ON ORTHOGONAL REVERSE DERIVATIONS OF SEMIPRIME RINGS

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Abstract

In this paper some results concerning two reverse derivations on semiprime rings are presented. These results are related to a result which is inspired by the classical result of E. Posner. This result is asserts that if R is a 2- torsion free semiprime ring, f and h are non-zero reverse derivations of R. Then $f h$ can not be a non-zero derivation. A notion of orthogonal reverse derivations arises here.
Key word and phrases: prime ring, semi-prime ring, derivation, reverse derivation, orthogonal reverse derivation.

حول تعامد المشتقات المتعاكسة في الحلقات شبه الاولية

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الخلاصة

قدمنا في هذا البحث بعض ا لنتائج حول المشتقات المتعاكسة على الحلقات شبه الاولية.هذه النتائج لها علاقة بنتيجة بوسنر المعروفة. احدى هذه النتائج تتضمن,اذاكانت R حلقةشبه اولية طليقة الالتواء من النمط 2 f و h مشتقات متعاكسة علىR . فان f h لايمكن ان تكون مشتقة غير صفرية.استخدمنا هنا مفهوم تعامد المشتقات.

1.Introduction

Throughout R will represent an associative ring. R is said to be 2 - torsion free if $2x = 0$, $x \in R$ implies $x = 0$. Recall that R is prime if $x R y = 0$ implies $x = 0$ or $y = 0$, and R is semiprime if $x R x = 0$ implies $x = 0$. An additive mapping $f : R \to R$ is called a derivation if

$f(xy)=f(x)\, y + xf(y)$ holds for all $x, y \in R$.

Breser and Vukman [1] have introduced the notion of a reverse derivation as, an additive mapping $f : R \to R$ satisfying

$f(xy) = f(y)\, x + y\, f(x)$ holds for all $x, y \in R$.

Other properties of derivations and reverse derivations can be found in ([2], [3], [4],[5],[6]and[7])

Two additive mapping $f, h: R \to R$ is said to be orthogonal if

$f(x)\, R\, h(y)=0=h(y)\, R\, f(x)$ for all $x, y \in R$.

In [8] Brešar and Vukman introduced the notion of orthogonality for two derivations f and h on a semiprime ring, and they presented several necessary and sufficient conditions for f and h to be orthogonal and they gave the related result to a classical result of E. Posner [9],which state that, if R is prime ring of characteristic not 2,and f, h are non-zero derivations of R, then fh can't be a derivation. In [10] Argaç Nakajima and

Albaş introduced orthogonal generalized derivations on a semiprime ring and they presented some results concerning two generalized derivations on a semiprime ring. Their results are a generalization of results of M. Brešar and J. Vukman in [8]. And in [11] Gölbaşi and Aydin, introduced the notion of orthogonal (σ, τ) – derivations and orthogonal generalized (σ, τ) – derivations. Their results abstracted some results of M. Brešar and J. Vukman[8].

In this paper, our aim is to introduce the notion of orthogonal for two reverse derivations *f* and *h* on a semiprime ring, and we presented several necessary and sufficient conditions for *f* and *h* to be orthogonal .Also we will give the same results of M. Brešar and J. Vukman [8] to orthogonal reverse derivations. We will show that if *R* is a 2 – torsion free semi-prime ring and *f*, *h* be reverse derivations of *R*. Then if *f* and *h* are orthogonal reverse derivations of *R*, then there exists an essential ideal *K* of *R* (i.e. $K \cap N \neq 0$ for every ideal *N* of *R*), such that the restrictions of *f* and *h* to *N* are appropriate direct sums.

For a semiprime ring *R* and an ideal *U* of *R*, it is well- known that the left and right annihilators of *U* in *R* coincide.We denote the annihilator of *U* by Ann (*U*). Note that $U \cap$ Ann $(U) = 0$ and $U \oplus$ Ann (U) is an essential ideal of *R* .

2. Orthogonal reverse derivations

Now we present the definition of orthogonal reverse derivations as follows

Definition . Two reverse derivations *f* and *h* of *R* are called orthogonal if

$$f(x)\, R\, h(y) = 0 = h(y)\, R\, f(x) \quad \text{for all } x, y \in R. \qquad (1)$$

It is obvious that a non-zero reverse derivation can not be orthogonal on itself.

