SOLUTION OF NEWTONIAN FLUID FLOW PROBLEM IN TWO DIMENSIONS SOLVED BY SIMPLE ALGORITHEM

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Abstract

In this paper, a consideration is given to viscose, incompressible and Newtonian fluid flowing in a pipe with square cross-section under the action of pressure gradient. In particular consideration is given to first order fluid flow which can be represented by the equation of state of the form:

$$T_{ij} = 2\eta\, e_{ij} \quad , \quad i,j = 1,2$$

Where η is constant of fluid, T_{ij} and e_{ij} are the stress and rate of strain respectively. Cartesian coordinate has been used to describe the fluid motion and it found that motion equations are controlled by Reynolds number. The motion equations are solved by a semi-implicit algorithm namely Semi-Implicit Method for Pressure Linked Equations (SIMPLE).

حل مسألة جريان مائع نيوتيني في المستوي بواسطة خوارزمية SIMPLE

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الخلاصه

يقدم هذه البحث دراسة جريان لمائع نيوتوني ,لزج, غير قابل للأنضغاط في مقطع عرضي مربع تحت تأثير الضغط. وبشكل خاص إعتبر المائع من الرتبة الاولى الذي يمكن أَن يمثّلَ بمعادلة حالة من النوع:

$$T_{ij} = 2\eta\, e_{ij} \quad , \quad i,j = 1,2$$

حيث η هو ثابت للمائع و T_{ij} و e_{ij} هما مركبات الاجهاد و مركبات معدل المرونة على التوالي. نظام الاحداثيات المتعامدة تم استخدامه لوصف حركة المائع وقد وجد أن معادلات الحركة مسيطر عليها من قبل معلمة وهي رقم رينولدز. إن معادلاتَ الحركةَ محلُولة بطريقة SIMPLE حيث أنها طريقة شبة ضمنية.

Introduction

A fluid is that state of matter which is capable to changing is shape and is capable of flowing. Both gases and liquids are classified as fluid, each fluid characterized by an equation that relates stress to rate of strain, known as "State Equation ".And the number of fluids engineering applications is enormous: breathing, blood flow, swimming, pumps, fans, turbines, airplanes, ships, pipes... etc. When you think about it, almost every thing on this planet either is a fluid or moves with respect to a fluid. Fluid mechanics is considered a branch of applied mathematics which deals with behavior of fluids either in motion (fluid dynamics) or at rest (fluid statics).

The first one who worked in the flow analysis of Newtonian fluids in curved pipes is Dean, (1927) [1]. He introduced a toroidal coordinate system to show that the relation between pressure gradient and the rate of flow through a curved pipe with circular cross-section of incompressible Newtonian fluid is dependent on the curvature. But he couldn't show this dependence and he will show in second paper (1928) [2].In his paper Dean modified his analysis by including higher order terms to be able to show that the rate of flow is slightly reduced by curvature.

Jones in (1960) makes a theoretical analysis of the flow of incompressible non-Newtonian viscose liquid in curved pipes with circular cross- section keeping only the first order terms. He shows that the secondary motion consists of two symmetrical vortices and the distance of the stream lines from the central plan decreases as the Non-Newtonian parameters increase, [3].

In (1961) Kawaguti [4] studied the flow of viscous fluid in a two-dimensional rectangular cavity. He assumed that the cavity is bounded by three rigid plan walls, and by a flat plate moving in its own plan. The Reynolds number of the flow is varied as 0 , 1, 2 ,4 , 8 , 16 , 32 , 62 , 128 and he find that in every case , there exits a circulation flow extending the whole length of the cavity, also he observed that no secondary flow seems to occur in the shallow cavities when the Reynolds number less than 64.

Greenspan, D. (1968) [5] used a new numerical method which is developed for the Navier-Stokes equation. Finite differences smoothing and a special boundary technique are fundamental. And this method is converges for all Reynolds numbers, he shows that the resulting stream curves exhibit only primary vortices.

