MODULES WHICH ARE SUBISOMORPHIC TO QUASI-INJECTIVE MODULES

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Abstract

Let R be a commutative ring with identity and let M be a unitary left $R-$module. We call the $R-$module M kerquasi-injective if for every monomorphism f from N into $Q(M)$, where N is a submodule of $Q(M)$ and $Q(M)$ is a quasi-injective hull of M and for every homomorphism g from N into M, there exists a homomorphism h from $Q(M)$ into M such that $\ker hf \subseteq \ker g$.

It is clear that every quasi-injective module is kerquasi-injective, however the converse is false. Also every ker-injective module is kerquasi-injective, however the converse is false. In this paper we give some characterizations of kerquasi-injective modules, we also study some conditions under which a kerqausi-injective module becomes quasi-injective. For example, if a kerquasi-injective module is a finitely generated, then it is a quasi-injective. We ought to mention that we were not able to give an example of a kerquasi-injective module which is not quasi-injective and ker-injective.

الموديلات التي تكون جزئية التقابل الى موديلات شبه اغمارية

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الخلاصة

لتكن R حلقة ابدالية وليكن M موديولا يساريا على R . نقول ان الموديول M شبه اغماري النواة اذا كان لكل تشاكل متباين f من N الى $Q(M)$ ، حيث N موديول جزئي من الغلاف شبه الاغماري للموديول M الذي يرمز له بالرمز $Q(M)$ ولكل تشاكل g من N الى M يوجد تشاكل h من $Q(M)$ الى M يحقق $\ker hf \subseteq \ker g$. من الواضح ان كل موديول شبه اغماري هو موديول شبه اغماري النواة. لقد بينا ان العكس غير صحيح بصورة عامة، كذلك كل موديول اغماري النواة هو شبه اغماري النواة ولقد بينا ان العكس غير صحيح ايضا. كذلك اعطينا بعض المكافئات الى تعريفنا. واخيرا لابد من الاشارة الى اننا لم نستطع اعطاء مثال على موديول شبه اغماري النواة لكن ليس شبه اغماري ولا اغماري النواة. وعليه نترك البرهنة على وجود (أو عدم وجود) مثال كهذا مسألة مفتوحة.

Introduction

Two modules are subisomorphic if each has a monomorphism into the other [1]. Bumby, [1] has shown that if two modules are subisomorphic then their quasi-injective hulls are isomorphic. The purpose of this paper is to initiate the study of modules which are subisomorphic to their quasi-injective hulls. We introduce the following definition: a module M is kerquasi-injective (KQI) if given any monomorphism $f: N \to Q(M)$, where N is any submodule of the quasi-injective hull $Q(M)$, and any homomorphism $g: N \to M$ there exists a homomorphism $h: Q(M) \to M$ such that $\ker hf \subseteq \ker g$. In section 1, we show that KQI modules are precisely those modules which are subisomorphic to their quasi-injective hulls. In section 2 of the paper we give various conditions under which a KQI module becomes quasi-injective, we would like point out that our results parallel the results in [2] of ker-injecctive modules. Finally, we remark that R in this paper stands for a commutative ring with 1 and a module means a unitary left $R-$module.

1. Characterization of kerquasi-injective modules

We start the section by the following:

Theorem 1.1. Let M be a $R-$module, then the following statements are equivalent:

(i) M is KQI.

(ii) M is subisomorphic to $Q(M)$.

(iii) Given any monomorphism $f: N \to Q(M)$, where N is any submodule of the quasi-injective hull $Q(M)$ and any homomorphism $g: N \to M$, there exists a monomorphism $k: M \to M$ and a homomorphism $h: Q(M) \to M$ such that $hf = kg$.

(iv) There exist two homomorphisms $f: M \to Q(M)$ and $h: Q(M) \to M$ such that hf is a monomorphism.