Let us consider an example of the non-zero orthogonal reverse derivations. Let R_1 and R_2 are prime rings, set $R = R_1 \oplus R_2$ .Then *R* is semiprime ring .Let d_1 be a non-zero reverse derivation of R_1 .A mapping $d:R \to R$ defined by $d((r_1, r_2)) = (d_1(r_1), 0)$ is a nonzero reverse derivation of *R*. We write *d* as $d_1 \oplus 0$. Similarly, let g_2 be a nonzero reverse derivation of R_2 and define $g:R \to R$ by $g(r_1, r_2) = (0, g_2(r_2))$,thus $g = 0_1 \oplus g_2$. Then *d* and *g* are orthogonal.

3. The Results

The main goal of this section is to prove the following theorem, which corresponds to ([8], Theorem 1).

Theorem 1. Let *R* be a 2-torsion free semiprime ring. Let *f* and *h* be reverse derivations of *R*. Then *f* and *h* are orthogonal if and only if one of the following conditions holds

(*i*) $fh = 0$.

(*ii*) $hf = 0$.

(*iii*) $fh + hf = 0$.

(*iv*) fh is a derivation.

(*v*) hf is a derivation.

(*vi*) $f(x)\, h(x) = 0$, for all $x \in R$.

(*vii*) $f(x)\, h(x) + h(x) f(x) = 0$,for all $x \in R$.

(*viii*) There exist ideals K_1 and K_2 of *R* such that:

(a) $K_1 \cap K_2 = 0$ and $K = K_1 \oplus K_2$ is an essential ideal of *R*.

(b) *f* maps *R* into K_1 and *h* maps *R* into K_2.

(c) The restriction of *f* to $K = K_1 \oplus K_2$ is a direct sum $f_1 \oplus 0_2$, where $f_1: K_1 \to K_1$ is a reverse derivation of K_1 and $0_2 : K_2 \to K_2$ is zero. If $f_1 = 0$ then $f = 0$.

(d) The restriction of *h* to $K = K_1 \oplus K_2$ is a direct sum $0_1 \oplus h_2$, where $0_1: K_1 \to K_1$ is zero and $h_2: K_2 \to K_2$ is a reverse derivation of K_2. If $h_2 = 0$ then $h = 0$.

For the proof of the Theorem 1 we need the following lemmas:

Lemma 1. ([8], Lemma 1). Let *R* be a 2-torsion free semiprime ring and *a*, *b* the elements of *R*. Then the following conditions are equivalent:

(*i*) $axb = 0$, for all $x \in R$.

(*ii*) $bxa = 0$, for all $x \in R$.

(*iii*) $axb + bxa = 0$, for all $x \in R$.

If one of these conditions is fulfilled then $ab = ba = 0$.

Lemma 2. ([8], Lemma 2). Let *R* be a semiprime ring. And suppose that additive mappings *f* and *h* of *R* into itself satisfy $f(x)\, R\, h(x) = 0$, for all $x \in R$. Then $f(x)\, R\, h(y) = 0$, for all $x, y \in R$.

Lemma 3. Let *R* be a 2-torsion free semiprime ring. Let *f* and *h* be reverse derivations of *R*. Then *f* and *h* are orthogonal if and only if $f(x)\, h(y) + h(x) f(y) = 0$, for all $x, y \in R$.

Proof. Suppose that

$f(x)\, h(y) + h(x) f(y) = 0$,

for all $x, y \in R$. (2)
Take $y = xy$ in (2). Then we obtain
$0 = f(x)\, h(xy) + h(x) f(xy)$
$0 = \{ f(x)\, h(y) + h(x) f(y)\}\, x + f(x)\, y\, h(x) + h(x)\, y f(x)$.
By (2) we get

$0 = f(x)\, y\, h(x) + h(x)\, y f(x)$.

And by Lemma 1, we get
$f(x)\, y\, h(x) = 0$, for all $x, y \in R$.

Hence
$f(x)\, R\, h(x) = 0$, for all $x \in R$.

By Lemma 2 we see that f and h are orthogonal.
Conversely, if f and h are orthogonal then by Lemma1, we get
$f(x)\, h(y) = h(x) f(y) = 0$, for all $x, y \in R$.

Thus

$f(x)\, h(y) + h(x) f(y) = 0$, for all $x, y \in R$.

