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Some Results of Ideals for Partial Transformation Semigroups

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Abstract:

Let $\mathcal{PT}_{\boldsymbol{n}}$ be the set of all partial functions from the set $\mathcal{N}$ to the set $\mathcal{M}$, where $\mathcal{N}, \mathcal{M} \subseteq X_n$ and $X_{\boldsymbol{n}} = \{1,2, \dots, n\}$. Then $\mathcal{PT}_{\boldsymbol{n}}$ is a semigroup subject to the composition of partial mapping or a monoid with identity $\mathcal{PI}_{\boldsymbol{n}}$. Here, the ideal of $\mathcal{PT}_{\boldsymbol{n}}$ in terms of an element $\xi \in \tilde{\mathcal{T}}_{\boldsymbol{n},\boldsymbol{0}}$ was studied, where $\tilde{\mathcal{T}}_{\boldsymbol{n},\boldsymbol{0}}$ is full transformation monoid from the set $X_{\boldsymbol{n},\boldsymbol{0}}$ to itself such that $X_{\boldsymbol{n},\boldsymbol{0}} = \{0\} \cup X_{\boldsymbol{n}}$. Finding that there are two kinds of ideals in $\mathcal{PT}_{\boldsymbol{n}}$, two-sided ideals, and a left ideal. As well as the ideals of partial transformation monoid of a free (left) G -act on n-generators, $\mathcal{PT}_{F_n(G)}$ where G is a finite group and $F_n(G) = \dot\cup_{i=1}^{n} Gx_i$ were considered. Finally, the number of elements of a two-sided ideal and left ideal of $\mathcal{PT}_{\boldsymbol{n}}$ and $\mathcal{PT}_{F_n(G)}$ were also found.

Keywords: Semigroup; Partial transformation semigroups; Free (left) G-act, Ideals.
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بعض نتائج المثاليات لتحويلات اشباه الزمر الجزئية

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الخلاصة:

لتكن $\mathcal{PT}_{\boldsymbol{n}}$ هي مجموعة كل الدوال الجزئية من المجموعة $\mathcal{N}$ الى المجموعة $\mathcal{M}$ عندما $\mathcal{N}, \mathcal{M}$ مجاميع جزئية من المجموعة $X_{\boldsymbol{n}}$ وان $X_{\boldsymbol{n}} = \{1,2, \dots, n\}$. لذلك $\mathcal{PT}_{\boldsymbol{n}}$ تحت تركيب الدوال الجزئية هي شبه زمرة او شبه زمرة مع محايد $\mathcal{PI}_{\boldsymbol{n}}$. في هذا البحث سندرس مثاليات المجموعة $\mathcal{PI}_{\boldsymbol{n}}$ في حالة كون العنصر $\xi \in \tilde{\mathcal{T}}_{\boldsymbol{n},\boldsymbol{0}}$ ، وان $\tilde{\mathcal{T}}_{n,0}$ هي تحويلات شبه الزمرة الكاملة للمجموعة $X_{\boldsymbol{n},\boldsymbol{0}}$ الى نفسها حيث ان $X_{\boldsymbol{n},\boldsymbol{0}} = \{0\} \cup X_{\boldsymbol{n}}$. في هذا البحث وجدنا نوعين من المثاليات للشبه الزمرة $\mathcal{PT}_{\boldsymbol{n}}$ وهي مثاليات ذات الجهتين ومثاليات الجهة اليسرى. وكذلك تم دراسة المثاليات لشبه الزمرة $\mathcal{PT}_{F_n(G)}$ عندما $F_n(G) = \dot\cup_{i=1}^{n} Gx_i$. وخيرا تم إيجاد عدد العناصر في مثاليات ذات الجهتين ومثاليات الجهة اليسرى لاشباه الزمر $\mathcal{PT}_{\boldsymbol{n}}$ و $\mathcal{PT}_{F_n(G)}$.

1. Introduction

The present article aims to study the ideals of partial transformation semigroups on a finite set. Suppose that $X_{\boldsymbol{n}} = \{1,2, \dots, n\}$ is a finite set of positive integers, which is fixed throughout this article. The partial transformation semigroups $\mathcal{PT}_{\boldsymbol{n}}$ is a semigroup of all partially defined functions from $X_{\boldsymbol{n}}$ to itself, i.e., $\mathcal{PT}_{\boldsymbol{n}} = \{\alpha | \alpha: \mathcal{N} \to \mathcal{M},$ where $\mathcal{N}, \mathcal{M} \subseteq$

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$X_n\}$ under the composition of partial functions. The subsemigroup set of $\mathcal{PT}_n$ includes the full transformation semigroups $\mathcal{T}_n$, which consists of all totally defined transformations on X_n (i.e., all functions $X_n \to X_n$), the symmetric inverse semigroups $\mathcal{IS}_n$, which consists of all injective partial transformations on X_n, and the symmetric group $\mathcal{S}_n$ which consists of all bijection maps and it represents the group of units of $\mathcal{PT}_n$ (and of $\mathcal{T}_n$ and $\mathcal{IS}_n$). The importance of these semigroups is due to the fact of fundamental results of Cayley's Theorem (for semigroups and groups), which states that every finite semigroup (or, finite group) may be embedded in some $\mathcal{T}_n$ (or, some $\mathcal{S}_n$), respectively. Also, the fundamental results of Vanger-Preston Theorem (for inverse semigroups), [1]. The corresponding transformation semigroups and other related semigroups have been studied in [2-37].

Hawie and McFadden [12] investigated the ideals of $\mathcal{T}_n$, these are the set $K(n,r) = \{\alpha \in \mathcal{T}_n : |Im\alpha| \le r\}$, where $1 \le r \le n$.

It is clear that $K(n,r)$ is a two-sided ideal, and $K(n,r) = \mathcal{T}_n$. James and R. D. Gray in [5] proved that where $1 \le r \le n-1$, then the ideal $K(n,r)$ is an idempotent generator and

$$rank\big(K(n,r)\big) = idrank(K(n,r)) = \begin{cases} n & if\ r = 1; \\ S(n,r) & if\ r > 1, \end{cases}$$

where, $S(n,r)$ is the Stirling number of the second kind. If S is a finite semigroup, then

$$rank(S) = min\{|A| : A \subseteq S, < A >= S\}.$$

If S is generated by its set $E(S)$ of idempotents, then the idempotent rank of S is defined by

$$idrank(S) = min\{|A| : A \subseteq E(S), < A > = S\}.$$

For $n \ge 3$, the rank of $\mathcal{PT}_n$ and $\mathcal{T}_n$ is 4, 3, respectively [11]. Garba [7] postulated the semigroup

$$P(n,r) = \{\alpha \in \mathcal{PT}_n : |Im\alpha| \le r\}.$$

And showed that, for $2 \le r \le n-1$, both the rank and idempotent rank are equal to $S(n+1, r+1)$. It is worth mentioning that for $\alpha \in \mathcal{T}_n$ (or, $\alpha \in \mathcal{PT}_n$) defining the *rank* of α to be the cardinality of $Im\alpha$ (i.e., $|Im\alpha| = rank(\alpha)$). In [5], the proper two-sided ideals of partition monoid $\alpha \in \mathcal{PT}_n$ are studied, and it was shown that each such ideal is an idempotent generated semigroup. The formula is given to generate the semigroup to find the minimal number of elements and the minimal number of idempotent elements.

The description of principal ideals has long been known for $\mathcal{T}_n$, $\mathcal{PT}_n$ and $\mathcal{IS}_n$ [8] which have the form $S\alpha S = \{\beta \in S : rank(\beta) \le rank(\alpha)\}$ where $S = \mathcal{T}_n$, $\mathcal{PT}_n$ or, $\mathcal{IS}_n$ and $\alpha \in S$. Moreover, all two-sided ideals in S are principal and generated by any element of the ideal, which has the maximal possible rank.

The full transformation semigroup of a free left G -act on n-generators $\mathcal{T}_{F_n(G)}$, where $F_n(G) = \dot{\cup}_{i=1}^{n} Gx_i$ has been considered in [36], and its ideal has been investigated in [35]. This work aims to describe the ideals in partial transformation semigroup $\mathcal{PT}_n$, and in partial transformation semigroup of free left G -act on n -generators $\mathcal{PT}_{F_n(G)}$.

