma\lambda(c).$$

Now, by using $h(K)\subseteq Z(G)$ and since (K, +) is an abelian, $\lambda\gamma h = h\gamma\lambda$, and $\delta\gamma h = h\gamma\delta$, it shows that

$$h(s)\beta\lambda(c)\gamma\lambda(r) - h(s)\beta\lambda(r)\gamma\lambda(c) = h(r)\beta\delta(s)\gamma\lambda(c)- h(r)\beta\lambda(c)\gamma\delta(s)$$

Then , $h(s)\beta\lambda([c,r]_\gamma)= h(r)\beta([\delta(r), \lambda(c)]_\gamma)$,$\forall s,r \in K$, $c\in G$ and $\gamma, \beta\in \Gamma$.
Suppose that K is not a central ideal.
By choosing $r\in K$ and $c\in G$, such that $[c ,r]_\gamma \neq 0$.
And since $h(K)\subseteq K$, let $s = h(x)\in Z(G)$, where $x\in K$, which gives

$$h^2(x)\beta\lambda([c,r]_\gamma) = h(r)\, \beta([\delta(r),\lambda(c)]_\gamma) \ ,\forall\ x, r\in K, c\in G \text{ and } \gamma, \beta\in \Gamma.$$

Then, $h^2(x)\beta\lambda([c,r]_\gamma) = 0$.
By Lemma 2.6 (i), the central element $h^2(x)$ cannot be a nonzero divisor of zero, then we conclude that $h^2(x) = 0$, $\forall x\in k$.
By using Lemma 2.8, we obtain that $h(x) = 0$
This contradicts that h is a nonzero Γ- (λ,δ)- derivation on G.
So, we obtain that $\lambda([c,r]_\gamma) = 0$, $\forall\ r\in K ,c\in G$.
Because $\lambda(K)=K$, this gives a contradiction with the assumption. Then K is a central ideal. □

Theorem 2.14. Let h be a nonzero Γ -(λ ,δ) – derivation on a prime Γ-near-ring G and K a semi-group ideal of G, so $\lambda\gamma h = h\gamma\lambda$,$\delta\gamma h = h\gamma\delta$ for every $\gamma\in\Gamma$, $\lambda(K)=K$, where $h(K)\subseteq Z(G)$, then (G, +) is an abelian. If G is a 2 – torsion free and $h(K)\subseteq K$, then G is a commutative ring.

Proof. By using Lemma 2.13, it gives that (K, +) is an abelian.
By using Lemma 2.10, it gives that (G,+) is an abelian.
Now, assume that G is a 2 – torsion free. The application of Lemma 2.13.
shows that K is a central ideal.
Thus, K is a commutative. By Lemma 2.9, it implies that
G is a commutative ring. □

Theorem 2.15. Let h be a nonzero Γ -(λ ,δ) – derivation on a prime Γ-near-ring G, and K is a nonzero semi-group ideal of G, so $h([s,r]_\rho) = -[s,r]_\rho$ where $t\gamma k\beta u = t\beta k\gamma u$ for every $t,k,u\in K$ and $\gamma,\beta\in\Gamma$, then (G, +) is an abelian. If G is a 2 –torsion free and $h(K)\subseteq K$, then G is a commutative ring.

Proof. Since $[s,s\gamma r]_\rho = s\gamma[s ,r]_\alpha$,$\forall\ s\in K, r\in G$ and $\rho ,\gamma\in \Gamma$.
By using Lemma 2.3(ii), we have

$$h(v)\beta s\gamma[s,r]_\rho = s\gamma[s,r]_\rho\beta h(v)= s\gamma h(v)\beta[s,r]_\rho ,\forall\ s,v\in K, r\in G \text{ and } \rho,\gamma,\beta\in \Gamma.$$

By using Lemma 2.4 (i), we obtain that

$$[s,r]_\rho\, \beta h(h(v)\gamma s - s\gamma h(v)) = 0, \forall s,v\in K\ , r\in G \text{ and } \rho,\gamma,\beta\in \Gamma.$$

Hence, $[s,r]_\rho\beta h([h(v), s]_\gamma) =0$
By Lemma 2.3 (i), we obtain that $[s,r]_\rho\beta([h(v), s]_\gamma) =0$
The application of Lemma 2.4 (ii) gives
either $[s,r]_\rho = 0$ or $[h(v),s]_\gamma=0$, $\forall s,v\in K$, $r\in G$ and $\rho,\gamma \in \Gamma$.
If $[h(v), s]_\gamma = 0$, $\forall$ $s,v\in K$ and $\gamma \in \Gamma$.
Hence, $h(K)\subseteq Z(K)$. By Lemma 2.5, we obtain that
$h(K)\subseteq Z(G)$. Then, by Theorem 2.14, we complete this theorem.
So, if $[s ,r]_\rho= 0$, $\forall\ s\in K$, $r\in G$ and $\rho\in \Gamma$.