Let f and h be reverse derivations of any ring R. By a direct computation, we verify the following identities:
$(f h)(x y) = (f h)(x)\, y + f(x)\, h(y) + h(x)\, f(y) + x\,(f h)(y)$ for all $x, y \in R$. (3)

We now have enough information to prove Theorem 1.

Proof of Theorem 1. (i) ⇒"f and h are orthogonal". Suppose that $f h = 0$. According to (3), we have
$f(x)\, h(y) + h(x) f(y) = 0$, for all $x, y \in R$.
Hence f and h are orthogonal by Lemma 3.

"f and h are orthogonal" ⇒(i). We have $f(x)\, y\, h(z) = 0$, for all $x, y, z \in R$.
Hence
$0 = f(f(x)\, y\, h(z))$
$= f(y\, h(z)) f(x) + (y\, h(z)) f^2(x)$
$= f h(z) y f(x)) + h(z) f(y) f(x) + y\, h(z)^2(x)$.
The second and third summands are zero since f and h are orthogonal.
Therefore this relation is reduced to
$(f h)(z)\, y\, f(x) = 0$
where x, y, z are arbitrary elements in R, so take $x=h(z)$ in the above relation, we get
$(f h)(z)\, R\, (f h)(z) = 0$, for all $z \in R$.
Since R is semiprime, we get
$(f h)(z) = 0$, for all $z \in R$.

(ii) similar way used in the proof of (i).

(iii) ⇒" f and h are orthogonal". If f and h are any reverse derivations, then we have by (3) and (4) that
$(f h + h f)(x y) = (f h)(x)\, y + f(x)\, h(y) + h(x) f(y) + x (f h)(y) (h f)(x)\, y + f(x)\, h(y) + h(x) f(y) + x\,(h f)(y)$
$= (f h + hf)(x)\, y + 2 f(x)\, h(y) + 2\, h(x) f(y) + x\,(f h + hf)(y)$.
Thus, if $f h + h f = 0$ then the above relation reduces to
$2(f(x)\, h(y) + h(x) f(y)) = 0$, for all $x, y \in R$.
Since R is 2–torsion free, we get
$f(x)\, h(y) + h(x) f(y) = 0$, for all $x, y \in R$.
By Lemma 3, we get f and h are orthogonal.

" f and h are orthogonal" ⇒(iii). From (i) and (ii), Theorem 1, we get $f h + h f = 0$.

(iv) ⇒"f and h are orthogonal". Since $f h$ is a derivation we have
$(f h)(x y) = (f h)(x)\, y + x\,(f h)(y)$.
Comparing this expression with (3), we obtain
$f(x)\, h(y) + h(x) f(y) = 0$
Now apply Lemma 3.

(i) ⇒(iv). Clear.

(v) ⇒"f and h are orthogonal". Similar way use in the proof of (v).

(ii) ⇒(v). Clear.

(vi) ⇒"f and h are orthogonal". A linearization of $f(x)\, h(x) = 0$ gives
$f(x)\, h(y) + f(y)\, h(x) = 0$,
for all $x, y \in R$. (5)
Take $y = y z$ in (5), we obtain
$f(x)\, h(z)\, y + f(x)\, z\, h(y) + f(z)\, y\, h(x) + z f(y)\, h(x) = 0$, for all $x, y, z \in R$.
By (5), $f(x)\, h(z) = - f(z)\, h(x)$ and $f(y)\, h(x) = - f(x)\, h(y)$ and so the above relation becomes
$- f(z)\, h(x)\, y + f(x)\, z\, h(y) + f(z)\, y\, h(x) - z f(x)\, h(y) = 0$, for all $x, y, z \in R$.
Hence we have
$f(z)\, [y , h(x)] + [f(x) , z]\, h(y) = 0$, Where $[u , v]$ denotes the commutator $uv - vu$.
Replacing z by $f(x)$ in the above relation, we obtain
$f^2(x)\, [y, h(x)] = 0$, for all $x, y \in R$.
Letting $y = y w$ in the last relation results in
$0 = f^2(x)\, [y w , h(x)]$
$= f^2(x)\, y [w , h(x)] + f^2(x)\, [y , h(x)]\, w$
$= f^2(x)\, z\, [w , h(x)]$.
Hence from Lemma 2 we obtain that