This paper is organized as follows: notation concerning preliminaries was set for full transformations $\mathcal{T}_n$, and partial transformations $\mathcal{PT}_n$, in Section 2. In Section 3, the description of the ideals in $\mathcal{PT}_n$ were considered. Also, enumerating the number of elements in a two-sided ideal and a left ideal of $\mathcal{PT}_n$. In Section 4, establishing the notation concerning partial transformations of free-left G-act, $\mathcal{PT}_{F_n(G)}$, and then describing the ideals of $\mathcal{PT}_{F_n(G)}$. Then, the number of elements in a two-sided ideal and a left ideal were found.

2. Preliminaries

Throughout this work, we compose (partial) functions from left to right and write them to the right of their arguments, which means if $\alpha, \beta \in \mathcal{PT}_{F_n(G)}$, then $x(\alpha\beta) = (x\alpha)\beta$ for any $x \in X_n$ for which both sides of the latter equality are defined. The domain of partial transformation α is denoted by $Dom\alpha$ and its image by $Im\alpha$. Furthermore, $Dom\alpha\beta = [Im\alpha \cap Dom\beta]\alpha^{-1}$ and $Im\alpha\beta = [Im\alpha \cap Dom\beta]\beta$, [13].

If $\alpha \in \mathcal{PT}_n$, then α can be illustrated as follows:

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ y_1 & y_2 & y_3 & \cdots & y_n \end{pmatrix},$$

where

$$y_i = \begin{cases} - & if \quad i \notin Dom\alpha; \\ i\alpha & if \quad i \in Dom\alpha. \end{cases}$$

Deducing from that $\mathcal{PT}_n$ has a zero element (the empty map) where for all $i \notin Dom\alpha = \{1,2,\dots,n\}$ and that can be represented as $\mathcal{P}0 = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ - & - & - & \cdots & - \end{pmatrix}$. As well as, for $i \in Dom\alpha$, then $\mathcal{PT}_n$ is a transformation monoid with identity $\mathcal{PI}_n = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 2 & 3 & \cdots & n \end{pmatrix}$.
Detailed information on the structure of $\mathcal{PT}_n$ is provided in [8]. It is well-known that the number of elements in $\mathcal{PT}_n$ is $(n+1)^n$, this can be found in [8]. In semigroup S, an idempotent element $e \in S$ with $e^2 = e$, and the set of all idempotents in S is denoted by $E(S)$. Let $\beta \in \mathcal{PT}_n$ and $\mathcal{Y} \subseteq X_n$, the restriction of β to $\mathcal{Y}$ is the function $\beta|_{\mathcal{Y}}$ such that $y(\beta|_{\mathcal{Y}}) = y\beta$ for all $y \in \mathcal{Y}$. It is clear that $\beta|_{\mathcal{Y}}$ has domain $\mathcal{Y}$ and co-domain $\mathcal{Y}\beta$.
The following theorem examines idempotents in $\mathcal{PT}_n$:
Theorem 2.1: [8] An $\alpha \in \mathcal{PT}_n$ is an idempotent if and only if $Im\alpha \subseteq Dom\alpha$ and the restriction $\alpha|_{Im\alpha} = \mathcal{I}_{n_{Im\alpha}}$.
Corollary 2.2: [8] The number $E(\mathcal{PT}_n)$ of idempotents in a partial transformation monoid $\mathcal{PT}_n$ is

$$E(\mathcal{PT}_n) = \sum_{k=0}^{n} \binom{n}{k} (k+1)^{n-k},$$

where $|Im\alpha| = k$.
Now, consider the set $X_n \cup \{0\}$ to be the set $X_{n,0} = \{0,1,2,\dots,n\}$, then the full transformation semigroup on $X_{n,0}$ will be defined as follows;

$$\mathcal{T}_{n,0} = \{\alpha : \alpha \text{ is transformation on } X_{n,0}\}.$$

Clearly, $\mathcal{T}_{n,0} \cong \mathcal{T}_{n+1}$. By letting $\tilde{\mathcal{T}}_{n,0} = \{\tilde{\alpha} : \alpha \in \mathcal{T}_{n,0}, 0\tilde{\alpha} = 0\}$. It is easy to show that the subset $\tilde{\mathcal{T}}_{n,0}$ is a submonoid of $\mathcal{T}_{n,0}$, where a submonoid means it is closed under multiplication. And for any $n \in \mathbb{N}$, the partial transformation monoid $\mathcal{PT}_n$ is isomorphic to $\tilde{\mathcal{T}}_{n,0}$. From that, it can be deduced that $\alpha \in \mathcal{PT}_n$ if and only if $\tilde{\alpha} \in \tilde{\mathcal{T}}_{n,0}$. Moreover, that can be illustrated as follows:
If

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & n-1 & n \\ 1\alpha & 2\alpha & - & - & 5\alpha & \cdots & (n-1)\alpha & - \end{pmatrix} \in \mathcal{PT}_n,$$

then

$$\tilde{\alpha} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & \cdots & n-1 & n \\ 0 & 1\alpha & 2\alpha & 0 & 0 & 5\alpha & \cdots & (n-1)\alpha & 0 \end{pmatrix} \in \tilde{\mathcal{T}}_{n,0}.$$

For the simplest, this work will focus on the set $\tilde{\mathcal{T}}_{n,0}$ instead of the set $\mathcal{PT}_n$.

3. Ideals of partial transformation monoids $\mathcal{PT}_n$

In this section, two kinds of ideals of partial transformation monoids $\mathcal{PT}_n \cong \tilde{\mathcal{T}}_{n,0}$ in terms of an element $\xi \in \tilde{\mathcal{T}}_{n,0}$ were describe; a two-sided ideal and a left ideal. The following definition is necessary for this work:

Definition 3.1: [13] Let $\emptyset \neq \mathcal{A} \subseteq S$, where S is a semigroup. A subset $\mathcal{A}$ of S is called a left ideal if $S\mathcal{A} \subseteq \mathcal{A}$, a right ideal if $\mathcal{A}S \subseteq \mathcal{A}$, that means for all $s \in S$ and $a \in \mathcal{A}$, then $sa \in \mathcal{A}$ $(as \in \mathcal{A})$. A subset $\mathcal{A}$ of S is a two-sided ideal if it is both a left and a right ideal.

The description of a two-sided ideal $\tilde{\mathcal{J}}_{n,0}$ of a full transformation monoid $\tilde{\mathcal{T}}_{n,0}$ is obtained in the following lemma, where $\tilde{\alpha} \in \tilde{\mathcal{J}}_{n,0}$ is an idempotent.

Lemma 3.2: Let $\tilde{\mathcal{J}}_{n,0}$ be a non-empty subsemigroup of $\tilde{\mathcal{T}}_{n,0}$. Assume $\tilde{\alpha} \in \tilde{\mathcal{J}}_{n,0}$, such that $\tilde{\alpha}$ is an idempotent and $|Im\tilde{\alpha}| = k$, $|Dom\tilde{\alpha}| = \ell$ where $0 \leq k \leq n$ and $k \leq \ell \leq n$. Define $\xi \in \tilde{\mathcal{T}}_{n,0}$ by

$$i\xi = \begin{cases} i & \text{for all} \quad i, \quad \text{such that} \quad i\tilde{\alpha} \neq 0. \\ 0 & \text{else.} \end{cases} \tag{1}$$

Then $\tilde{\alpha}\xi = \xi\tilde{\alpha} = \tilde{\alpha}$, and $\tilde{\mathcal{J}}_{n,0}$ is a two-sided ideal of $\tilde{\mathcal{T}}_{n,0}$.

Proof: Let $\tilde{\alpha} \in \tilde{\mathcal{J}}_{n,0}$ such that $\tilde{\alpha}$ is an idempotent, and let $\xi \in \tilde{\mathcal{T}}_{n,0}$. If $i\tilde{\alpha} \neq 0$, then from the definition of ξ having $i\xi = i$, for all i, and hence $(i\tilde{\alpha})\xi = i\tilde{\alpha}$, this gives $\tilde{\alpha}\xi = \tilde{\alpha}$, as well as, $(i\xi)\tilde{\alpha} = i\tilde{\alpha}$, so $\xi\tilde{\alpha} = \tilde{\alpha}$.
Now, as $\tilde{\alpha} \in \tilde{\mathcal{J}}_{n,0}$ which is a subsemigroup of $\tilde{\mathcal{T}}_{n,0}$ then $0\tilde{\alpha} = 0$, for all i, so from the definition of ξ obtaining $0\xi = 0$. This gives $(0\tilde{\alpha})\xi = 0\xi = 0 = (0\xi)\tilde{\alpha}$. If $i\tilde{\alpha} = 0$, for some i, then $i\xi = 0$, this implies $(i\tilde{\alpha})\xi = 0\xi = 0 = 0\tilde{\alpha} = (i\xi)\tilde{\alpha}$. Therefore, $\tilde{\mathcal{J}}_{n,0}$ is a two-sided ideal of $\tilde{\mathcal{T}}_{n,0}$.

