SIMPLE NUMERICAL SOLUTION FOR THE MASTER EQUATION OF NUCLEAR STATISTICAL MODEL

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Abstract

The preequilibrium statistical model, which is used to analyze nuclear reaction data, depends basically on number of parameters including the master equation. A numerical solution for the master equation of the preequilibrium statistical model is presented in this work. One-component Fermi gas model is assumed. This numerical method is shown to give the same accuracy for the simple as well as the more advanced schemes. The present method shows fast convergence for a wide range of physical system parameters. Such a method is applied for two reaction examples that include neutron and proton induced reactions with ^{54}Fe nucleus for entrance energies 20 and 80 MeV. Comparisons with standard solution methods show that the method suggested here reveals its accuracy and simplicity for practical applications and calculations. The results indicated that the present method is proper for low reaction energies, but with worse accuracy at higher energies. The behavior of the present method was examined against the energy, exciton numbers, and time variation.

طريقة عددية بسيطة لحل المعادلة الأساسية في الأنموذج الإحصائي للتفاعلات النووية

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الخلاصة

إن الأنموذج الإحصائي للتفاعلات النووية يعتمد على عدد من العوامل من ضمنها المعادلة الأساسية (The Master Equation). في البحث الحالي نقترح طريقة عددية لحل المعادلة الأساسية لأنموذج التفاعلات النووية الإحصائي بافتراض أن النواة تتألف من غاز فيرمي متكون من نوع واحد من الجسيمات. باستخدام الطريقة الحالية وجدنا أن دقة النتائج مقبولة عند استخدام أنموذج حسابي بسيط وكذلك نفس الدقة قد وجدت عند تطبيق نموذج أكثر تعقيدا، مما يدل على كفاءة الطريقة المقترحة في هذا البحث. أثبتت الطريقة الحالية أنها ملتمة (convergent) لمدى واسع من مواصفات النظام المدروس. تم تطبيق هذه الطريقة على مثالين للتفاعلات النووية المحتثة بواسطة النيترون والبروتون (nucleon induced reactions) مع نواة الحديد ^{54}Fe لطاقات تفاعل عند قناة البدء 20 و 80 ميغا إلكترون فولت. مقارنة النتائج الحالية مع الطرق المعتمدة لحل المعادلة الأساسية أشارت إلى أن الطريقة الحالية تجمع كلا من الدقة والبساطة مما يجعلها طريقة ملائمة للتطبيق في الحسابات العملية. بينت النتائج الحالية أن هذه الطريقة

تمتاز بدقة مناسبة للأغراض الحسابية في الطاقات الواطئة، لكن الدقة تسوء عند الطاقات العالية. تم اختبار تصرف الطريقة الحالية مع الطاقة، عدد الجسيمات المؤقتة والزمن.

Introduction

The preequilibrium statistical models are a group of models that are based on statistical approach to describe the various nuclear reactions at intermediate energies. This group includes exciton model [1], Hybrid Model (HM) and Geometry Dependent Hybrid (GDH) model [2, 3, 4] and the Monte Carlo Hybrid (MCH) model [5, 6]. These models have been developed rapidly since Griffin [1] announced the exciton model as a semi-classical theory to explain the precompound nuclear emission (PE). Extended from the exciton model, the theory of PE was developed in order to include more dynamically related parameters such as the transition matrix and scattering cross-section. The family of Hybrid models [2-6] are assumed as an improved version of Griffin's model where the basic ideas were combined with the Master Equation Model (MEM) due to Harp, Miller and Berne [7, 8].

All these models, and many others that came after, deal with the intermediate states of the nuclear reactions. Instead of the highly complicated quantum-mechanical treatment for nuclear reactions, the statistical models provided a suitable solution, and the exciton model seems an ideal model among them.

The process of nuclear equilibration is thought to be responsible of the precompound nuclear emissions, where the energy of the projectile is shared with the nuclear constituents via successive process of two-body collisions. At each state of this "preequilibrium phase", there will exist a small but important probability of nuclear decay from these excited states to the continuum. Such decay is observed as a continuous spectrum laying between the direct reaction (fast emission) and the evaporation (slow) emission. The residual two-body interaction was first assumed to occur between identical particles, i.e., there is only one type of particles in the nucleus. This is the "One-component System", or "One-Component Fermi Gas System". The "Two-Component System" therefore will distinguish between protons and neutrons inside the nucleus. The one-component system is still used, however, to explain the details of the model and its basic theory in a clear manner.

The excitation development, when based on the residual two-body interaction, will ensure that the number of excitons characterizing each stage in the equilibration process will change by ±2 or zero. Equilibration process will cause creation of particle-hole pairs, and these pairs will be the reason that carries out the basic mechanism of energy share between the constituents of the nucleus. This will lead to successive creation of excitons during the equilibration process, and therefore, each stage in the equilibration process can be specified well by the exciton number n and excitation energy E. Decay may take place from some of these stages by a certain probability λ, and this probability depends mainly on the specifications of these stages, n and E.

The transition between adjacent stages is characterized, more generally, by the transition rate, $\lambda_{x\text{-}y}$, between stages x and y as (Fermi golden rule),

$$\lambda_{x-y} = \frac{2\pi}{\hbar} M^2_{x-y}\, \omega_y \qquad (1),$$

where (x-y) represents (initial–final) destination, ω_y is the final accessible density of states. $M_{x\text{-}y}$ is the matrix element of the specific interaction, $M_{x-y} = < y | \hat{M} | x >$, with $\hat{M}$ being the operator of the effective potential causing the system transition from (x) initial state to (y) final state. This quantity, $M_{x\text{-}y}$ is used by approximate formulae that are based on data extracted from experimental data [9].