Ali M. M. (2005) [6] concerned with the study of unsteady flow of Non-Newtonian, viscose, incompressible fluid in a curved pipe with rectangular cross-section, under the action of pressure gradient. He used variational method namely, Galerkin method after eliminating the dependence term on time.

Mathematical Formulation

Unsteady flow of fluid in the xy- plane is considered. The Newtonian fluid is characterized by equation of state of the form:

$T_{ij} = 2\eta\, e_{ij}$ i, j=1,2 .... (1)

Where T_{ij} , e_{ij} and η are stress , rate of strain and viscosity coefficient respectively.

The coordinate systems in the cross-section are related to coordinates (x, y) by the equations:

$X = \boldsymbol{x}$, $Y = \boldsymbol{y}$ ………………………………...... (2)

And the line element is

$(ds)^2 = (dx)^2 + (dy)^2$ ……………………….......(3)

To drive the line element, let us denoted these by y^i to distinguish them from the general curvilinear coordinate x^i .where y^1,y^2,x^1 and x^2 is X,Y**, *x*** and ***y*** respectively .

The distance between two points P and Q with coordinate y^i and y^i+dy^i is ds where

$$(ds)^2 = \sum_{k=1}^{2} dy^k dy^k \qquad \text{………………. (4)}$$

However $dy^k = \dfrac{\partial y^k}{\partial x^i} dx^i$

Hence we get

$$(ds)^2 = \sum_{k=1}^{2}\left(\frac{\partial y^k}{\partial x^i}dx^i\right)\left(\frac{\partial y^k}{\partial x^j}dx^j\right) \quad \text{Where}$$

$$= g_{ij}\, dx^i dx^j$$

Where

$$g_{ij} = \sum_{k=1}^{2}\left(\frac{\partial y^k}{\partial x^i}\right)\left(\frac{\partial y^k}{\partial x^j}\right)$$

And g_{ij} is called the metric tensor. Since it relates distance to the infinitesimal coordinate increment. Where only the diagonal terms are nonzero i.e. (g_{ii}) the coordinate system are orthogonal. Then

$(ds)^2 = g_{11}(d\boldsymbol{x}^1)^2 + g_{22}(d\boldsymbol{x}^2)^2$ …………………. (5)

We can compute g_{11},g_{22} which is $g_{11}=1$,$g_{22}=1$, if we put these value in eq. (5) we get (3).

Since any line element ds in any curvilinear coordinates may be written in the form:

$$(ds)^2 = h_1^2 (dx^1)^2 + h_2^2 (dx^2)^2 \text{ ………….. (6)}$$

Where h_i are called scale factor. The comparing equation (6) with (3) gives us that; $h_1=h_2=1$.

The Motion Equations and Continuity Equation in Curvilinear Coordinates [7]

The motion equations for two dimensional flow in Cartesian coordinates can be written as:

$$\rho\left(\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x_1}+V\frac{\partial U}{\partial y_1}\right)= -\frac{\partial P^*}{\partial x_1}+\frac{\partial T_{x_1x_1}}{\partial x_1}+\frac{\partial T_{y_1x_1}}{\partial y_1} \quad \ldots(7)$$

$$\rho\left(\frac{\partial V}{\partial t}+U\frac{\partial V}{\partial x_1}+V\frac{\partial V}{\partial y_1}\right)= -\frac{\partial P^*}{\partial y_1}+\frac{\partial T_{y_1y_1}}{\partial y_1}+\frac{\partial T_{x_1y_1}}{\partial x_1} \quad \ldots(8)$$

$$\frac{\partial U}{\partial x_1}+\frac{\partial V}{\partial y_1}=0 \quad \ldots.(9)$$

In the above equations, we assume that the fluid is incompressible (ρ = constant).