The following lemma for a left ideal $\tilde{\mathcal{J}}'_{n,0}$ of $\tilde{\mathcal{T}}_{n,0}$ obtained where $\tilde{\alpha} \in \tilde{\mathcal{J}}'_{n,0}$, and $\tilde{\alpha}$ is not an idempotent

Lemma 3.3: Let $\tilde{\mathcal{J}}'_{n,0}$ be a non-empty subsemigroup of $\tilde{\mathcal{T}}_{n,0}$. If $\tilde{\alpha} \in \tilde{\mathcal{J}}'_{n,0}$ such that $\tilde{\alpha}$ is not an idempotent and $|Im\tilde{\alpha}| = k$, $|Dom\tilde{\alpha}| = \ell$ where $1 \leq k \leq n-1$ and $k \leq \ell \leq n-1$. Let $\xi \in \tilde{\mathcal{T}}_{n,0}$, then $\xi\tilde{\alpha} = \tilde{\alpha} \in \tilde{\mathcal{J}}'_{n,0}$ and $\tilde{\mathcal{J}}'_{n,0}$ is a left ideal, where ξ defined in (1).

Proof: Let $\tilde{\alpha} \in \tilde{\mathcal{J}}'_{n,0}$ such that $\tilde{\alpha}$ is not an idempotent, and let $\in \tilde{\mathcal{T}}_{n,0}$. If $i\tilde{\alpha} \neq 0$, then $i\xi = i$ for all i, and from that, having $(i\xi)\tilde{\alpha} = i\,\tilde{\alpha}$, that means $\xi\tilde{\alpha} = \tilde{\alpha}$, and therefore $\tilde{\mathcal{J}}'_{n,0}$ will be a left ideal of $\tilde{\mathcal{T}}_{n,0}$.
As $\tilde{\alpha} \in \tilde{\mathcal{J}}'_{n,0}$ which is a subsemigroup of $\tilde{\mathcal{T}}_{n,0}$ then $0\tilde{\alpha} = 0$, where $i = 0$. From that, having $i\xi = 0\xi = 0$, then $(i\xi)\tilde{\alpha} = 0\tilde{\alpha} = 0 = i\tilde{\alpha}$. When $i\tilde{\alpha} = 0$, for some i, then form (1) obtaining $i\xi = 0$, this gives $(i\xi)\tilde{\alpha} = 0\tilde{\alpha} = 0 = i\tilde{\alpha}$. Therefore, $\tilde{\mathcal{J}}'_{n,0}$ is a left ideal of $\tilde{\mathcal{T}}_{n,0}$.

Now, where, $\tilde{\alpha} \in \tilde{\mathcal{J}}_{n,0}$, and $\tilde{\alpha}$ is not an idempotent such that $|Im\tilde{\alpha}| = |Dom\tilde{\alpha}| = n$ and $\tilde{\alpha} \neq \tilde{\mathfrak{I}}_n$, ($\tilde{\mathfrak{I}}_n \cong \mathcal{P}\mathfrak{I}_n$) the identity transformation of $\tilde{\mathcal{T}}_{n,0}$, another description of a two-sided ideal is given in the following lemma.

Lemma 3.4: Let $\tilde{\mathcal{J}}_{n,0}$ be a non-empty subsemigroup of $\tilde{\mathcal{T}}_{n,0}$. If $\tilde{\alpha} \in \tilde{\mathcal{J}}_{n,0}$ such that $\tilde{\alpha}$ is not an idempotent with $|Im\tilde{\alpha}| = |Dom\tilde{\alpha}| = n$ and $\tilde{\alpha} \neq \tilde{\mathfrak{I}}_n$. Define ξ as before in (1), then $\tilde{\alpha}\xi = \xi\tilde{\alpha} = \tilde{\alpha}$, and $\tilde{\mathcal{J}}_{n,0}$ is a two-sided ideal of $\tilde{\mathcal{T}}_{n,0}$.

Proof: Let $\tilde{\alpha} \in \tilde{\mathcal{J}}_{n,0}$ such that $\tilde{\alpha}$ is not an idempotent with $|Im\tilde{\alpha}| = |Dom\tilde{\alpha}| = n$ and $\tilde{\alpha} \neq \tilde{\mathfrak{I}}_n$,
and let $\xi \in \tilde{\mathcal{T}}_{n,0}$. From the hypotheses and the definition of ξ it is easy to show that $\xi = \tilde{\mathfrak{I}}_n$ the identity transformation of $\tilde{\mathcal{T}}_{n,0}$. Hence, $(i\tilde{\alpha})\xi = i(\tilde{\alpha}\xi) = i(\xi\tilde{\alpha}) = (i\xi)\tilde{\alpha}$. That means $\tilde{\mathcal{J}}_{n,0}$ is a two-sided ideal of $\tilde{\mathcal{T}}_{n,0}$.

The number of elements in a two-sided ideal $\tilde{\mathcal{J}}_{\boldsymbol{n},\mathbf{0}}$ of $\tilde{\mathcal{T}}_{\boldsymbol{n},\mathbf{0}}$ will be counted by the following lemma:

Lemma 3.5: The number $\mathcal{N}(\tilde{\alpha}, \tilde{\mathcal{J}}_{\boldsymbol{n},\mathbf{0}})$ of elements in a two-sided ideal $\tilde{\mathcal{J}}_{\boldsymbol{n},\mathbf{0}}$ of $\tilde{\mathcal{T}}_{\boldsymbol{n},\mathbf{0}}$ is

$$\mathcal{N}(\tilde{\alpha}, \tilde{\mathcal{J}}_{\boldsymbol{n},\mathbf{0}}) = \sum_{k=0}^{n} \binom{n}{k} (k+1)^{n-k} + (n! - 1),$$

where $|Im\tilde{\alpha}| = k$.

Proof: Let $\tilde{\alpha} \in \tilde{\mathcal{J}}_{\boldsymbol{n},\mathbf{0}}$ such that $|Im\tilde{\alpha}| = k$, where $0 \leq k \leq n$. Let $\xi \in \tilde{\mathcal{T}}_{\boldsymbol{n},\mathbf{0}}$ defined by (1). From Lemma 3.2 and Lemma 3.4, there are two cases for $\tilde{\mathcal{J}}_{\boldsymbol{n},\mathbf{0}}$ to be a two-sided ideal:

Case 1: This case comes from Lemma 3.2. It is already known from Corollary 2.2 that there are $\sum_{k=0}^{n} \binom{n}{k} (k+1)^{n-k}$, idempotent elements in $\tilde{\mathcal{T}}_{\boldsymbol{n},\mathbf{0}}$, which is the number of elements in $\tilde{\mathcal{J}}_{\boldsymbol{n},\mathbf{0}}$, as every element in $\tilde{\mathcal{J}}_{\boldsymbol{n},\mathbf{0}}$ is an idempotent, in this case.

Case 2: This case comes from Lemma 3.4, where $\tilde{\alpha} \in \tilde{\mathcal{J}}_{\boldsymbol{n},\mathbf{0}}$ is not an idempotent and $|Im\tilde{\alpha}| = |Dom\tilde{\alpha}| = n$ with $\tilde{\alpha} \neq \tilde{\mathcal{J}}_{\boldsymbol{n}}$. Clearly, $\tilde{\alpha} \in \tilde{S}_{\boldsymbol{n},\mathbf{0}}$, where $\tilde{S}_{\boldsymbol{n},\mathbf{0}}$ is a symmetric group (a group of units of $\tilde{\mathcal{T}}_{\boldsymbol{n},\mathbf{0}}$). Since there are $n!$ elements in $\tilde{S}_{\boldsymbol{n},\mathbf{0}}$ [8], and as $\tilde{\alpha} \neq \tilde{\mathcal{J}}_{\boldsymbol{n}}$, then there are $(n! - 1)$ elements of $\tilde{\alpha} \in \tilde{\mathcal{J}}_{\boldsymbol{n},\mathbf{0}}$ in this case.