$f^2(x) R [w, h(y)] = 0,$ for all $x, y, w \in R$ (6)

Replacing x by $x\,u$ in (6) and using (3) yields that

$(f^2(x)\,u + 2 f(x) f(u) + x\, f^2(u))\, R\, [w, h(y)] = 0.$

By (6) the above relation reduces to

$2\, f(x) f(u)\, R\, [w, h(y)] = 0.$

Since R is 2–torsion free, we have

$f(x) f(u)\, R\, [w, h(y)] = 0,$ for all $x \in R$. (7)

Taking $x = x\,z$ in (7), we get

$f(z)\, x\, f(u)\, R\, [w, h(y)] + z\, f(x) f(u)\, R\, [w, h(y)] = 0,$

and by using (7), we get

$f(z)\, x\, f(u)\, R\, [w, h(y)] = 0.$

In particular,

$f(x)\, R\, [w, h(y)]\, R\, f(x)\, R[w, h(y)] = 0$

since R is semiprime,which implies

$f(x)\, R\, [w, h(y)] = 0.$

But then also

$[f(x),h(y)]\, R\, [f(x),h(y)] = 0$, for all $x, y \in R$.

Hence

$f(x)\, h(y) = h(y) f(x)$, for all $x, y \in R$.

Thus (5) can be written in the form

$h(y) f(x) + f(y)\, h(x) = 0$, for all $x, y \in R$.

Now use Lemma 3.

" f and h are orthogonal " $\Rightarrow$(*vi*). If f and h are orthogonal then we have

$f(x)\, R\, h(x) = 0$, for all $x \in R$.

Then by Lemma 1, we get

$f(x)\, h(x) = 0$, for all $x \in R$.

(*vii*) $\Rightarrow$(*iv*). Take $y = x$ in (3). Then we see that

$(f h)(x^2) = (f h)(x)\, x + f(x)\, h(x) + h(x)\, f(x) + x\, (f h)(x).$

Thus we have

$(f h)(x^2) = (f h)(x)x + x(f h)(x)$, for all $x \in R$.

The above relation implies that $f\,h$ is a Jordan derivation. Then $f\,h$ is a derivation by [[2], Theorem 1].

" f and h are orthogonal " $\Rightarrow$(*vii*). This follows immediately from Lemma 3.

(*viii*) $\Rightarrow$" f and h are orthogonal ". Clear.

" f and h are orthogonal " $\Rightarrow$(*viii*). Let K_1 be an ideal of R generated by all $f(x)$, $x \in R$, and let K_2 be $Ann(K_1)$, the annihilator of K_1. From (1), we see that $h(x) \in K_2$, for all $x \in R$. Whenever K_1 is an ideal in a semiprime ring we have $K_1 \cap K_2 = 0$ and $K = K_1 \oplus K_2$ is an essential ideal. Thus (a) and (b) are proved.

Our next goal is to show that f is zero on K_2. Take $k_2 \in K_2$. Then $k_1\, k_2 = 0$, for all $k_1 \in K_1$. Hence

$0 = f(k_1\, k_2) = f(k_2)\, k_1 + k_2\, f(k_1).$

It is obvious from the definition of K that f leaves K_1 invariant and , hence $k_2\, f(k_1) = 0$. Then the above relation reduces to $f(k_2)\, k_1 = 0$. Since in a semiprime ring the left and right and two–sided annihilators of an ideal coincide, we then have $f(k_2) \in Ann(K_1) = K_2$. But on the other hand $f(k_2)$ belongs to the set of generating elements of K_1. Thus $f(k_2) \in K_1 \cap K_2 = 0$, which means that f is zero on K_2 .

As we have mentioned above f leaves K_1 invariant. Therefore we may define a mapping $f_1 : K_1 \rightarrow K_1$ as a restriction of f to K_1.

Suppose that $f_1 = 0$. Then f is zero on $K = K_1 \oplus K_2$.

Take $k \in K$ and $y \in R$, we have

$f(y\,k) = f(k)\, y + k\, f(y)$

But $f(y\,k) = f(k) = 0$ since $k\,y, k \in K$.

Consequently $k\, f(y) = 0$, for all $y \in R$. Thus $f(y) \in Ann(K)$. But ideal K is essential and therefore $Ann(K) = 0$ by . Hence $f(y) = 0$, for all $y \in R$.