Each stage actually has two transition possibilities, up and down, which means that the change of the exciton number $\Delta n = -2$ or $+2$, and the inter-substages has $\Delta n = 0$. There is also a transition probability to the continuum which represents the decay of each stage.

The Master Equation (ME) describes the occupation probabilities for each stage at a given time t, n and E. The ME will be given as, and in terms of one-component Fermi system [9],

$$\frac{dP(n,E,t)}{dt} = \lambda^{-}_{n+2}(n,E)\,P(n+2,E,t) + \lambda^{+}_{n-2}(n,E)\,P(n-2,E,t) - \frac{P(n,E,t)}{\tau_n(n,E)} \qquad (2),$$

where $P(n,E,t)$ is the occupation probability of the n^{th} stage with excitation energy E at time t, and τ_n is the mean lifetime of this stage, defined as [9],

$$\tau_n(n,E) = \left(\lambda_n^-(n,E) + \lambda_n^+(n,E) + W(n,E)\right)^{-1} \quad (3),$$

and $W(n,E)$ is the rate of decay to the continuum. Eq.(2) is the simple version of the two-component ME,

$$\begin{aligned}
\frac{dP(N,h_\pi,t)}{dt} = & \\
& \left[\lambda_\nu^{++}(E,N-1,h_\pi) + \lambda_\pi^{++}(E,N-1,h_\pi)\right]P(N-1,h_\pi-1,t) \\
& + \left[\lambda_\pi^{+0}(E,N-1,h_\pi) + \lambda_\nu^{+0}(E,N-1,h_\pi)\right]P(N-1,h_\pi,t) \\
& + \left[\lambda_\nu^{-0}(E,N+1,h_\pi) + \lambda_\pi^{-0}(E,N+1,h_\pi)\right]P(N+1,h_\pi,t) \\
& + \left[\lambda_\pi^{--}(E,N+1,h_\pi+1) + \lambda_\pi^{--}(E,N+1,h_\pi+1)\right]P(N+1,h_\pi+1,t) \\
& + \left[\lambda_\nu^{0+}(E,N,h_\pi-1)\right]P(N,h_\pi-1,t) \\
& + \left[\lambda_\pi^{0-}(E,N,h_\pi+1)\right]P(N,h_\pi+1,t) \\
& - \left[\lambda_\pi^{+0}(E,N,h_\pi) + \lambda_\nu^{+0}(E,N,h_\pi) + \lambda_\nu^{++}(E,N,h_\pi) + \lambda_\pi^{++}(E,N,h_\pi)\right. \\
& \quad + \lambda_\nu^{-0}(E,N,h_\pi) + \lambda_\pi^{-0}(E,N,h_\pi) + \lambda_\pi^{--}(E,N,h_\pi) + \lambda_\pi^{--}(E,N,h_\pi) \\
& \quad \left. + \lambda_\nu^{0+}(E,N,h_\pi) + \lambda_\pi^{0-}(E,N,h_\pi) + W(E,N,h_\pi)\right]P(N,h_\pi,t) \qquad (4),
\end{aligned}$$

where n_π *and* n_ν are the exciton numbers for protons and neutrons, respectively, they are related to n as $n = n_\pi + n_\nu$ and the occupation probability, $P(N,h_\pi,t)$, of n_π *and* n_ν number of excitons at time t and of excitation energy E. N is a function of n_π *and* n_ν number as: $n = p_\pi + h_\pi + p_\nu + h_\nu = 2(N+1) + n_o$, and n_o is the initial exciton number. h and p represent hole and particle numbers, and the subscripts n_π *and* n_ν represent proton and neutron types, respectively. Eq.(4) basically distinguishes between different transition types for different interactions, namely, the changes of λ's are according to the general form: $\lambda^{\Delta N,\ \Delta h_\pi}_{type1,\ type2}$

In this paper we shall use eq.(2) to explain the numerical procedure rather than eq.(4). Hopefully, the method applied here for one-component ME can be extended in the future for the more complicated two-component system.

The total lifetime for each of the states described by eq.(2) is given as [9],

$$T(n,E) = \int_0^\infty P(n,E,t)dt \qquad (5),$$

and if we assume that the emission will occur for a particle of type β then the spectrum expected will be given by the following,

$$\frac{d\sigma_\beta(\varepsilon_\beta)}{d\varepsilon_\beta} = \sigma_\alpha \sum_{n_\pi,n_\nu} T(n,E)\,\lambda_\beta^c(\varepsilon_\beta;n,E) \qquad (6),$$

where σ_α is the formation cross-section of the composite nucleus in the exit channel α (the inverse cross-section). The decay rate λ_β^c represents the partial emission rate of the particle β from the state described by n and E into the final channel.

From the above, we can see the importance of the ME in preequilibrium calculations of particle emission during nuclear reactions.

Methods of Solutions for the Master Equation

There are many suggested methods to find the solutions of the master equation, although the mathematical form seen from eq.(2) might appear simple. The difficulty in the ME is that each stage depends dynamically on the adjacent stages at the same time, t. The system of equations, eq.(2) represents coupled differential equations of the first order and it has been a subject for many studies before. The most

important and popular methods are reviewed below.