By putting the two cases together, having the number of elements in a two-side ideal $\tilde{\mathcal{J}}_{\boldsymbol{n},\mathbf{0}}$.

Lemma 3.6: The number $\mathcal{N}(\tilde{\alpha}, \tilde{\mathcal{J}}'_{\boldsymbol{n},\mathbf{0}})$ of elements in a left ideal $\tilde{\mathcal{J}}'_{\boldsymbol{n},\mathbf{0}}$ of $\tilde{\mathcal{T}}_{\boldsymbol{n},\mathbf{0}}$ is

$$\mathcal{N}(\tilde{\alpha}, \tilde{\mathcal{J}}'_{\boldsymbol{n},\mathbf{0}}) = (n+1)^n - \mathcal{N}(\tilde{\alpha}, \tilde{\mathcal{J}}_{\boldsymbol{n},\mathbf{0}}).$$

Proof: Let $\tilde{\mathcal{J}}'_{\boldsymbol{n},\mathbf{0}}$ be a left ideal, and let $\tilde{\alpha} \in \tilde{\mathcal{J}}'_{\boldsymbol{n},\mathbf{0}}$ such that $\tilde{\alpha}$ is not an idempotent and $|Im\tilde{\alpha}| = k$, $|Dom\tilde{\alpha}| = \ell$ where $1 \leq k \leq n-1$ and $k \leq \ell \leq n-1$. Since $\mathcal{PT}_{\boldsymbol{n}} \cong \tilde{\mathcal{T}}_{\boldsymbol{n},\mathbf{0}}$ there are $(n+1)^n$ elements in $\tilde{\mathcal{T}}_{\boldsymbol{n},\mathbf{0}}$. Then, the result is obtained by subtracting the value of a two-sided ideal in Lemma 3.5 from the above value.

It is helpful to illustrate the formula in Lemma 3.6 by having the following example:

Example 3.7: For counting the elements in a left ideal $\tilde{\mathcal{J}}'_{\boldsymbol{n},\mathbf{0}}$ of $\tilde{\mathcal{T}}_{\boldsymbol{n},\mathbf{0}}$, where $\boldsymbol{n} = 3$. Let $\xi \in \tilde{\mathcal{T}}_{\boldsymbol{n},\mathbf{0}}$ be defined by (1). In the binging, we will find the number of elements in a two-sided ideal $\tilde{\mathcal{J}}_{\mathbf{3},\mathbf{0}}$. As known from Lemma 3.2 and Lemma 3.4, there are two cases for $\tilde{\mathcal{J}}_{\mathbf{3},\mathbf{0}}$ to be a two-sided ideal:

Case 1: This case comes from Lemma 3.2, which is if $\tilde{\alpha} \in \tilde{\mathcal{J}}_{\mathbf{3},\mathbf{0}}$ and $\tilde{\alpha}$ is an idempotent such that $|Im\tilde{\alpha}| = k$, $|Dom\tilde{\alpha}| = \ell$ where $0 \leq k \leq n$ and $k \leq \ell \leq n$. By Corollary 2.2, there are 23 idempotents in $\tilde{\mathcal{J}}_{\mathbf{3},\mathbf{0}}$. These idempotents are divided into 4 cases:

Case I: If $|Im\tilde{\alpha}| = k = 1$. There are 3 positions for $|Dom\tilde{\alpha}| = \ell$, that are either 1, 2, or 3.

1. If $|Dom\tilde{\alpha}| = \ell$=1. There are 3 idempotents in $\tilde{\mathcal{J}}_{\mathbf{3},\mathbf{0}}$ having $k = \ell = 1$, and they are

$$\tilde{\alpha}_1 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \tilde{\alpha}_2 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 0 \end{pmatrix}, \quad \tilde{\alpha}_3 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Then for all $\tilde{\alpha} \in \tilde{\mathcal{J}}_{\mathbf{3},\mathbf{0}}$ having $k = \ell = 1$, and from (1) we have $\tilde{\alpha}\xi = \xi\tilde{\alpha} = \tilde{\alpha}$.
Therefore, there are 3 idempotents in $\tilde{\mathcal{J}}_{\mathbf{3},\mathbf{0}}$ in terms of $\xi \in \tilde{\mathcal{T}}_{\mathbf{3},\mathbf{0}}$.

2. If $|Dom\tilde{\alpha}| = \ell$=2. There are 6 idempotents in $\tilde{\mathcal{J}}_{\mathbf{3},\mathbf{0}}$ having $|Im\tilde{\alpha}| = 1$, $|Dom\tilde{\alpha}| = 2$ and they are as follows:

$$\tilde{\alpha}_4 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \quad \tilde{\alpha}_5 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad \tilde{\alpha}_6 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 2 & 2 & 0 \end{pmatrix},$$

$\tilde{\alpha}_7 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 2 \end{pmatrix}$, $\tilde{\alpha}_8 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 3 & 0 & 3 \end{pmatrix}$, $\tilde{\alpha}_9 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 3 \end{pmatrix}$.

Then for all $\tilde{\alpha} \in \tilde{\mathcal{J}}_{3,0}$ having $k = 1$, $\ell = 2$, and from (1) we have $\tilde{\alpha}\xi = \xi\tilde{\alpha} = \tilde{\alpha}$.

That means there are 6 idempotents in $\tilde{\mathcal{J}}_{3,0}$ in terms of $\xi \in \tilde{\mathcal{T}}_{3,0}$.

3. If $|Dom\tilde{\alpha}| = \ell$=3. In this case, there are 3 idempotents in $\tilde{\mathcal{J}}_{3,0}$ having $|Im\tilde{\alpha}| = 1$, $|Dom\tilde{\alpha}| = 3$, which are as follows:

$\tilde{\alpha}_{10} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \end{pmatrix}$, $\tilde{\alpha}_{11} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 2 & 2 & 2 \end{pmatrix}$, $\tilde{\alpha}_{12} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 3 & 3 & 3 \end{pmatrix}$.

For all $\tilde{\alpha} \in \tilde{\mathcal{J}}_{3,0}$ having $k = 1$, $\ell = 3$, and from (1) we have $\tilde{\alpha}\xi = \xi\tilde{\alpha} = \tilde{\alpha}$. From that, there are 3 idempotents in $\tilde{\mathcal{J}}_{3,0}$ having $|Im\tilde{\alpha}| = 1$, $|Dom\tilde{\alpha}| = 3$ in terms of $\xi \in \tilde{\mathcal{T}}_{3,0}$.

Case II: If $|Im\tilde{\alpha}| = k = 2$. There are 2 cases for $|Dom\tilde{\alpha}| = \ell$, and they are either 2 or, 3.

1. If $|Dom\tilde{\alpha}| = \ell$=2. In this case, there are 3 idempotents in $\tilde{\mathcal{J}}_{3,0}$ having $|Im\tilde{\alpha}| = 2$, $|Dom\tilde{\alpha}| = 2$, and they are

$\tilde{\alpha}_{13} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 0 \end{pmatrix}$, $\tilde{\alpha}_{14} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 3 \end{pmatrix}$, $\tilde{\alpha}_{15} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 3 \end{pmatrix}$.

It is clear that for all $\tilde{\alpha} \in \tilde{\mathcal{J}}_{3,0}$ having $k = \ell = 2$, and from (1) we have $\tilde{\alpha}\xi = \xi\tilde{\alpha} = \tilde{\alpha}$. That means there are 3 idempotents in $\tilde{\mathcal{J}}_{3,0}$ in terms of $\xi \in \tilde{\mathcal{T}}_{3,0}$.

2. If $|Dom\tilde{\alpha}| = \ell$=3. In this case, there are 6 idempotents in $\tilde{\mathcal{J}}_{3,0}$ having $|Im\tilde{\alpha}| = 2$, $|Dom\tilde{\alpha}| = 3$, and they are

$\tilde{\alpha}_{16} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 2 \end{pmatrix}$, $\tilde{\alpha}_{17} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 1 \end{pmatrix}$, $\tilde{\alpha}_{18} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 3 & 3 \end{pmatrix}$,

$\tilde{\alpha}_{19} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 1 & 3 \end{pmatrix}$, $\tilde{\alpha}_{20} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 2 & 2 & 3 \end{pmatrix}$, $\tilde{\alpha}_{21} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 3 & 2 & 3 \end{pmatrix}$.