Then (c) is thereby proved.

It remains to prove (d). First we show that h is zero on K_1. Take $x, y, z \in R$ and set $k_1 = z\, f(y)\, x$. Then

$h(k_1) = h(x)\, (z\, f(y)) + x\, h(z\, f(y))$

$= h(x)\, z\, f(y) + x(h\, f)(y) z + x\, f(y)\, h(z).$

Since f and h are orthogonal we have $h(x)\, z\, f(y) = 0$, $f(y) h(z) = 0$ and $hf = 0$. Hence $h(k_1) = 0$. In a similar fashion we see that $h(z\, f(y)) = 0$, $h(f(y)\, x) = 0$ and $h(f(y)) = 0$. Then h is zero on K_1. Recall that h maps R into K_2. In particular, it leaves K_2 invariant. Thus we may define $h_2 : K_2 \rightarrow K_2$ as a restriction of h to K_2. The proof that $h_2 = 0$ implies $h = 0$ is the same as the proof that $f_1 = 0$ implies $f = 0$. The proof of the theorem is complete.

A well known result of E. Posner [9] states that if R is a prime ring of characteristic not equal 2, f and h are non-zero derivations of R, then $f\,h$ can not be a non-zero derivation.

The result which is inspired by a theorem of E. Posner, states that, if R is a 2–torsion free semiprime ring, f and h are non-zero reverse derivations of R. Then $f\,h$ can not be a non-zero derivation. One can consider (*iv*) and (*i*), Theorem 1 as a proof of this result.

We now state some consequences of Theorem 1.

Corollary 1. Let R be a prime ring of characteristic not equal 2. Let f and h be reverse derivations of R. If f and h are satisfy one of the conditions of Theorem 1, then either $f = 0$ or $h = 0$.

Since a non-zero reverse derivation can not be orthogonal on itself we see that (*iv*) of Theorem 1 yields the following result.

Corollary 2. Let R be a 2 – torsion free semiprime ring and let f be a reverse derivation of R. If f^2 is also a derivation, then $f = 0$.

Similarly, using (*vi*) of Theorem 1, we obtain

Corollary 3. Let R be a 2–torsion free semiprime ring and let f be a reverse derivation of R. If $f(x)^2 = 0$ for all $x \in R$, then $f = 0$.

It is natural to ask if there is any connection between reverse derivations f and h of a ring R. If $f^2 = h^2$ or if $f(x)^2 = h(x)^2$, for every $x \in R$.Theorem 1 enables the consideration of these problems.
In the following theorems, we answer this question.

Theorem 2. Let R be a 2-torsion free semiprime ring. Let f and h be reverse derivations of R. Suppose that $f^2 = h^2$, then $f + h$ and $f - h$ are orthogonal. Thus, there exist ideals K_1 and K_2 of R such that $K = K_1 \oplus K_2$ is an essential direct sum in R, $f = h$ on K_1 and $f = -h$ on K_2.

Proof . From $f^2 = h^2$ it follows immediately that

$$(f + h)(f - h) + (f - h)(f + h) = 0.$$

Hence $f + h$ and $f - h$ are orthogonal by (*iii*), Theorem 1. Another part of the Theorem 2, follows from (*viii*),Theorem 1.

Corollary 4. Let R be a prime ring of characteristic not equal 2. Let f and h be derivations of R . If $f^2 = h^2$ then either $f = -h$ or $f = h$.

Theorem 3. Let R be a 2–torsion free semiprime ring. Let f and h be reverse derivations of R . If $f(x)^2 = h(x)^2$, for all $x \in R$, then $f + h$ and $f - h$ are orthogonal. Thus, there exist ideals K_1 and K_2 of R such that $K = K_1 \oplus K_2$ is an essential direct sum in R, $f = h$ on K_1 and $f = -h$ on K_2.

Proof . Note that $(f + h)(x)(f - h)(x) + (f - h)(x)(f + h)(x) = 0$, for all $x \in R$. Now apply (*vii*) and (*viii*), Theorem 1.

Corollary 5. Let R be a prime ring of characteristic not equal 2. Let f and h be reverse derivations of R . If $f(x)^2 = h(x)^2$, for all $x \in R$, then either $f = -h$ or $f = h$.

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