Luider [10], assuming one-component Fermi system, showed that the ME can be given in the matrix form as,

$$[\dot{x}] = [A][x] \quad (7),$$

where [*A*] is the matrix representing the decay rates,

$$[A] = \begin{bmatrix} \lambda_1^+ & -1/\tau_1 & \lambda_1^- & 0 & \dots & 0 & 0 \\ 0 & \lambda_2^+ & -1/\tau_2 & \lambda_2^- & & & 0 \\ & & & & & & \cdot \\ 0 & 0 & \dots & \dots & & & \cdot \\ & & & & & & \cdot \\ 0 & 0 & 0 & \dots & \lambda_n^+ & -1/\tau_n & \lambda_n^- \end{bmatrix} \quad (8),$$

and the solution [*x*] can then be represented as,

$$[A][T] = [x(t)] - [x_o] \quad (9),$$

which assumes that $[x(t)] \neq 0$ at $t = \infty$. The form of *T* in eq.(9) is given by eq.(5).

This method was the first extended method that try to deal with ME for any given exciton number. However, for large *n* values, this method tends to need complicated computational scheme [9]. Dobeš and Běták [9] suggested further that the ME can be solved by iterative method for both one- and two-component systems, where eq,(2) is written as,

$$-D_n = x_{n-2}\,\tau_{n-2}\,\lambda_{n-2}^+ + x_{n+2}\,\tau_{n+2}\,\lambda_{n+2}^- - x_n \quad (10),$$

which is divergent as long as the term $\left(\tau_{n-2}\,\lambda_{n-2}^+ + \tau_{n+2}\,\lambda_{n+2}^-\right) < 1$, where,

$$x_n = \frac{T_n}{\tau_n} \quad (11)$$

D_n is the initial condition. The iteration in this method starts from the zeroth approximation formula given by,

$$-D_n = T_{n-2}^{[0]}\,\lambda_{n-2}^+ + T_n^{[0]}/\tau_n \quad (12),$$

and for n_o, and *n* states we have,

$$\left.\begin{aligned} T_{n_o}^{[0]} &= \tau_{n_o}\,D_{n_o} \\ T_n^{[0]} &= \tau_n\left(D_n + T_{n-2}^{[0]}\,\lambda_{n-2}^+\right) \end{aligned}\right\} \quad (13),$$

n_o is the initial exciton number. However, this method is inconvenient for reactions with low preequilibrium fraction [9] and other method are required at these limits.

The explicit method of Akkermans [11] was also proposed. As in the method of Luider, one can write eq.(2) as a matrix form and the matrix [A], for one-component system, will be in the form of tri-diagonal matrix. This property gives a direct method for finding the solution of eq.(2) above, and hence the master equation can be solved if the transition matrix [A] explicitly inverted by making use of its tridiagonality property. The solution in this method is [11],

$$[t] = [B]\,[C]\,[q_o] \quad (14).$$

This can be expanded in terms of power series to give [11],

$$T_n = \tau_n\,h_n \sum_{\substack{j=n_o \\ \Delta j=2}}^{n} (q_o)_j \left(\prod_{\substack{i=j \\ \Delta i=2}}^{n-2} \lambda_i^+\;\tau_i\;h_i \right) \quad (15).$$

where if *j=n*, then we put the expression within the Π to be equal to unity.

The method above was further modified by Chatterjee and Gupta [12] who used element elimination method to solve the ME for two-component. However, their method is assumed to be impractical for direct applications [13, 14]. Dobeš and Běták [13] suggested yet another modified iterative method for two-component system where the ME solves to the following,

$$\begin{aligned} &T^{[j+1]}(n_\pi, n_\nu) = \\ &\tau(n_\pi, n_\nu)\Big\{ T^{[j]}(n_\pi+2, n_\nu)\,\lambda^-(n_\pi+2, n_\nu) \\ &+ T^{[j]}(n_\pi, n_\nu+2)\,\lambda^-(n_\pi, n_\nu+2) + \\ &T^{[j]}(n_\pi+2, n_\nu-2)\,\lambda_{\pi\nu}^0(n_\pi+2, n_\nu-2) \\ &+ T^{[j]}(n_\pi-2, n_\nu+2)\,\lambda_{\nu\pi}^0(n_\pi-2, n_\nu+2) + \\ &T^{[j]}(n_\pi-2, n_\nu)\,\lambda^+(n_\pi-2, n_\nu) \\ &+ T^{[j]}(n_\pi, n_\nu-2)\,\lambda^+(n_\pi, n_\nu-2) + D(n_\pi, n_\nu)\Big\} \end{aligned} \quad (16).$$

The parameters suggested in [9] were used to find the transition rates, namely,

$$|M_{\pi\nu}|^2 = \frac{K}{A\,N\,Z\,E} \quad (17-a),$$

$$|M_{\nu\nu}|^2 = \frac{K}{A\,N^2\,R\,E} \quad (17-b),$$

$$|M_{\pi\pi}|^2 = \frac{K}{A\,Z^2\,R\,E} \qquad (17-c),$$

where:
K=fitting parameter,
R=a numerical factor that accounts for different ways of interaction between like and unlike types of particles. Its value is ~3. Ref.[9] used *R*=2.89 for proper treatment.
A, *Z* and *N*= Atomic mass number, atomic number and neutrons number of the nucleus, and *E*=excitation energy of the nucleus.
Other methods such as those due to Kalbach [14] and Herman et al. [15] use explicit treatments to find the solution of the ME for two-component system. All these methods compete among each other to solve the ME depending on the selected case under study. However, there is still a need to have a more general and easier method.