Clearly, for all $\tilde{\alpha} \in \tilde{\mathcal{J}}_{3,0}$ having $k = 2, \ell = 3$, and from (1) we have $\tilde{\alpha}\xi = \xi\tilde{\alpha} = \tilde{\alpha}$. That means there are 6 idempotents in $\tilde{\mathcal{J}}_{3,0}$ in terms of $\xi \in \tilde{\mathcal{T}}_{3,0}$.

Case III: If $|Im\tilde{\alpha}| = k = 3$. There is just one case for $|Dom\tilde{\alpha}|$, which is $\ell = 3$.

Here, there is only one idempotent in $\tilde{\mathcal{J}}_{3,0}$, which is

$$\tilde{\mathfrak{I}}_{3,0} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix},$$

the identity transformation of $\tilde{\mathcal{T}}_{3,0}$. As $Im\tilde{\mathfrak{I}}_{n} = \{1, 2, 3\}$, it is clear from (1) that $\xi = \tilde{\mathfrak{I}}_{3,0}$, and then $\xi\tilde{\mathfrak{I}}_{3,0} = \tilde{\mathfrak{I}}_{3,0}\xi = \tilde{\mathfrak{I}}_{3,0}$. This means there is only one element in $\tilde{\mathcal{J}}_{3,0}$ in terms of ξ having 3 elements in its image and in its domain.

Case IV: If $|Im\tilde{\alpha}| = k = 0$. Here, there is also one matter for $|Dom\tilde{\alpha}| = \ell$, that is $\ell = 0$.

There is only one idempotent in $\tilde{\mathcal{J}}_{3,0}$ such that $k = \ell = 0$, that is, the zero element

$$\tilde{0} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \tilde{c}_0,$$

where $\tilde{c}_0$ is a constant function with image zero.

As $|Im\tilde{0}| = |Dom\tilde{0}| = 0$, then giving from the definition of ξ that

$$\xi = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

That means $\xi = \tilde{0}$, and there is only one element in $\tilde{\mathcal{J}}_{3,0}$ with $k = \ell = 0$ in terms of $\xi \in \tilde{\mathcal{T}}_{3,0}$.

Deducing from **Case 1** that there are 23 elements in a two-sided ideal $\tilde{\mathcal{J}}_{3,0}$.

Case 2: If $\tilde{\alpha} \in \tilde{\mathcal{J}}_{3,0}$, and $\tilde{\alpha}$ is not an idempotent where $|Im\tilde{\alpha}| = |Dom\tilde{\alpha}| = 3$, and $\tilde{\alpha} \neq \tilde{\mathfrak{I}}_{3,0}$.

Clearly, $\tilde{\alpha} \in \tilde{S}_{3,0}$. It is well-known that there are 3! elements in $\tilde{S}_{3,0}$, [8]. But $\tilde{\alpha} \neq \tilde{\mathfrak{I}}_{3,0}$, this gives $3! - 1 = 6 - 1 = 5$ elements in $\tilde{\mathcal{J}}_{3,0}$. Now, from the definition of ξ (1), implies

$$\xi = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix} = \tilde{\mathfrak{I}}_{3,0},$$

this gives $\tilde{\alpha}\xi = \xi\tilde{\alpha} = \tilde{\alpha}$. That means there are 5 elements in $\tilde{\mathcal{J}}_{\mathbf{3,0}}$ in terms of $\xi \in \tilde{\mathcal{T}}_{\mathbf{3,0}}$.
Deducing from **Case 1** and **Case 2** that there are 23+5=28 elements in a two-sided ideal $\tilde{\mathcal{J}}_{\mathbf{3,0}}$.
Observe that the same result was obtained using the formula in Lemma 3.5.
Now, as known, there are $(n+1)^n$ elements in $\tilde{\mathcal{T}}_{\boldsymbol{n},\mathbf{0}}$, hence there are $(3+1)^3 = 64$ elements in $\tilde{\mathcal{T}}_{\mathbf{3,0}}$.
By subtracting the value of a two-sided ideal $\tilde{\mathcal{J}}_{\mathbf{3,0}}$ from 64, obtain $64 - 28 = 36$, which represents the number of elements in a left ideal $\tilde{\mathcal{J}}'_{\mathbf{3,0}}$. By applying Lemma 3.6, we have the similar result.

4. Ideals of partial transformation monoids $\mathcal{PT}_{\mathrm{F}_n(\mathrm{G})}$

This section aims to study and describe the ideals of partial transformation semigroup $\mathcal{PT}_{F_n(G)}$, where G is a finite group and $F_n(G) = \dot{\cup}_{i=1}^{n} G_{x_i}$ is rank (n) free left G-act. Denoting the semigroup of all morphisms $\boldsymbol{A} \longrightarrow \boldsymbol{B}$, where $\boldsymbol{A}$ and $\boldsymbol{B}$ are subalgebras of $F_n(G)$ by $\mathcal{PT}_{F_n(G)}$.
If $\alpha \in \mathcal{PT}_{\boldsymbol{F_n(G)}}$ then

$$\alpha = \begin{pmatrix} x_{i_1} & \cdots & x_{i_t} \\ g_{i_1}^{\alpha} x_{i_{1\tilde{\alpha}}} & \cdots & g_{i_t}^{\alpha} x_{i_{t\tilde{\alpha}}} \end{pmatrix},$$

where $\tilde{\alpha} \in \mathcal{PT}_{\boldsymbol{n}}$, $g_{i_1}^{\alpha}, \dots, g_{i_t}^{\alpha} \in G$, and $x_{i_l}\alpha = g_{i_l}^{\alpha}\ x_{i_{l\tilde{\alpha}}}$. Furthermore, for every selection of $\tilde{\mu} \in \mathcal{PT}_{\boldsymbol{n}}$ with $Dom\tilde{\mu} = \{j_1, \dots, j_k\}$, where $1 \le j_1 < \cdots < j_k \le n$, $k \ge 0$ and $h_{j_1}^{\mu}, \dots, h_{j_k}^{\mu} \in G$ gives

$$\mu = \begin{pmatrix} x_{j_1} & \cdots & x_{j_k} \\ h_{j_1}^{\mu} x_{j_{1\tilde{\mu}}} & \cdots & h_{j_k}^{\mu} x_{j_{k\tilde{\mu}}} \end{pmatrix} \in \mathcal{PT}_{\boldsymbol{F_n(G)}}.$$

If $G = \{e\}$ (i.e., G is a trivial set) that implies $\mathcal{PT}_{\boldsymbol{F_n(G)}}$ is isomorphic to $\mathcal{PT}_{\boldsymbol{n}}$, and if $G \neq \{e\}$ then $\mathcal{PT}_{\boldsymbol{F_n(G)}}$ isomorphic to $EndF_n(G)^0$, the endomorphism monoid of free left G-act that is given by $F_n(G)^0 = F_n(G) \dot{\cup} \{0\}$, when $\{0\}$ is a trivial left G-act. Notice, when $\alpha \in EndF_n(G)^0$ then

$$x_i\alpha = g_i^{\alpha} x_{i_{\tilde{\alpha}}},$$

for some $g_i^{\alpha} \in G$, such that g_i^{α} is defined uniquely if $i\tilde{\alpha} \neq 0$, $\forall\ i$ when $i \in \{1,2,\dots,n\}$. Observe that, as $0\tilde{\alpha} = 0$ so that $\tilde{\alpha} \in \tilde{\mathcal{T}}_{\boldsymbol{n},\mathbf{0}}$. It was already proved that $\mathcal{PT}_{\boldsymbol{F_n(G)}}$ is embedded via ψ in
$G^0 \wr_{n+1} \mathcal{T}_{\boldsymbol{n},\mathbf{0}}$, where G^0 is a group adjoining with 0 and $Im\psi = \widetilde{K}_n(G)^0$ such that $\widetilde{K}_n(G)^0 = \{(0, g_1, \dots, g_n, \tilde{\alpha}) : i\tilde{\alpha} = 0 \text{ if and only if } g_i = 0, \text{where } 1 \le i \le n \text{ and } \tilde{\alpha} \in \tilde{\mathcal{T}}_{\boldsymbol{n},\mathbf{0}}\}$. See [36] and the references therein.

From the definition of $\widetilde{K}_n(G)^0$, now having
Lemma 4.1: The number of elements in $\widetilde{K}_n(G)^0$ is equal to $|G|^n(n+1)^n$.
Proof: There are $|G|^n$ choices for arbitrary elements $g_i \in G$, $1 \le i \le n$. Further, as $\tilde{\alpha} \in \tilde{\mathcal{T}}_{\boldsymbol{n},\mathbf{0}}$ and there are $(n+1)^n$ elements in $\tilde{\mathcal{T}}_{\boldsymbol{n},\mathbf{0}}$, then there are $|G|^n(n+1)^n$ elements in $\widetilde{K}_n(G)^0$.