Non-dimensional Form of Motion and Continuity Equations

We can write down the motion and continuity equation (7)-(9) in non-dimensional form through using scaling and order-of-magnitude analysis. See [8]

This can be done through introducing the following new quantities;

$$x=\frac{x_1}{a}\ ,\ y=\frac{y_1}{a}\ ,\tau=\frac{V_0 t}{a},u=\frac{U}{V_0}\ ,\ v=\frac{V}{V_0}\ ,\mathrm{P}=\frac{P^*}{\rho V_0^2}$$

The substituting of these quantities into equations gives the motion and continuity equations in dimensionless form which are:

$$\frac{\partial u}{\partial \tau}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{\partial P}{\partial x}=\frac{1}{\mathrm{Re}}\left(\nabla^2 u\right) \quad \ldots(10)$$

$$\frac{\partial v}{\partial \tau}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\frac{\partial P}{\partial y}=\frac{1}{\mathrm{Re}}\left(\nabla^2 v\right) \quad \ldots(11)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad \ldots(12)$$

The above equations are controlled by a parameter namely the Reynold's Number $\mathrm{Re}=aV_0/\nu$, where ν is kinamtic viscosity.

SIMPLE Formulation and Discretization

On staggered grid, difference control volumes are used to difference equations. Thus the physical location of $P_{j+1/2,k}$ and $u_{j,k}$ are the same physical location of $P_{j,k+1/2}$ and $v_{j,k}$. For sake of simplification, we will use equidistance grid points.

In order to obtain discrete equation corresponding to the continuity equation (12) we will apply the FVM over the control volume.

The continuity equation (12) is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

The integration of the first term gives

$$\int_{\tau}^{\tau+\Delta\tau}\int_s^n\int_w^e\left(\frac{\partial u}{\partial x}\right)dx\,dy\,d\tau_1=(u)_e\Delta y\Delta\tau-(u)_w\Delta y\Delta\tau=(F_e-F_w)\Delta\tau \quad \ldots(13)$$

Where $(u)^e_w = u_e - u_w$ and $F_e = u_e\,\Delta y$, $F_w = u_w\,\Delta y$, similarly , for the second term

$$\int_{\tau}^{\tau+\Delta\tau}\int_w^e\int_s^n\left(\frac{\partial v}{\partial y}\right)dy\,dx\,d\tau_1=(v)_n\Delta x\Delta\tau-(v)_s\Delta x\Delta\tau=(F_n-F_s)\Delta\tau \quad \ldots(14)$$

Where $(v)^n_s = v_n - v_s$ and $F_n = v_n\,\Delta x$, $F_s = v_s\,\Delta x$.

Substitute in to the continuity equation we obtain:-

$$(F_e - F_w)+(F_n - F_s) = 0 \quad \ldots\ldots.(15)$$

the last equation can be written as;

$$(u_e - u_w)\,\Delta y + (v_n - v_s)\,\Delta x = 0 \quad \ldots\ldots(16)$$

from the Fig (11) we have $u_e = u_{j,k}$, $u_w = u_{j-1,k}$, $v_n = v_{j,k}$ and $v_s = v_{j,k-1}$

Then the equation (16) becomes

$$\left(u_{j,k}^{n+1}-u_{j-1,k}^{n+1}\right)\Delta y+\left(v_{j,k}^{n+1}-v_{j,k-1}^{n+1}\right)\Delta x=0 \quad \ldots\ldots(17)$$

The a application of the FVM to the *x*-momentum equation (13), using the control volume leads to the following discrete equation

$$\left(\frac{\Delta x\Delta y}{\Delta\tau}\right)\left(u_{j,k}^{n+1}-u_{j,k}^{n}\right)+\left(F_{j+1/2,k}^{(1)}-F_{j-1/2,k}^{(1)}\right)\Delta y+\left(G_{j,k+1/2}^{(1)}-G_{j,k-1/2}^{(1)}\right)\Delta x+\left(P_{j+1,k}^{n+1}-P_{j,k}^{n+1}\right)\Delta y=0 \quad \ldots(18)$$

The last equation can be obtained by applying the finite volume method when we choose

$$\bar{q}=u,\ \bar{F}=u^2+P-\frac{1}{\mathrm{Re}}\frac{\partial u}{\partial x} \text{ and } \bar{G}=uv-\frac{1}{\mathrm{Re}}\frac{\partial u}{\partial y}$$