Description of the Present Iterative Method

In the present paper, we suggest another method that we try to combine the iterative methods of Dobeš and Běták [9, 13] and Luider [10] and Akkermans [11] methods, all in one simple numerical method.
Suppose we have the ME written in the form,

$$\frac{dP_n}{dt} = \lambda^-_{n+2} P_{n+2} + \lambda^+_{n-2} P_{n-2} - \frac{P_n}{\tau_n} \qquad (18),$$

which is the same as eq.(2) but we omitted the dependence on *E* and *t* for simplicity. Using the finite difference scheme for replacing the time derivative, and writing the equations from n=1 to the n^{th} scheme, after defining the following factors,

$$\left.\begin{aligned} a_{j,1} &= \Delta t\,\lambda^-_{n+2} \\ a_{j,2} &= \left(1 - \frac{\Delta t}{\tau_n}\right) \\ a_{j,3} &= \Delta t\,\lambda^+_{n-2} \end{aligned}\right\} \qquad (19),$$

where the index j is related to n by the simple relation, $j = \frac{n+1}{2}$. Then, the ME simply solves to the following,

$$P_i^{n+1} = \sum_{k=1}^{3} a_{j,k} P_i^n \qquad (20).$$

This simple equation will converge with initial conditions,

$$\left.\begin{aligned} P_{i=0}^{n_o} &= 1 \\ P_{i=0}^{n} &= 0 \end{aligned}\right\} \qquad (21),$$

which is convergent as long as the factors a's are convergent.
This simple method is actually based on Euler difference scheme. One may argue that this simple method is of less accuracy than the above mentioned methods, however, a practical comparison shows that the difference of the occupation probability between this method and the earlier method is less than ~ 5%. Combining this level of accuracy with the relatively simple programming effort, the method suggested in the present paper shows its importance. Beside this, the method can be easily improved to include any number of excitons *n*. Other methods of difference schemes, such as Runge-Kutta method, was also examined numerically against the present method and the difference was also promising as will be seen in the next paragraph, where numerical calculations are described for selected examples and comparisons are presented to test this method.
It should be mentioned that, regardless its simplicity, the method suggested here should be carefully applied with suitable choice of the time step, Δt, just as in any ordinary differential equation when one seeks numerical solution. Convergence might never be reached if this parameter were of order of β or larger. However, the simplicity of this method suggests that one can try a set of values for $\beta\,t$, and select the best (leading to a suitably convergent) solution. Details about the numerical solution are presented below.
Furthermore, the present method need not to be simplified by ignoring transitions other than β^-_{n-2} as in Refs.[1, 2, 9] because the simplicity of this scheme make it programmable with ease. However, in order to be consistent with Kalbach approximation [14, 16], we shall ignore here the dependence on $W(n,E)$.

Results, Discussion of Numerical Calculations, and Comparisons

In the present paper, a numerical method is suggested to solve the master equation needed in preequilibrium nuclear reaction analysis. The present method depends on simple iterative

numerical scheme that joins both precise and fast estimation of the occupation probability. The current method is given in general by eqs.(18–21). Two different numerical methods were used in order to find the time derivative of the occupation probability, namely, Euler (centered) method and Runge-Kutta method. In order to perform numerical calculations, the transition rates given in the ME are used assuming two approaches, the first is the experimentally evaluated transition rate (according to the most recent formulae due to Kalbach [17, 18]), namely,

$$|M|^2 = K\frac{A_\beta}{g_o^3}\left(\frac{E}{3A_\beta}+20.9\right)^{-3} \quad \text{(MeV)} \qquad (22),$$

where A_β is the mass number of the incident projectile, g_o is the single-particle state density, and K is a constant that has the value 900 MeV^2 for proton-proton and 2200 MeV^2 for neutron-neutron reaction. In this paper, we assumed that the excitation is mainly due to neutron-neutron and proton-proton reaction only, and we shall ignore neutron-proton one.

The second approach is to assume that the transition rates are all equal in either ways, i.e., with β n= +2 and -2, and are replaced by unity in order to obtain the relative occupation probabilities in each exciton configuration. These details will lead to four different applications of the method suggested in this paper. Furthermore, the entire set above was applied for four examples, namely, (n+^{54}Fe) and (p+^{54}Fe) reactions at nucleon incident energies 20 and 80 MeV. This verity of cases will ensure careful investigation of the results of this work with other references for practical comparison. All of the numerical examples described below are written in Matlab. The value of the single-particle density g_o was fixed at 14 MeV^{-1} in all calculations of the present paper. The occupation probabilities which represent the solution of the ME are given in percentage for the purpose of clear comparisons. Below are the applications of the suggested method.

1.Euler Scheme with Energy-Dependent Matrix Elements

The results are shown in Figures (1-A and B), for neutron at incident energies 20 and 80 MeV, respectively, for exciton numbers 1, 3 and 5. From these figures, one immediately notices how the system changes with time, *t*, exciton number *n* and energy of the incident particle. The initial conditions imposed are according to Dobeš and Běták [9]. Here we shall discuss the physics of these figures because this behavior will be repeated in the entire set of results presented here. Later, only the effects of numerical methods will be discussed.