Proof. $(i) \Rightarrow (ii)$. Consider the following diagram:

$$\begin{array}{ccc} 0 \to M & \xrightarrow{i} & Q(M) \\ I \downarrow & \swarrow h & \\ M & & \end{array}$$

where i is the inclusion homomorphism and I is the identity homomorphism. Since M is KQI, there exists a homomorphism $h: Q(M) \to M$ such that $\ker hi = \ker I = 0$. This implies that hi is a monomorphism, we claim that h is a monomorphism. In fact, if $h(x) = 0$ and $x \neq 0$, there exists $r \in R$ such that $0 \neq rx \in M$, since $M \leq_e Q(M)$ (M is an essential extension of $Q(M)$), $h(rx) = rh(x) = 0$, but $h(rx) = hi(rx) = 0$, hence $rx = 0$ contradiction. Therefore h is a momomorphism and hence M is subisomorphic to $Q(M)$.

$(ii) \Rightarrow (iii)$. Consider the following diagram:

$$\begin{array}{ccccc} & 0 \to N & \xrightarrow{f} & Q(M) & \\ & g \downarrow & & t \downarrow & \searrow h \\ O \dashrightarrow & M & \xrightarrow{I} & Q(M) & \longrightarrow M \end{array}$$

where N is a submodule of the quasi-injective hull of $Q(M)$, f is any monomorphism and g is any homomorphism. Let $i: M \to Q(M)$ be the inclusion homomorphism. Since $Q(M)$ is quasi-injective there exists a homomorphism $t: Q(M) \to Q(M)$ such that $tf = ig$ [3]. From part (ii) there exists a monomorphism $s: Q(M) \to M$. Let $st = h$ and $si = k$. Hence $hf = kg$.

$(iii) \Rightarrow (iv)$. Let $N = M$ and $g =$ identity homomorphism.

$(iv) \Rightarrow (i)$. Let $f: N \to Q(M)$ be any monomorphism and $g: N \to M$ be any homomorphism. Let $i: M \to Q(M)$ be the inclusion homomorphism. Since $Q(M)$ is quasi-injective there exists a homomorphism $t: Q(M) \to Q(M)$ such that $tf = ig$. From part (iv) there exists a homomorphism s such that si is a monomorphism. Let $st = h$. We claim that $\ker hf \subseteq \ker g$. In fact let $x \in \ker hf$ then $0 = (hf)(x) = h(f(x)) = st(f(x)) = s(ig)(x) = (si)(g(x))$. But si is a monomorphism therefore $g(x) = 0$ and hence $x \in \ker g$.

Corollary 1.2. If M is a KQI module, then for every diagram of the form

$$\begin{array}{l} 0 \to N \xrightarrow{f} Q(M) \\ \quad \downarrow \; g \downarrow \\ \quad M \quad M \end{array}$$

there exists a homomorphism $h : Q(M) \to M$ such that $\ker hf = \ker g$.

Proof. From the proof of theorem 1.1 part $(iv) \Rightarrow (i)$, if $x \in \ker g$ then $g(x) = 0$ $= si(g(x)) = s(ig(x)) = (st)(f(x)) = h(f(x))$ $= (hf)(x)$ and hence $x \in \ker hf$, that is $\ker g \subseteq \ker hf$. Thus $\ker hf = \ker g$.

Remarks and examples 1.3

(1) If $Q(M)$ is subisomorphic to $E(M)$($E(M)$ is an injective hull of M) then $E(M) = Q(M)$.

Proof. By [1], $E(M) \subseteq Q(M)$,but $Q(M) \subseteq E(M)$, therefore $E(M) = Q(M)$.

(2) If M_1 is subisomorphic to M_2 and M_1 is KQI, then M_2 is also.

Proof. There exists a monomorphism $f : M_1 \to M_2$ and a monomorphism $g : Q(M_1) \to M_1$ and hence fg is also a monomorphism. By [1] $Q(M_1) \cong Q(M_2)$ .Thus there exists a monomorphism from $Q(M_2)$ into M_2.

(3) It is easy to see that if $\ker g \subseteq \ker hf$ in the definition of KQI, then the module M is not necessarily KQI (take $M = Z$ as $Z-$module).