The following lemma has already been proved in [36] (see the references).
Lemma 4.2: Assume that $\tilde{I}$ be an ideal of $\widetilde{K}_n(G)^0$ such that

$$\tilde{I}' = \{\tilde{\alpha} : \text{there exists } (0, g_1, \dots, g_n, \tilde{\alpha}) \in \tilde{I}\}.$$

So that $\tilde{I}'$ is an ideal of $\tilde{\mathcal{T}}_{\boldsymbol{n},\mathbf{0}}$.
Conversely, if $\tilde{J}$ is an ideal of $\tilde{\mathcal{T}}_{\boldsymbol{n},\mathbf{0}}$, then when

$$\tilde{J}' = \{(0, g_1, \dots, g_n, \tilde{\alpha}) \in \widetilde{K}_n(G)^0 : \tilde{\alpha} \in \tilde{J}\},$$

obtaining $\tilde{J}'$ is an ideal of $\widetilde{K}_n(G)^0$.

Theorem 4.3: Let $\tilde{I}_n(G)^0$ be a two-sided ideal of $\widetilde{K}_n(G)^0$. Let $\tilde{\alpha} \in \tilde{\mathcal{J}}_{\boldsymbol{n},\mathbf{0}}$ be an idempotent such that $|Im\tilde{\alpha}| = k$, $|Dom\tilde{\alpha}| = \ell$, where $1 \le k \le n$, $k \le \ell \le n$ and $\tilde{\mathcal{J}}_{\boldsymbol{n},\mathbf{0}}$ be a two-sided of $\tilde{\mathcal{T}}_{\boldsymbol{n},\mathbf{0}}$.
Then $(0, g_1, \dots, g_n, \tilde{\alpha}) \in \tilde{I}_n(G)^0$ if and only if $(0, g'_1, \dots, g'_n, \tilde{\alpha}) \in \tilde{I}_n(G)^0$ for any $g'_i \in G^0$ with $g'_i = 0$ if and only if $i\tilde{\alpha} = 0$.
Proof: Let $(0, g_1, \dots, g_n, \tilde{\alpha}) \in \tilde{I}_n(G)^0$, such that $\tilde{\alpha}$ be given as above. Define $\xi \in \tilde{\mathcal{T}}_{\boldsymbol{n},\mathbf{0}}$ by (1).
Let $g'_1, \dots, g'_n \in G^0$ be $g'_i = 0$ if and only if $i\tilde{\alpha} = 0$. Hence, $(0, g_1^{-1}g'_1, \dots, g_n^{-1}g'_n, \xi) \in \widetilde{K}_n(G)^0$, where $0^{-1} = 0$. As $\tilde{I}_n(G)^0$ is a two-sided ideal of $\widetilde{K}_n(G)^0$ then

$$(0, g_1, \dots, g_n, \tilde{\alpha})(0, g_1^{-1}g'_1, \dots, g_n^{-1}g'_n, \xi) = \left(0, g_1 g_{1\tilde{\alpha}}^{-1} g'_{1\tilde{\alpha}}, \dots, g_n g_{n\tilde{\alpha}}^{-1} g'_{n\tilde{\alpha}}, \tilde{\alpha}\xi\right).$$

Now, since $\tilde{\alpha}$ is an idempotent, implies $g_i g_{i\tilde{\alpha}}^{-1} g'_{i\tilde{\alpha}} = g_i g_i^{-1} g'_i = g'_i$, for all $i \in \{1, \dots, n\}$, and as $\tilde{\alpha}\xi \in \tilde{\mathcal{J}}_{\boldsymbol{n},\mathbf{0}}$, then from Lemma 3.2, obtain $\tilde{\alpha}\xi = \xi\tilde{\alpha} = \tilde{\alpha}$. Therefore,

$$(0, g_1, \dots, g_n, \tilde{\alpha})\left(0, g_1^{-1}g'_1, \dots, g_n^{-1}g'_n, \xi\right) = \left(0, g'_1, \dots, g'_n, \tilde{\alpha}\right) \in \tilde{I}_n(G)^0.$$

Furthermore, $(0, g'_1 g_1^{-1} \dots, g'_n g_n^{-1}, \xi) \in \widetilde{K}_n(G)^0$, such that $0^{-1} = 0$. As $\tilde{I}_n(G)^0$ is a two-sided ideal then

$$(0, g'_1 g_1^{-1}, \dots, g'_n g_n^{-1}, \xi)(0, g_1, \dots, g_n, \tilde{\alpha}) = \left(0, g'_1 g_1^{-1} g_{1\xi}, \dots, g'_n g_n^{-1} g_{n\xi}, \xi\tilde{\alpha}\right).$$

Notice, if $i\tilde{\alpha} \neq 0$, then $g'_i \neq 0$, and from the definition of ξ gives $i\xi = i$. So, $g'_i g_i^{-1} g_{i\xi} = g'_i g_i^{-1} g_i = g'_i$, for all $i \in \{1, \dots, n\}$. If $i\tilde{\alpha} = 0$, then $g'_i = 0 = g'_i g_i^{-1} g_{i\xi}$.
Since $\xi\tilde{\alpha} \in \tilde{\mathcal{J}}_{\boldsymbol{n},\mathbf{0}}$ and from Lemma 3.2, implies $\xi\tilde{\alpha} = \tilde{\alpha}\xi = \tilde{\alpha}$, this gives

$$(0, g'_1 g_1^{-1}, \dots, g'_n g_n^{-1}, \xi)(0, g_1, \dots, g_n, \tilde{\alpha}) = \left(0, g'_1, \dots, g'_n, \tilde{\alpha}\right) \in \tilde{I}_n(G)^0.$$

Corollary 4.4: Let $\tilde{I}_n(G)^0$ be a right (left) ideal of $\widetilde{K}_n(G)^0$, and let $\tilde{\alpha} \in \tilde{\mathcal{J}}_{\boldsymbol{n},\mathbf{0}}$ be an idempotent such that $|Im\tilde{\alpha}| = k$, $|Dom\tilde{\alpha}| = \ell$, where $1 \le k \le n$, $k \le \ell \le n$ and $\tilde{\mathcal{J}}_{\boldsymbol{n},\mathbf{0}}$ is a two-sided of $\tilde{\mathcal{T}}_{\boldsymbol{n},\mathbf{0}}$.Then $(0, g_1, \dots, g_n, \tilde{\alpha}) \in \tilde{I}_n(G)^0$ if and only if $(0, g'_1, \dots, g'_n, \tilde{\alpha}) \in \tilde{I}_n(G)^0$ for any $g'_i \in G^0$ with $g'_i = 0$ if and only if $i\tilde{\alpha} = 0$.
Proof: Clear form Theorem 4.3.

It is worth mentioning that if $\tilde{\alpha}$ is not an idempotent, then Theorem 4.3 works only if $\tilde{I}_n(G)^0$ is a left ideal, and this can be illustrated by the following lemma:

Lemma 4.5: Suppose $\tilde{I}'_n(G)^0$ is a left ideal of $\widetilde{K}_n(G)^0$.Then $(0, g_1, \dots, g_n, \tilde{\alpha}) \in \tilde{I}'_n(G)^0$ if and only if $(0, g'_1, \dots, g'_n, \tilde{\alpha}) \in \tilde{I}'_n(G)^0$ for any $g'_i \in G^0$ with $g'_i = 0$ if and only if $i\tilde{\alpha} = 0$, and $\tilde{\alpha} \in \tilde{\mathcal{T}}_{\boldsymbol{n},\mathbf{0}}$, such that $\tilde{\alpha}$ is not an idempotent.
Proof: See [36].