First, as the exciton number increases, one can see that the distribution peak shifts towards longer time. Equilibration before these maxima is not possible because the occupation probabilities lay in a region of large differences (non-equilibrated behavior). In addition, it is seen that the decrement of the distribution tail becomes less dependent on time as *n* increases. These specifications of system behavior indicate that as equilibration process progresses in time (with larger exciton numbers) then the system will approach the most probable (equilibrated) state. Due to these specific observations of the ME, it was suggested [2, 4] that there is a certain exciton number that is called "the most probable exciton number $\overline{n}$" from which nuclear reaction is most expected to terminate by forming the compound nucleus. In the examples above, only few exciton numbers were considered, *n*=1, 3 and 5. Although these exciton numbers are far less than $\overline{n}$, and still important because nuclear preequilibrium decay is most probable from these low exciton numbers [9,11,14-16].

An interesting observation is that at *n*=1 the occupation probability (hence the total lifetime *T*) of the system behaves ideally as the usual nuclear (exponential) decay law. This behavior, in fact, is a direct consequence of the initial condition imposed by the program, that is at *t*=0, P_1=1. Still, this result might indicate that there is a possibility to find analytical solution that depends exponentially on time and transition rates. This interesting point will be studied in the future. The analytical solutions presented due to Luider [10], Akkermans [11] and Dobeš and Běták [9] have quite level of complexity, so a simple analytical solution, if exist, will highly simplify the problem of PE calculations. Also, from Figures (1-A and B), the effect of energy on the total behavior is quite obvious. As the energy increases from 20 to 80 MeV, the system reaches faster to balanced (equilibrated) state even at these low exciton numbers. Energy effect can be seen to effect the entire behavior where: (*a*) the probabilities decay faster to equilibrated states, (*b*) the centers corresponding to the maxima happen at shorter lifetime, and (*c*) the tail of all the curves becomes more dependent on *t*.

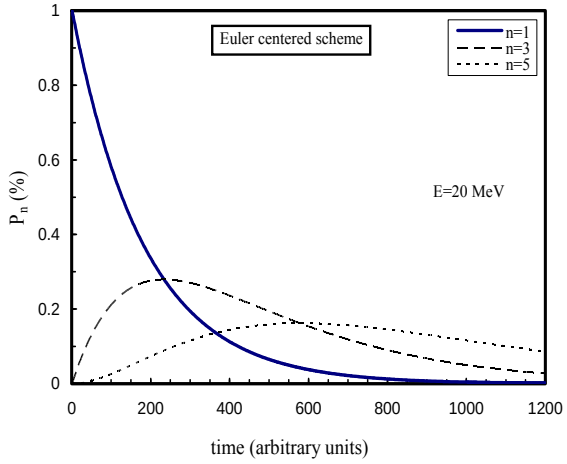

Figure (1-A): The time-distribution of the occupation probabilities for *n*=1, 3 and 5, using Euler centered scheme for (n+^{54}Fe) reaction at incident energy 20 MeV.

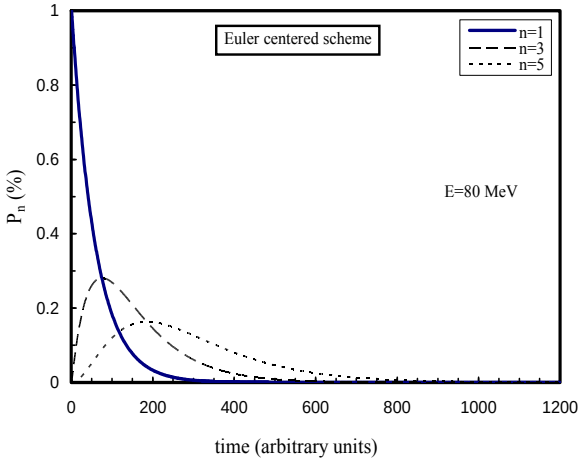

Figure (1-B): The same as Fig.(1-A) for incident energy 80 MeV.

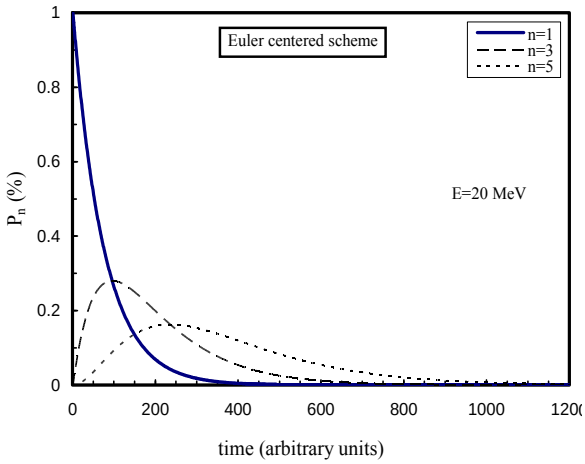

Figure (1-C): The same as Fig.(1-A) for (p+^{54}Fe) reaction at incident energy 20 MeV.