(4) It is clear that every quasi-injective is KQI. But the converse is not true in general. For example let $M = Z \oplus \Pi E(Z)$ as $Z-$module. $E(M)$ is subisomorphic to M and by (1) $Q(M)$ is subisomorphic to M. But M is not quasi-injective since Z is not quasi-injective.

(5) It is clear that every ker-injective module M ($E(M)$ is subisomorphic to M) [2] is KQI, but the converse is not true in general. For example take $M = Z_2$ as $Z-$module, $Q(Z_2) = Z_2$ that is Z_2 is quasi-injective and hence by (4) Z_2 is KQI, but Z_2 is not ker-injective because if Z_2 is ker-injeective module, then Z_2 is subisomorphic to $E(Z_2) \cong Z_{2^\infty}$ and by [1] $Z_{2^\infty} \cong Z_2$,this is contradiction. Thus Z_2 is KQI but not ker-injective.

Proposition 1.4. If M is KQI then for every monomorphism $f : N \to M$ and for every homomorphism $g : N \to M$ there exists a homomorphism $h : M \to M$ such that $\ker hf = \ker g$.

Proof. Consider the following diagram:

$$\begin{array}{l} 0 \to N \xrightarrow{f} M \xrightarrow{i} Q(M) \\ \quad g \downarrow \quad h \quad t \\ \quad M \end{array}$$

Since M is KQI, there exists a homomorphism $t : Q(M) \to Q(M)$ such that $\ker tif = \ker g$. Let $ti = h$. Thus $\ker hf = \ker tif = \ker g$.

Lemma 1.5. If R is principle ideal domain then every nonzero homomorphism $f : I \to R$, where I is a nonzero ideal of R is a monmorphism.

Proof. Let $I = (r)$ for some $0 \neq r \in R$ and let $f(x) = 0$, $x \in I$, this implies that $f(sr) = 0$, for some $s \in R$ and hence $sf(r) = 0$. But R is an integral domain, therefore either $s = 0$ or $f(r) = 0$. If $f(r) = 0$, then f is the zero homomorphism and this is impossible. Thus $s = 0$ and hence $x = 0$ that is f is a monomorphism.

The converse of the proposition 1.4 is not true. For example let $M = Z$ as $Z-$module. Consider the following diagram:

$$\begin{array}{l} 0 \to (n) \xrightarrow{f} Z \\ \quad g \downarrow \quad h \\ \quad Z \end{array}$$

Now we suppose that $n \neq 0$ and $g \neq 0$. Since Z is principle ideal domain therefore by lemma 1.5, any homomorphism $h : Z \to Z$ is a monomorphism also g is a monomorphism, thus hf is also a monomorphism. Hence $\ker hf = \ker g$ =0

But Z as $Z-$module is not KQI.

2. Quasi-injective modules and kerquasi-injective modules

We start with the following:

Proposition 2.1. If M is a KQI module, then there exists an epimorphism $h: M \to Q(M)$.

Proof. Since M is KQI, there exists a monomorphism $f: Q(M) \to M$. This implies that $Q(M) \cong f(M)$. Consider the following diagram:

$$\begin{array}{ccccc} 0 \to f(M) & \xrightarrow{i} & M & \xrightarrow{j} & Q(M) \\ k \downarrow & & & \swarrow t & \\ Q(M) & & & & \end{array}$$

where i, j are inclusion homomorphisms and k is an isomorphism. Since $Q(M)$ is a quasi-injective module, there exists a homomorphism $t: Q(M) \to Q(M)$ such that $tij = k$. This implies t is an epimorphism. Let $tj = h$ and let $x \in Q(M)$, since k is an isomorphism, there exists $y \in f(M)$ such that $k(y) = x$. Thus $h(y) = hi(y) = k(y) = x$. That is h is an epimorphism.

Corollary 2.2. If M is KQI, then $ann_R(M) = ann_R(Q(M))$.