We substitute $s\gamma h(v)$ for s, where $v \in K$, which gives

$$[s\gamma h(v), r]_\rho = 0 = s\gamma[h(v), r]_\rho, \ \forall\ s,v\in K, r\in G \text{ and } \rho,\gamma \in \Gamma$$

Which gives, $K\gamma[h(v), r]_\rho = 0, \ \forall\ v\in K, r \in G$ and $\rho,\gamma \in \Gamma$

Because K is a nonzero semi-group ideal of G and G is a prime Γ-near–ring, this implies that $[h(v), r]_\rho = 0, \ \forall v\in K, r\in G$ and $\rho\in \Gamma$.

So, $h(K)\subseteq Z(G)$. By Theorem 2.14, the proof will be complete. □

Theorem 2.16. Let h be a nonzero Γ -(λ,δ)– derivation on a prime Γ-near-ring G, and K is a nonzero semi-group ideal of G, so $h([s,r]_\rho)= [s,r]_\rho$ and $t\gamma k\beta u = t\beta k\gamma u$ for every $t,k,u\in K$ and $\gamma,\beta\in \Gamma$, $\lambda(K)=K$, then G is a commutative ring.

Proof . Since $h([s,r]_\rho) = - s\rho r + r\rho s, \ \forall\ s\in K, r\in G$ and $\rho \in \Gamma$.

By replacing r by $r\gamma s$ in this equation, we get

$h([s,r\gamma s]_\rho) = - s\rho r\gamma s + r\gamma s\rho s = (-s\rho r + r\rho s)\gamma s$, $\forall s\in K, r\in G$ and $\rho,\gamma \in \Gamma$

And,

$$\begin{aligned} h\left(\left[s, r\gamma s\right]_\rho\right) = h\left(\left[s, r\right]_\rho \gamma s\right) &= h\left(\left[s, r\right]_\rho\right)\gamma\lambda(s) + \delta(\left[s, r\right]_\rho)\gamma h(s) \\ &= \left(-s\rho r + r\rho s\right)\gamma\lambda(s) + \delta(\left[s, r\right]_\rho)\gamma h(s) \end{aligned}$$

It follows from the two expressions for $h([s, r\gamma s]_\rho)$ that

$$\delta(s)\rho\delta(r)\gamma h(s) = \delta(r)\rho\delta(s)\gamma h(s), \ \forall s\in K, r\in G \text{ and } \rho ,\gamma\in \Gamma \qquad \ldots\ldots\ldots\ldots\ldots (4)$$

Replacing r by $v\beta r$ in equation (4), where $v\in K$ and $\beta\in \Gamma$, we obtain that

$$\delta(s)\rho\delta(v\beta r)\gamma h(s) = \delta(v\beta r)\rho\delta(s)\gamma h(s), \forall\ s, v\in K, r\in G \text{ and } \rho,\gamma ,\beta\in \Gamma \qquad \ldots\ldots\ldots\ldots(5)$$

Left multiplicative equation (4) by $\delta(v)\beta$, gives

$$\delta(v)\beta\delta(s)\rho\delta(r)\gamma h(s) = \delta(v)\beta\delta(r)\rho\delta(s)\gamma h(s), \forall s ,v\in K, r\in G \text{ and } \rho,\gamma,\beta\in \Gamma \qquad \ldots\ldots\ldots\ldots(6)$$

The combining of equation (5) and equation (6) gives

$$\delta([s,v]_\beta)\rho\delta(r)\gamma h(s) = 0, \forall s,v\in K , r\in G \text{ and } \rho,\gamma,\beta\in\Gamma.$$

Since $\delta(K) = K$, and δ is automorphism on G.

Hence, $[s,v]_\beta\, \Gamma G\Gamma h(s) = 0$.

Because G is a prime Γ-near–ring, this implies that $h(s) = 0$ or $[s,v]_\beta= 0$.

Since $h \neq 0$, therefore $[s,v]_\beta = 0, \ \forall\ s,v\in K$ and $\beta\in \Gamma$.

Thus , K is a commutative. By Lemma 2.9, we obtain that G is a commutative ring. □

Lemma 2.17:- Let h be a nonzero Γ -(λ,δ)– derivation on a prime Γ-near-ring G, and K be a nonzero semi-group ideal of G, if $[s,r]_\rho = [h(s), h(r)]_\rho$, then the constant in K is in $Z(G)$.

Proof: Let s be a constant in K. i.e. $h(s) = 0$, then

$$[s ,r]_\rho = [h(s),h(r)]_\rho = [0, h(r)]_\rho = 0, \ \forall\ r\in K \text{ and } \rho\in \Gamma.$$

Then, $s\in Z(K)$. By Lemma 2.5, we obtain that $s\in Z(G)$.