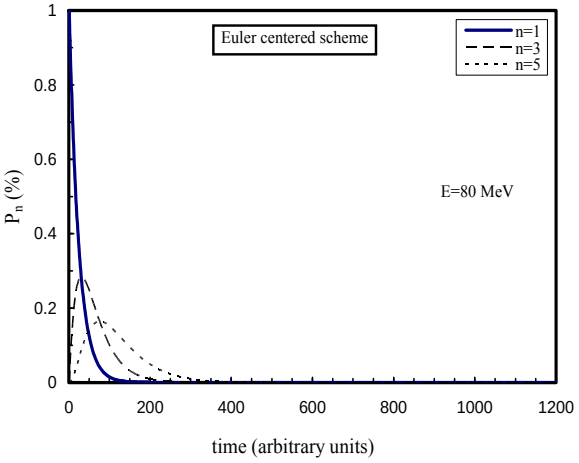

Figure (1-D): The same as Fig.(1-C) at incident energy 80 MeV.

From Figures (1-C and D), the same effect is seen for energy change for the same type of incident particle. In these examples, the system tends to be equilibrated even faster that in the case for neutron reaction. Such results, which indicate the rapid energy share between the incident particle and the nucleus, show that proton reactions have, in general, more details that should be taken into account when investigating any of the statistical models applications for nuclear reactions.

2.Euler Scheme with Energy-Independent Matrix Elements

The results of these calculations are shown in Figures (2-A and B). In this case, proton and neutron reactions will read the same as seen from eq.(22). Thus, the major difference will be due to energy change of the incident particle, not the *type* of the particle itself. Of course, energy-independent matrix element is considered as a crude approximation in the physics of nuclear reaction. This case, therefore, seems to be not precise in practical calculation; nevertheless, it can be used for the present comparison for clarity of the method suggested herein. These figures show that the probabilities become less dependent on *t* if the matrix elements were energy-independent. The spread of the curves in the case of 20 MeV is even wider than that of Figure (1-A). This signifies the importance of energy dependence for various cases that needed to be taken under investigation during compound nucleus formation.

A glance on these two figures, as in the case of the preceding subsection, indicates that as the incident energy increases the occupation probabilities fall faster with time. In fact, the falling in this case is faster than in the previous one. In other words, at energy 20 MeV the occupation probability is less dependent on t, and at energy 80 MeV, these probabilities are more dependent on t. This means that in this approximation the method adopted in this paper might fail to reproduce the same accuracy when not dealing with the more realistic energy-dependent matrix element during calculating the transition rates. So this point adds another restriction on the present method. However, most sensible PE calculations deal with the realistic case, i.e., that these matrix elements depend on E in one way or another. Therefore, the numerical method presented here is still of meaning to be considered in the cross-section calculations.

3. Runge-Kutta Method with Energy-Dependent Matrix Elements

The same examples above were applied using Runge-Kutta method in order to test the accuracy of the present method. Runge-Kutta method is of high accuracy than the simple Euler method and, as one may expect, Runge-Kutta method needs much more complicated numerical scheme and programming effort.

The results of comparison of the two method against each other is presented in Figures (3-A and B). This example is intended to be compared with the results of Figures (1-C and D), i.e., for ($p+^{54}Fe$) reaction at energies 20 and 80 MeV, respectively. Similar input parameters were used in all cases of Figures (3-A and B). The consistency of the method suggested in this paper, eq.(21), is shown to be acceptable to an approximate degree at low energy, while atenergy 80 MeV the difference is quite obvious. Therefore, it is emphasized here that the simple numerical scheme proposed above should be applied at energies around 20 MeV. At higher energies, more complicated methods must be used. The results of the other examples were seen to have the same behavior for other examples as in Figures (3-A and B), i.e., as the energy increased from 20 to 80 MeV, the difference between the two methods becomes more clear. Therefore, no other comparisons will be made here.

4.The Dependence of the Occupation Probability on E and n

It is convenient in this paper to investigate how the occupation probability will depend on *E* and *n* in the case of numerical solution. To achieve this, the calculations with Euler method were repeated for various range of these parameters.

The results of the dependence of *E* are shown in the group of Figures (4) for the energy range from 0 to 100 MeV. In each of these figures, the time was fixed at a certain value (shown in the figures) and the dependence of the results on energy is plotted. The phenomena seen before in subsection 2 is apparent now, where as the energy increases, the values of the occupation probabilities tend to drop faster with time.

On the other hand, the dependence on *n* is shown in Figure (5), where the calculation was made for the range *n*=1 to 15. Obviously, the systems with higher exciton numbers have their peaks shifted toward longer lifetimes even when the energy is assumed to be constant. This is joined with crude reduction of the maxima at the centers.

From Figure (5) one can also expect that there is a certain value of the exciton number at which the change of the cooptation probability is very small for the entire time range. Such a value represents the final exciton number possible for the system to attain before equilibrium.

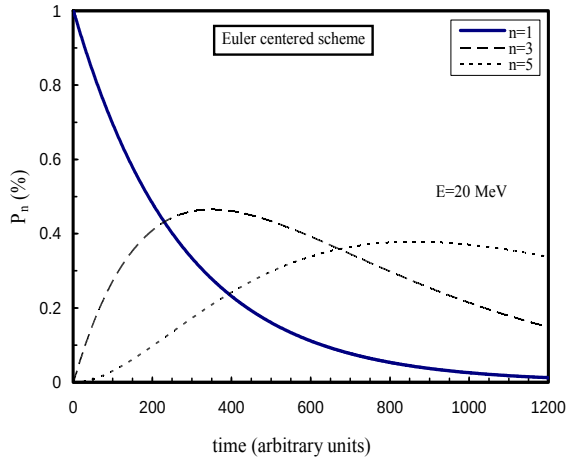

Figure (2-A): The same as Fig.(1-A) for ($p+^{54}Fe$) reaction at incident energy 20 MeV.