We will obtain the above discretization (18) where;

$$F^{(1)}=u^2-\frac{1}{\mathrm{Re}}\frac{\partial u}{\partial x} \quad \ldots(19)$$

$$G^{(1)}=uv-\frac{1}{\mathrm{Re}}\frac{\partial u}{\partial y} \quad \ldots(20)$$

$\bar{G}=G^{(1)}$ and $\bar{F}=F^{(1)}+P$ Then the discretization form corresponding to(19)and (20) are

$$F^{(1)}_{j+1/2,k}=0.25\left(u^n_{j,k}+u^n_{j+1,k}\right)\left(u^{n+1}_{j,k}+u^{n+1}_{j+1,k}\right)-\frac{1}{\text{Re}}\frac{u^{n+1}_{j+1,k}-u^{n+1}_{j,k}}{\Delta x} \quad ..(21)$$

$$F^{(1)}_{j-1/2,k}=0.25\left(u^n_{j-1,k}+u^n_{j,k}\right)\left(u^{n+1}_{j-1,k}+u^{n+1}_{j,k}\right)-\frac{1}{\text{Re}}\frac{u^{n+1}_{j,k}-u^{n+1}_{j-1,k}}{\Delta x} \quad ...(22)$$

$$G^{(1)}_{j,k+1/2}=0.25\left(v^n_{j,k}+v^n_{j-1,k}\right)\left(u^{n+1}_{j,k}+u^{n+1}_{j,k+1}\right)-\frac{1}{\text{Re}}\frac{u^{n+1}_{j,k+1}-u^{n+1}_{j,k}}{\Delta y} \quad ...(23)$$

$$G^{(1)}_{j,k-1/2}=0.25\left(v^n_{j,k-1}+v^n_{j-1,k-1}\right)\left(u^{n+1}_{j,k-1}+u^{n+1}_{j,k}\right)-\frac{1}{\text{Re}}\frac{u^{n+1}_{j,k}-u^{n+1}_{j,k-1}}{\Delta y} \quad ....(24)$$

The substitution of (21)-(24) into (18) and rearranging the terms, leads to the discrete equation in the *x*- direction, this is given by

$$\frac{\Delta x\Delta y}{\Delta\tau}u^{n+1}_{j,k}+b^u+\left[\left(0.25\left(u^{n+1}_{j+1,k}-u^{n+1}_{j-1,k}\right)\Delta y+\frac{2\Delta y}{\text{Re}\Delta x}\right)u^{n+1}_{j,k}+a^u_{j+1,k}u^{n+1}_{j+1,k}+a^u_{j-1,k}u^{n+1}_{j-1,k}\right]+\left[\left(0.25\left(-v^n_{j,k-1}-v^n_{j+1,k-1}\right)\Delta x+\frac{2\Delta x}{\text{Re}\Delta y}\right)u^{n+1}_{j,k}+a^u_{j,k+1}u^{n+1}_{j,k+1}+a^u_{j,k-1}u^{n+1}_{j,k-1}\right]+\left(P^{n+1}_{j+1,k}-P^{n+1}_{j,k}\right)\Delta y=0 \quad ...(25)$$

Equation (25) can be rearranged as;

$$\left(\frac{\Delta x\Delta y}{\Delta\tau}+a^u_{j,k}\right)u^{n+1}_{j,k}+b^u+a^u_{j+1,k}u^{n+1}_{j+1,k}+a^u_{j-1,k}u^{n+1}_{j-1,k}+a^u_{j,k+1}u^{n+1}_{j,k+1}+a^u_{j,k-1}u^{n+1}_{j,k-1}+\left(P^{n+1}_{j+1,k}-P^{n+1}_{j,k}\right)\Delta y=0 \quad ....(26)$$