Theorem 4.6: Let $G \neq \{0\}$. The number $E(n, \tilde{I}_n(G)^0)$ of idempotents in a two-sided ideal $\tilde{I}_n(G)^0$ of the set $\widetilde{K}_n(G)^0$is

$$E(n, \tilde{I}_n(G)^0) = \left[\sum_{k=1}^{n}\sum_{\ell=k}^{n}\binom{n}{\ell}\binom{\ell}{k}k^{\ell-k}|G|^{\ell-k}\right] + 1,$$

where $|Im\tilde{\alpha}| = k$, $|Dom\tilde{\alpha}| = \ell$ where $1 \le k \le n$, and $k \le \ell \le n$.
Proof: We want to count all the possibilities for an element $= (0, g_1, \dots, g_n, \tilde{\alpha}) \in \tilde{I}_n(G)^0$, where $\tilde{\alpha} \in \tilde{\mathcal{J}}_{\boldsymbol{n},\mathbf{0}}$ be an idempotent. It is clear that A is an idempotent if and only if $\tilde{\alpha}$ is an idempotent, and for all $1 \le i \le n$, $i\tilde{\alpha} \neq 0$, $g_i g_{i\tilde{\alpha}} = g_i$ and $g_{i\tilde{\alpha}} = 1$. As $\tilde{\alpha} \in \tilde{\mathcal{J}}_{\boldsymbol{n},\mathbf{0}}$ is an idempotent, then the number of idempotent elements in $\tilde{I}_n(G)^0$ will equal to all the idempotent elements in $\widetilde{K}_n(G)^0$, [36] (see the references there).

Corollary 4.7: Let $G \neq \{e\}$. The number of elements in a two-sided ideal $\tilde{I}_n(G)^0$ of $\widetilde{K}_n(G)^0$ is equal to the number of its idempotent.
Proof: As all the elements in $\tilde{I}_n(G)^0$ is an idempotent, so it was done.

Theorem 4.8: Let $G \neq \{e\}$. The number $\mathcal{N}(n, \tilde{I}'_n(G)^0)$ of elements in a left ideal $\tilde{I}'_n(G)^0$ of $\widetilde{K}_n(G)^0$ is

$$\mathcal{N}(n, \tilde{I}'_n(G)^0) = |G|^n (n+1)^n - E(n, \tilde{I}_n(G)^0).$$

Proof: Let $\tilde{\alpha} \in \tilde{\mathcal{T}}_{\boldsymbol{n},\mathbf{0}}$, such that $\tilde{\alpha}$ is not an idempotent. In virtue of Lemma 4.1, the number of elements in $\widetilde{K}_n(G)^0$ is $|G|^n(n+1)^n$, then by subtracting the number of elements of $\tilde{I}_n(G)^0$ in Corollary 4.7 from $|\widetilde{K}_n(G)^0|$, we get the result.

Example 4.9: Let $G = \{1, a\}$. Wanting to calculate the number of elements of a left ideal $\tilde{I}'_3(G)^0$ of the set $\widetilde{K}_3(G)^0$.
To apply the formula in Theorem 4.8, first, needing to count the number of elements in a two-sided ideal $\tilde{I}_3(G)^0$.
Deducing from Corollary 4.7 that the number of elements in a two-sided ideal $\tilde{I}_3(G)^0$ of $\widetilde{K}_3(G)^0$ is equal to the number of its idempotents. So, for counting the number of idempotents in $\tilde{I}_3(G)^0$, remember that (if $A \in \tilde{I}_3(G)^0 \subseteq \widetilde{K}_3(G)^0$, means from the definition of $\widetilde{K}_3(G)^0$ that $A = (0, g_1, g_2, g_3, \tilde{\alpha})$, where $g_1, g_2, g_3 \in G$, and $\tilde{\alpha} \in \tilde{\mathcal{J}}_{\mathbf{3},\mathbf{0}}$ such that $\tilde{\alpha}$ is an idempotent and $\tilde{\mathcal{J}}_{\mathbf{3},\mathbf{0}}$ is a two-sided ideal of $\tilde{\mathcal{T}}_{\mathbf{3},\mathbf{0}}$). It is clear that
$G^3 = \{(1,1,1), (a,a,a), (a,1,1), (1,a,1), (1,1,a), (a,a,1), (a,1,a), (1,a,a)\}$.
So, $|G|^3 = |G^3| = 2^3 = 8$.
Suppose $|Im\tilde{\alpha}| = k$, $|Dom\tilde{\alpha}| = \ell$ such that $1 \leq k \leq n$, $k \leq \ell \leq n$.
From Example 3.7, there are 23 idempotents in $\tilde{\mathcal{J}}_{\mathbf{3},\mathbf{0}}$ in terms of $\xi \in \tilde{\mathcal{T}}_{\mathbf{3},\mathbf{0}}$, and those elements are divided into 4 cases, all of which correspond to $|Im\tilde{\alpha}| = k$. Also, there are 4 cases to find the idempotents in $\tilde{I}_3(G)^0$ all of them also corresponds to k.
Case I: If $|Im\tilde{\alpha}| = k = 1$. Three cases for $|Dom\tilde{\alpha}| = \ell$, that are either 1, 2, or 3.
Case A: If $|Dom\tilde{\alpha}| = \ell = 1$. From Example 3.7 that there are 3 idempotents in $\tilde{\mathcal{J}}_{\mathbf{3},\mathbf{0}}$ in terms of $\xi \in \tilde{\mathcal{T}}_{\mathbf{3},\mathbf{0}}$ where $k = \ell = 1$. Observe that, in this case, there are 3 ways to choose $Im\tilde{\alpha}$ for each of these, there is only one way to select $Dom\tilde{\alpha}$. Notice that if $(0, g_1, g_2, g_3, \tilde{\alpha}_1) \in \tilde{I}_3(G)^0$, then $g_1 = 1$, and from the definition of $\widetilde{K}_3(G)^0$ implies $g_2 = g_3 = 0$, as $2, 3 \notin Dom\tilde{\alpha}_1$. So, that there is only one idempotent in $\tilde{I}_3(G)^0$ of the form $(0, g_1, g_2, g_3, \tilde{\alpha}_1)$ that is $(0, 1, 0, 0, \tilde{\alpha}_1)$. Similarly, when the idempotents have the form $(0, g_1, g_2, g_3, \tilde{\alpha}_2)$ or, $(0, g_1, g_2, g_3, \tilde{\alpha}_3)$. So, there will be three idempotents in $\tilde{I}_3(G)^0$ where $k = \ell = 1$.
Case B: If $|Dom\tilde{\alpha}| = \ell = 2$. know from Example 3.7 that there are 6 idempotents in $\tilde{\mathcal{J}}_{\mathbf{3},\mathbf{0}}$ in terms of $\xi \in \tilde{\mathcal{T}}_{\mathbf{3},\mathbf{0}}$. Notice that there are 3 ways to choose $Im\tilde{\alpha}$ for each of these, there are two ways to select $Dom\tilde{\alpha}$. Now, if $Im\tilde{\alpha} = \{1\}$, and if $(0, g_1, g_2, g_3, \tilde{\alpha}_4) \in \tilde{I}_3(G)^0$, then $g_1 = 1$ and as $2 \in Dom\tilde{\alpha}_4$, there are two choices for g_2 which are either 1 or a.
Moreover, by the definition of $\widetilde{K}_3(G)^0$ having $g_3 = 0$, as $3 \notin Dom\tilde{\alpha}_4$. From that, there are 2 idempotents in $\tilde{I}_3(G)^0$ of the form $(0, g_1, g_2, g_3, \tilde{\alpha}_4)$ and they are $(0, 1, 1, 0, \tilde{\alpha}_4)$, $(0, 1, a, 0, \tilde{\alpha}_4)$. Similarly, if the idempotent has the form $(0, g_1, g_2, g_3, \tilde{\alpha}_5)$. Hence, there are 4 idempotents in $\tilde{I}_3(G)^0$ where $k = 1$, $\ell = 2$, and $Im\tilde{\alpha} = \{1\}$.
As there are 3 ways to choose $Im\tilde{\alpha}$, then there are $3 \times 4 = 12$ idempotents in this position.
Case C: If $|Dom\tilde{\alpha}| = \ell = 3$. It is obvious from Example 3.7 that there are 3 ways to choose $Im\tilde{\alpha}$, for each of these, there is only one way to select $Dom\tilde{\alpha}$. In each way of choosing $Im\tilde{\alpha}$

there are 4 idempotents in $\tilde{I}_3(G)^0$ of the form $(0, g_1, g_2, g_3, \tilde{\alpha})$ such that, $g_j = 1$, where $j \in Im\tilde{\alpha}$, hence by the definition of $\widetilde{K}_3(G)^0$ obtain $g_i = 0 \;\; \forall i \notin Dom\tilde{\alpha}$.
As there are 3 ways to choose $Im\tilde{\alpha}$, there are $3 \times 4 = 12$ idempotents in this case.
Now, by putting everything of case $k = 1$, there are $3 + 12 + 12 = 27$ idempotents in this matter.
Case II: If $|Im\tilde{\alpha}| = k = 2$. In this case, there are two cases for $|Dom\tilde{\alpha}|$, which are either 2 or, 3.
Case A: If $|Dom\tilde{\alpha}| = \ell = 2$. From Example 3.7, there are 3 idempotents in $\tilde{\mathcal{J}}_{\mathbf{3,0}}$ in terms of $\xi \in \tilde{\mathcal{T}}_{\mathbf{3,0}}$ and they are

$$\tilde{\alpha}_{13} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 0 \end{pmatrix}, \quad \tilde{\alpha}_{14} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 3 \end{pmatrix}, \quad \tilde{\alpha}_{15} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 3 \end{pmatrix}.$$