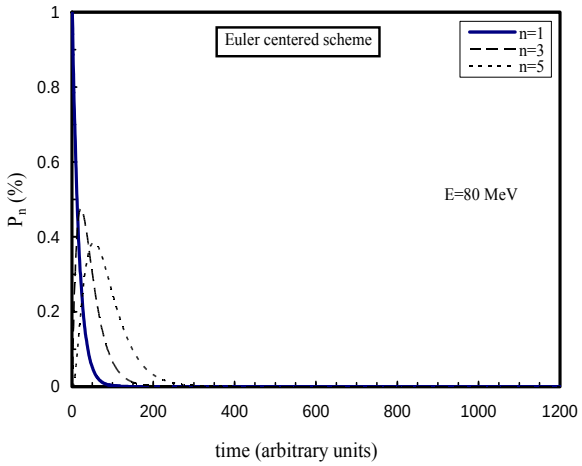

Figure (2-B): The same as Fig.(2-A) at incident energy 80 MeV.

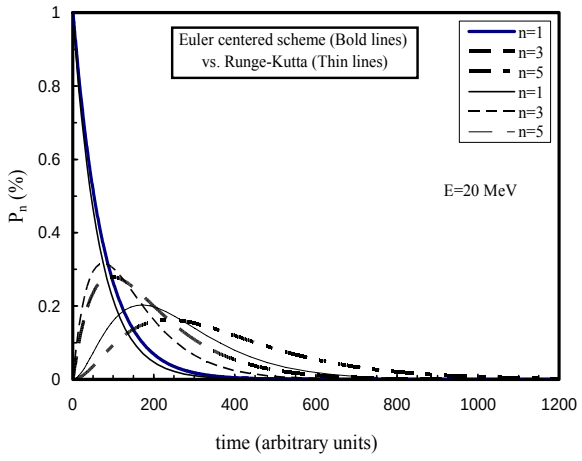

Figure (3-A): The solution of the ME for ($p+^{54}Fe$) reaction at incident energy 20 MeV using both Euler method (bold lines) and Runge-Kutta method (thin lines).

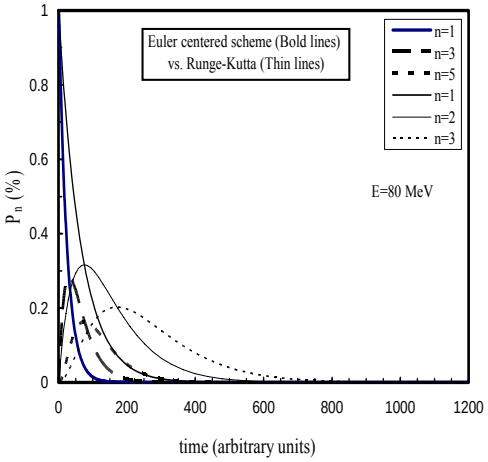

Figure (3-B): The same as Fig.(3-A) at incident energy 80 MeV.

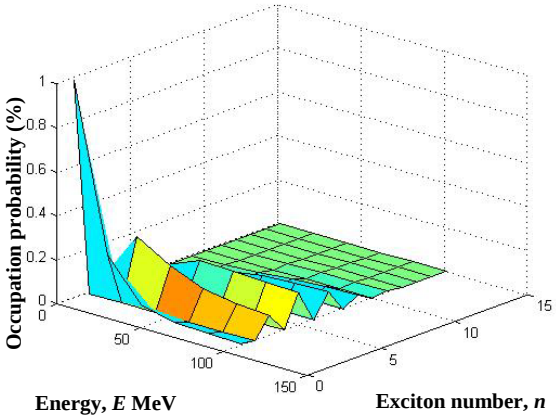

Figure (4-A): The dependence of P on *E* at energy time 100 (arbitrary).

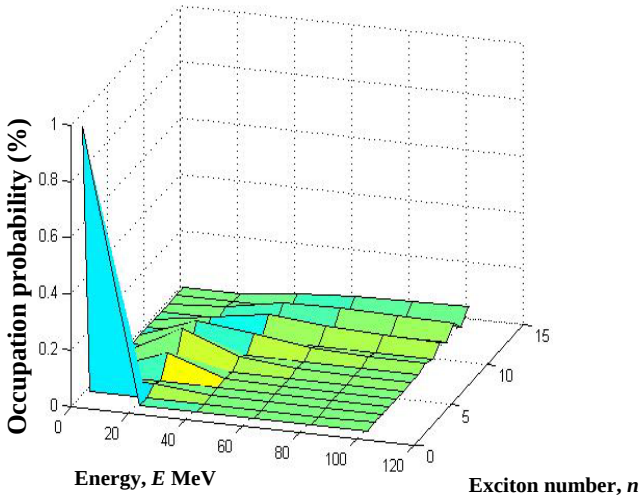

Figure (4-B): The dependence of P on *E* at energy time 500 (arbitrary).