Theorem 2.18 . Let h be a nonzero Γ -(λ,δ)– derivation on a prime Γ - near - ring G, and K is a nonzero semi-group ideal of G that has no nonzero divisors of zero. If K has a right cancellation and $[h(s),h(r)]_\alpha = [s ,r]_\alpha$, $t\gamma k\beta u = t\beta k\gamma u$ for every $t,k,u\in K$ and $\gamma,\beta\in \Gamma$, and $h(K) \subseteq K$, then h is commuting and (G, +) is an abelian.

Proof. $\forall s\in K, [s, s\gamma h(s)]_\rho = [h(s), h(s\gamma h(s))]_\rho \qquad \ldots\ldots\ldots\ldots\ldots\ldots (7)$

By using Lemma 2.1 and 2.2, the right – hand side of equation (7) equals

$$\begin{aligned} \left[h(s), h(s\gamma h(s))\right]_\rho &= h(s)\rho h(s)\gamma\lambda(h(s)) + h(s)\rho\delta(s)\gamma h^2(s) - h(s)\gamma\lambda(h(s))\rho h(s) - \delta(s)\gamma h^2(s)\rho h(s) \\ &= h(s)\rho\delta(s)\gamma h^2(s) - \delta(s)\gamma h^2(s)\rho h(s) \end{aligned}$$

The left – hand side of equation (7) equals

$$s\gamma[s,h(s)]_\rho= s\gamma[h(s),h^2(s)]_\rho= s\gamma h(s)\rho h^2(s)- s\gamma h^2(s)\rho h(s), \ \forall\ s\in K \text{ and } \rho ,\gamma\in \Gamma$$

It follows from equation (7), since $\delta(K)= K$, it implies that

$$s\gamma h(s)\rho h^2(s) = h(s)\rho s\gamma h^2(s), \ \forall\ s\in K \text{ and } \rho, \gamma\in\Gamma. \qquad \ldots\ldots\ \ldots..(8)$$

Hence , by using the hypotheses, we obtain that $[s,h(s)]_\gamma\, \rho h^2(s) = 0 ,\forall\ s\in K$ and $\rho,\gamma\in\Gamma$.

If $h^2(s) = 0, \forall s\in K$.

Then, $h(s)$ is constant in K, by using Lemma 2.17, we obtain that h is central. Thus, h is commuting in K.

By Theorem 2.12, we obtain that (G, +) is an abelian.
Otherwise, $h^2(x)$ can be cancelled on the right in equation (8).
In either event, $[s,h(s)]_\gamma = 0$, $\forall$ s$\in$ K and $\gamma\in\Gamma$.
Then, by using Theorem 2.12, we obtain that (G, +) is an abelian. □

Theorem 2.19. Let h be a nonzero Γ -(λ,δ)– derivation on a prime Γ-near-ring G, and K be a nonzero semi-group ideal of G that has no nonzero divisors of zero. If h is commuting on K and $[s, r]_\rho = [h(s),h(r)]_\rho$, then G is a commutative ring.

Proof. For every s,r $\in$ K, we have

$$[s, s\beta r]_\rho = [h(s),h(s\beta r)]_\rho = [h(s), h(s)\beta\lambda(r) + \delta(s)\beta h(r)]_\rho$$

By using Lemma 2.2, we have

$$[h(s),h(s\beta r)]_\rho = h(s)\beta[h(s), \lambda(r)]_\rho + \delta(s)\beta[h(s), h(r)]_\rho$$

Since h is commuting, by using Theorem 2.12, (G, +) is an abelian, and $\delta(K) = K$,
we obtain that

$$s\beta[s, r]_\rho= s\beta[h(s),h(r)]_\rho= h(s)\beta[h(s),\lambda(r)]_\rho + \delta(s)\beta[h(s), h(r)]_\rho ,\forall\ s,r\in K \text{ and } \rho, \beta\in \Gamma.$$

Hence, $h(s)\beta[h(s),r]_\rho = 0$.
Since K has no nonzero divisors and h is a nonzero, we conclude that

$$[h(s),r]_\rho = 0, \ \forall s,r\in K \text{ and } \rho\in\Gamma.$$

In particular, $\forall$ s,r,t$\in$K, we have

$$[h(s),t\beta h(r)]_\rho = 0 = t\beta[h(s),h(r)]_\rho.$$

Hence,

$$K\beta[h(s),h(r)]_\rho = 0, \forall\ s,r\in K \text{ and } \rho, \beta\in\Gamma.$$

Since K is a nonzero semi-group ideal of G and G is a prime Γ-near–ring, we obtain that $[h(s),h(r)]_\rho = 0$, $\forall$ s,r $\in$ K and $\rho\in\Gamma$.
Then, we conclude that $[s,r]_\rho= 0$, $\forall$ s,r $\in$K and $\rho\in\Gamma$.
Hence, K is commutative. By Lemma 2.9, we have G is a commutative ring.

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