It is clear that there are three ways for choosing $Im\tilde{\alpha}$, and for each way, there is only one way to select $Dom\tilde{\alpha}$. Now, if $(0, g_1, g_2, g_3, \tilde{\alpha}_{13}) \in \tilde{I}_3(G)^0$. Then, $g_1 = g_2 = 1$ as $1, 2 \in Dom\tilde{\alpha}_{13}$. As well as $g_3 = 0$ since $g_3 \notin Dom\tilde{\alpha}_{13}$. Hence, there is only one idempotent in $\tilde{I}_3(G)^0$ of the form $(0, g_1, g_2, g_3, \tilde{\alpha}_{13})$, which is $(0, 1, 1, 0, \tilde{\alpha}_{13})$.
In the same way, there is only one idempotent in $\tilde{I}_3(G)^0$ of the form $(0, g_1, g_2, g_3, \tilde{\alpha}_{14})$ and $(0, g_1, g_2, g_3, \tilde{\alpha}_{15})$.
Now, since there are three ways for selecting $Im\tilde{\alpha}$ then $3 \times 1 = 3$ idempotents in this case.
Case B: If $|Dom\tilde{\alpha}| = \ell = 3$. It has been known from Example 3.7 that there are six idempotents in $\tilde{\mathcal{J}}_{\mathbf{3,0}}$ in terms of $\xi \in \tilde{\mathcal{T}}_{\mathbf{3,0}}$. Observe that, in this matter, there are 3 ways to select $Im\tilde{\alpha}$, and for each of these, there are 2 ways to select $Dom\tilde{\alpha}$. Furthermore, in each way of selecting $Dom\tilde{\alpha}$ there are 2 idempotents in $\tilde{I}_3(G)^0$ having the form $(0, g_1, g_2, g_3, \tilde{\alpha})$ such that $g_j = 1$, where $j \in Im\tilde{\alpha}$, and $g_i = 0$ for all $i \notin Dom\tilde{\alpha}$ by the virtue of the definition of $\widetilde{K}_3(G)^0$. Hence, there are $2 \times 2 = 4$ idempotents in $\tilde{I}_3(G)^0$ in every way of selecting $Im\tilde{\alpha}$. Since we have 3 ways of choosing $Im\tilde{\alpha}$, therefore, there are $3 \times 4 = 12$ idempotents in this case.
By putting all the cases corresponding to $k = 2$ together, obtaining $3 + 12 = 15$ idempotents in $\tilde{I}_3(G)^0$.
Case III: If $|Im\tilde{\alpha}| = k = 3$. There is only one case for $|Dom\tilde{\alpha}| = \ell$, that is $\ell = 3$. From Example 3.7, there is only one idempotent in $\tilde{\mathcal{J}}_{\mathbf{3,0}}$ in terms of $\xi \in \tilde{\mathcal{T}}_{\mathbf{3,0}}$ having three elements in its image and in its domain, which is

$$\tilde{\jmath}_{\mathbf{3,0}} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix}.$$

There is only one idempotent in $\tilde{I}_3(G)^0$ of the form $(0, g_1, g_2, g_3, \tilde{\jmath}_{\mathbf{3,0}})$ and that it is $(0, 1, 1, 1, \tilde{\jmath}_{\mathbf{3,0}})$. Therefore, $g_j = 1$, where $j \in Im\tilde{\alpha}$.
Case IV: If $|Im\tilde{\alpha}| = k = 0$. There is only one matter for $|Dom\tilde{\alpha}| = \ell$, that is $|Dom\tilde{\alpha}| = \ell = 0$.
Obviously, there is one idempotent in $\tilde{\mathcal{J}}_{\mathbf{3,0}}$ in terms of $\xi \in \tilde{\mathcal{T}}_{\mathbf{3,0}}$ having $|Im\tilde{\alpha}| = |Dom\tilde{\alpha}| = 0$, which is

$$\tilde{c}_0 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

From that, there is only one idempotent in $\tilde{I}_3(G)^0$ of the form $(0, g_1, g_2, g_3, \tilde{c}_0)$, which is $(0, 0, 0, 0, \tilde{c}_0)$, this is by using the definition of $\widetilde{K}_3(G)^0$ such that have $g_i = 0$ for all $i \notin Dom\tilde{\alpha}$.
For counting all the idempotents in $\tilde{I}_3(G)^0$ need to count all the idempotents in the four cases to have $27 + 15 + 1 + 1 = 44$ idempotents in $\tilde{I}_3(G)^0$. So, there are 44 elements in a two-sided ideal $\tilde{I}_3(G)^0$ of $\widetilde{K}_3(G)^0$. The result will be obtained using the formula in Theorem 4.6. Now, we are ready to use the formula Theorem 4.8 for counting the number of elements

in a left $\tilde{I}'_3(G)^0$ of $\widetilde{K}_3(G)^0$. From Corollary 4.1, know $|\widetilde{K}_3(G)^0| = |G|^3(3+1)^3 = 2^3(3+1)^3 = 512$.
By subtracting the number of elements in a two-sided ideal $\tilde{I}_3(G)^0$ from the above result, obtaining $512 - 44 = 468$ elements in a left $\tilde{I}'_3(G)^0$ of $\widetilde{K}_3(G)^0$.

5. Conclusions

The ideals of partial transformations semigroup $\mathcal{PT}_{\boldsymbol{n}}$ were considered such that there are two kinds of ideals in $\mathcal{PT}_{\boldsymbol{n}}$. Firstly, (a two-sided ideal $\tilde{\mathcal{J}}_{\boldsymbol{n},\boldsymbol{0}}$) which have two cases if $\tilde{\alpha} \in \tilde{\mathcal{J}}_{\boldsymbol{n},\boldsymbol{0}}$, such that $\tilde{\alpha}$ is an idempotent and $|Im\tilde{\alpha}| = k$, $|Dom\tilde{\alpha}| = \ell$ where $0 \le k \le n$ and $k \le \ell \le n$, and if $\tilde{\alpha} \in \tilde{\mathcal{J}}_{\boldsymbol{n},\boldsymbol{0}}$ such that $\tilde{\alpha}$ is not an idempotent with $|Im\tilde{\alpha}| = |Dom\tilde{\alpha}| = n$ and $\tilde{\alpha} \neq \tilde{\mathcal{I}}_{\boldsymbol{n}}$. Secondly, (left ideal $\tilde{\mathcal{J}}'_{\boldsymbol{n},\boldsymbol{0}}$), where $\tilde{\alpha} \in \tilde{\mathcal{J}}'_{\boldsymbol{n},\boldsymbol{0}}$ such that $\tilde{\alpha}$ is not an idempotent and $|Im\tilde{\alpha}| = k$, $|Dom\tilde{\alpha}| = \ell$ where $1 \le k \le n-1$ and $k \le \ell \le n-1$. As well as the number of elements of a two-sided ideal $\tilde{\mathcal{J}}_{\boldsymbol{n},\boldsymbol{0}}$ and a left ideal $\tilde{\mathcal{J}}'_{\boldsymbol{n},\boldsymbol{0}}$ were calculated. Proceeded by describing the ideals of partial transformation monoid of a free (left) G -act on n-generators, $\mathcal{PT}_{F_n(G)}$ where G is a finite group and $F_n(G) = \dot{\cup}_{i=1}^{n} Gx_i$. The extension of the current work to cover the problem of finding the number of nilpotent elements of a two-sided ideal $\tilde{I}_n(G)^0$ and a left ideal $\tilde{I}'_n(G)^0$ of $\widetilde{K}_n(G)^0$, is deferred as a future work.

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