Where

$$a^u_{j,k}=0.25\left(u^n_{j+1,k}-u^n_{j-1,k}\right)\Delta y+0.25\left(-v^n_{j,k-1}-v^n_{j+1,k-1}\right)\Delta x+\frac{2}{\text{Re}}\left(\frac{\Delta x}{\Delta y}+\frac{\Delta y}{\Delta x}\right)$$

$$a^u_{j+1,k}=0.25\left(u^n_{j+1,k}+u^n_{j,k}\right)\Delta y-\frac{1}{\text{Re}}\left(\frac{\Delta y}{\Delta x}\right)$$

$$a^u_{j-1,k}=-0.25\left(u^n_{j-1,k}+u^n_{j,k}\right)\Delta y-\frac{1}{\text{Re}}\left(\frac{\Delta y}{\Delta x}\right)$$

$$a^u_{j,k+1}=0.25\left(v^n_{j,k}+v^n_{j,k+1}\right)\Delta x-\frac{1}{\text{Re}}\left(\frac{\Delta x}{\Delta y}\right)$$

$$a^u_{j,k-1}=-0.25\left(v^n_{j,k}+v^n_{j,k-1}\right)\Delta x-\frac{1}{\text{Re}}\left(\frac{\Delta x}{\Delta y}\right)$$

and $b^u=-\frac{\Delta x\Delta y}{\Delta\tau}u^n_{j,k}$

The above equation may be written in more convent form by use of summation

$$\left(\frac{\Delta x\Delta y}{\Delta\tau}+a^u_{j,k}\right)u^{n+1}_{j,k}+\sum a^u_{nb}u^{n+1}_{nb}+b^u+\left(P^{n+1}_{j+1,k}-P^{n+1}_{j,k}\right)\Delta y=0 \quad ...(27)$$

Where $\sum a^u_{nb}u^{n+1}_{nb}$ denoted all the convection and diffusion contributions from neighboring nodes denoted by nb .The coefficients $a^u_{j,k}$ and a^u_{nb} depend on the grid sizes and the solution u, v at the n-th time level. It may be noted that some terms in $F^{(1)}$ and $G^{(1)}$ have been evaluated at the n-th time level to ensure that (27) is linear in u^{n+1} .

Using the FVM, the discredited form of the y-momentum equation (11) can be written when we choose;

$\bar{q}=v,\bar{F}=uv-\frac{1}{\text{Re}}\frac{\partial v}{\partial x}$ and $\bar{G}=v^2+P-\frac{1}{\text{Re}}\frac{\partial v}{\partial y}$ We have;

$$\left(\frac{\Delta x\Delta y}{\Delta\tau}\right)\left(v^{n+1}_{j,k}-v^n_{j,k}\right)+\left(F^{(2)}_{j+1/2,k}-F^{(2)}_{j-1/2,k}\right)\Delta y+\left(G^{(2)}_{j,k+1/2}-G^{(2)}_{j,k-1/2}\right)+\left(P^{n+1}_{j,k+1}-P^{n+1}_{j,k}\right)\Delta x=0 \quad ....(28)$$

Where

$$F^{(2)}=uv-\frac{1}{\text{Re}}\frac{\partial v}{\partial x}\quad,\quad G^{(2)}=v^2-\frac{1}{\text{Re}}\frac{\partial v}{\partial y}$$

thus $\bar{F}=F^{(1)}$ and $\bar{G}=G^{(1)}+P$

The discretization form for $F^{(2)}$ and $G^{(2)}$are:

$$F^{(2)}_{j+1/2,k}=0.25\left(v^{n+1}_{j,k}+v^{n+1}_{j+1,k}\right)\left(u^n_{j,k}+u^n_{j,k+1}\right)-\frac{1}{\text{Re}}\frac{v^{n+1}_{j+1,k}-v^{n+1}_{j,k}}{\Delta x} \quad ...(29)$$

$$F^{(2)}_{j-1/2,k}=0.25\left(v^{n+1}_{j-1,k}+v^{n+1}_{j,k}\right)\left(u^{n+1}_{j-1,k}+u^n_{j-1,k+1}\right)-\frac{1}{\text{Re}}\frac{v^{n+1}_{j,k}-v^{n+1}_{j-1,k}}{\Delta x} \quad ...(30)$$