Proof. Clearly $ann_R(Q(M)) \subseteq ann_R(M)$ .Let $r \in ann_R(M)$, by proposition 2.1 there exists an epimorphism $h: M \to Q(M)$. But $r \in ann_R(M)$, therefore $0 = h(0) = h(rM) = rh(M) = rQ(M)$. This implies $r \in ann_R(Q(M))$, thus $ann_R(M) \subseteq ann_R(Q(M))$.

Corollary 2.3. For every KQI module, there exists a proper submodule N of M such that M/N is quasi-injective. Moreover $M/N \cong Q(M)$.

Proof. By proposition 2.1 there exists an epimorphism $h: M \to Q(M)$ and hence $M/\ker h \cong Q(M)$, put $N = \ker h$.

A module is hopfian if every onto endomorphism is an automorphism.

Theorem 2.4. If M is a KQI module and $Q(M)$ is hopfian, then M is quasi-injective.

Proof. Suppose $Q(M)$ is hopfian, by the proof of proposition 2.1 there exists an epimorphism $t: Q(M) \to Q(M)$ such that $tji = k$. Since $Q(M)$ is hopfian, therefore t is an isomorphism and hence ji is an isomorphism, this implies j is an epimorphism and hence $M = Q(M)$.

Theorem 2.5. If M is a KQI module and M or $Q(M)$ is finitely generated, then M is quasi-injective.

Proof. Suppose $Q(M)$ is finitely generated, then by [4] $Q(M)$ is hopfian and we use theorem 2.4 to have the requirement .Now suppose M is finitely generated. By proposition 2.1 there exists an epimorphisim $h: M \to Q(M)$ and hence $Q(M)$ is finitely generated and by [4] $Q(M)$ is hopfian and by proposition 2.4 M is quasi-injective.

A module is cohopfian if every one to one endomorphism is an automorphism.

Theorem 2.6. If M is KQI and M or $Q(M)$ is cohopfian, then M is quasi-injective.

Proof. Suppose M is cohopfian and $i: M \to Q(M)$ is the inclusion homomorphism and $f: Q(M) \to M$ is a monomorphism . Then fi is also a monomorphism, this implies fi is an isomorphism and hence f is an epimorphism. Thus M is quasi-injective. While if $Q(M)$ is cohopfian, if is a monomorphism this implies if is an isomorphism. Thus i is an epimorphism, that is M is quasi-injective.

Corollary 2.7. If M is KQI and $Q(M)$ is directly finite (that is a module not isomorphic to any proper direct summand of it self [5, P.165]) then M is quasi-injective.

Proof. If $Q(M)$ is directly finite then by [6] $Q(M)$ is cohopfian and by theorem 2.6 M is quasi-injective.

Proposition 2.8. If R is an integral domain and ${}_R R$ has a submodule which is KQI, then R is a field.

Proof. By [7] ${}_R R$ is subisomorphic to every submodule of ${}_R R$. Since ${}_R R$ has KQI

submodule say I, then by remarks and examples 1.3 (2) $_R R$ is KQI . By proposition

2.1 there exists an epimorphism $f : R \to Q(R)$. Since R is finitely generated, then by theorem 2.5 R is quasi-injective and hence R is self injective. Thus by [7] R is a field.

Next we give a proposition that shows that if every KQI on the ring R is a quasi-injective, then that ring must be a semi-simple artinian and conversely.

Proposition 2.9. R is a semi-simple artinian *iff* every KQI $R-$module is quasi-injective.

Proof. $(\Rightarrow)$ By [8] every module is a quasi-injective. Conversely let M be any module, this implies $M \oplus \Pi E(M)$ is a ker-injective module and hence by remarks and examples 1.3 (5) $M \oplus \Pi E(M)$ is KQI .Thus $M \oplus \Pi E(M)$ is quasi-injective. Since a direct summand of a quasi-injective is a quasi-injective [5] then M is a quasi-injective and by [8] R is a semi-injective artinian.

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