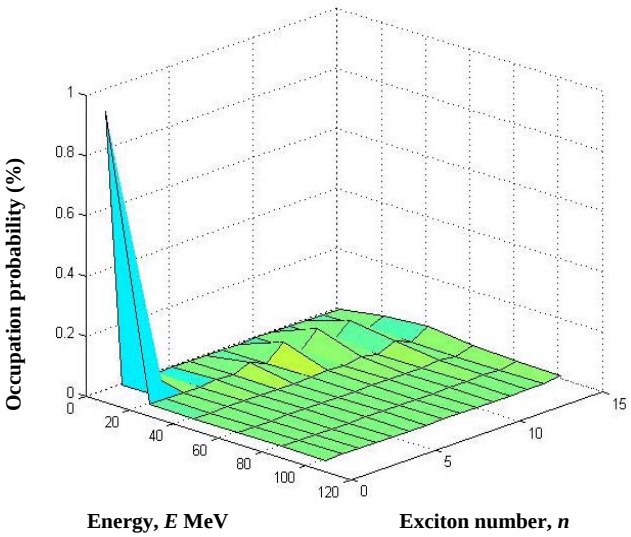

Figure (4-C): The dependence of P on *E* at energy time 1000 (arbitrary).

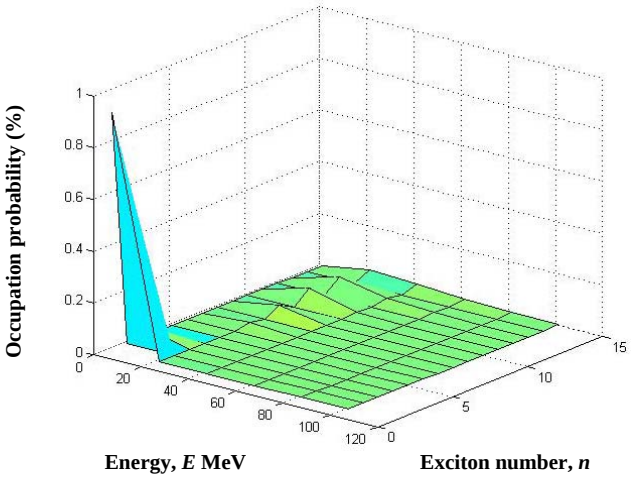

Figure (4-D): The dependence of P on *E* at energy time 1200 (arbitrary).

Figure 6 is the same as Figure 5 but in this case we fixed time and varied the energy. Also Figure 6 is meant to be in a semi-log in order to compare the results of the present work with the earlier work. To compare the results of the present method, we refer to Figure 1 of Ref.[2], Figure 2 of Ref.[16] and Figures 2 and 3 of Ref.[20]. The behavior of the present method, Figure 5 above and Figure 6, is consistent in general with the references above. These figures indicate and almost summarize the entire performance of the present method.

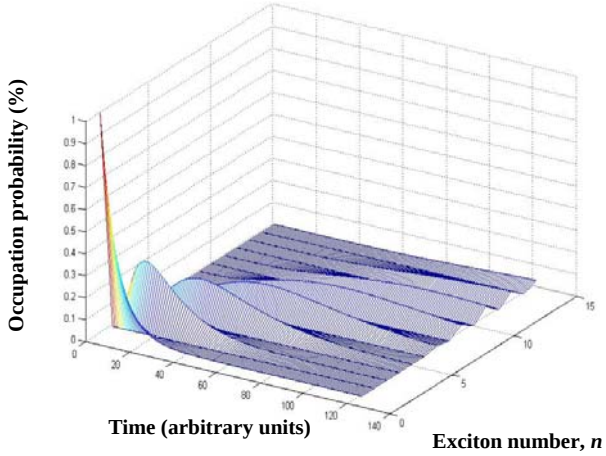

Figure (5): The dependence of P on *n* and *t* at energy 20 MeV.

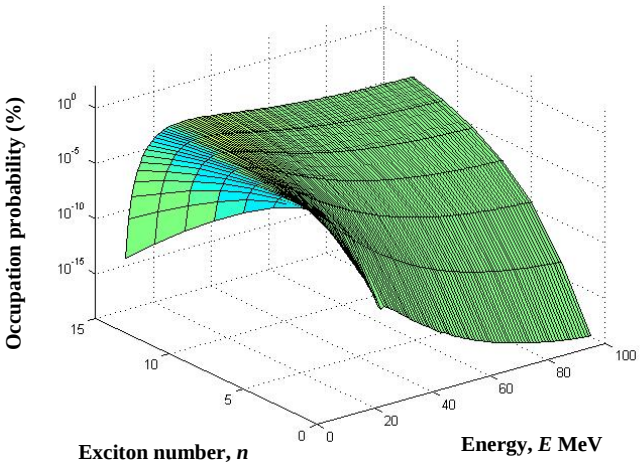

Figure (6): The dependence of P on *n* and E.

V. Conclusions

The important conclusions can be summarized as follows:

(a) the master equation can be solved numerically using simple version of Euler scheme, but can only be applied at low energies,

(b) other complicated approaches such as Runge-Kutta method, or any of the numerical methods described earlier in, is needed if the problem extends to high reaction energies,

(c) the occupation probabilities behave as slow functions of time at low energies, and tend to drop faster with more dependence on time at high energies,

(d) as the exciton number increases above unity, the peaks of the occupation probabilities are shifted toward longer lifetimes, whereas at n=1 the occupation probability behaves almost in exponentially decreasing manner.

(e) at large exciton number the dependence of the occupation probability on time and energy becomes insignificant which suggests that there is a maximum exciton number possible to be accessed by the system before statistical and thermal nuclear equilibrium.

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