$$G^{(2)}_{j,k+1/2}=0.25\left(v^{n+1}_{j,k}+v^{n+1}_{j,k+1}\right)\left(v^n_{j,k}+v^n_{j,k+1}\right)-\frac{1}{\text{Re}}\frac{v^{n+1}_{j,k+1}-v^{n+1}_{j,k}}{\Delta y} \quad ...(31)$$

$$G^{(2)}_{j,k-1/2}=0.25\left(v^{n+1}_{j,k-1}+v^{n+1}_{j,k}\right)\left(v^n_{j,k-1}+v^n_{j,k}\right)-\frac{1}{\text{Re}}\frac{v^{n+1}_{j,k}-v^{n+1}_{j,k-1}}{\Delta y} \quad ...(32)$$

The substitution of (29)-(32) into (28) we have;

$$\frac{\Delta x\Delta y}{\Delta\tau}v^{n+1}_{j,k}+b^v+\left[\left(0.25\left(-u^{n+1}_{j-1,k}-u^{n+1}_{j-1,k-1}\right)\Delta y+\frac{2\Delta y}{\text{Re}\Delta x}\right)v^{n+1}_{j,k}+a^v_{j+1,k}v^{n+1}_{j+1,k}+a^v_{j-1,k}v^{n+1}_{j-1,k}\right]+\left[\left(0.25\left(v^n_{j,k+1}-v^n_{j,k-1}\right)\Delta x+\frac{2\Delta x}{\text{Re}\Delta y}\right)v^{n+1}_{j,k}+a^v_{j,k+1}v^{n+1}_{j,k+1}+a^v_{j,k-1}v^{n+1}_{j,k-1}\right]+\left(P^{n+1}_{j,k+1}-P^{n+1}_{j,k}\right)\Delta x=0 \quad ...(33)$$

Equation (33) can be rearranged as

$$\left(\frac{\Delta x\Delta y}{\Delta\tau}+a^v_{j,k}\right)v^{n+1}_{j,k}+b^v+a^v_{j+1,k}v^{n+1}_{j+1,k}+a^v_{j-1,k}v^{n+1}_{j-1,k}+a^v_{j,k+1}v^{n+1}_{j,k+1}$$
$$+a^v_{j,k-1}v^{n+1}_{j,k-1}+\left(P^{n+1}_{j,k+1}-P^{n+1}_{j,k}\right)\Delta x=0$$

…………………………………………….. (34)

Where

$$a^v_{j,k}=0.25\left(-u^n_{j-1,k}-u^n_{j-1,k+1}\right)\Delta y+0.25\left(v^n_{j,k+1}-v^n_{j,k-1}\right)\Delta x+\frac{2}{\mathrm{Re}}\left(\frac{\Delta x}{\Delta y}+\frac{\Delta y}{\Delta x}\right)$$

$$a^v_{j+1,k}=0.25\left(u^n_{j,k}+u^n_{j,k+1}\right)\Delta y-\frac{1}{\mathrm{Re}}\left(\frac{\Delta y}{\Delta x}\right)$$

$$a^v_{j-1,k}=-0.25\left(u^n_{j-1,k}+u^n_{j-1,k+1}\right)\Delta y-\frac{1}{\mathrm{Re}}\left(\frac{\Delta y}{\Delta x}\right)$$

$$a^v_{j,k+1}=0.25\left(v^n_{j,k}+v^n_{j,k+1}\right)\Delta x-\frac{1}{\mathrm{Re}}\left(\frac{\Delta x}{\Delta y}\right)$$

$$a^v_{j,k-1}=-0.25\left(v^n_{j,k}+v^n_{j,k-1}\right)\Delta x-\frac{1}{\mathrm{Re}}\left(\frac{\Delta x}{\Delta y}\right)$$

and

$$b^v=-\frac{\Delta x\Delta y}{\Delta\tau}v^n_{j,k}$$

The above equation can be written in more convent form by use of summation

$$\left(\frac{\Delta x\Delta y}{\Delta\tau}+a^v_{j,k}\right)v^{n+1}_{j,k}+\sum a^v_{nb}v^{n+1}_{nb}+b^v \quad (35)$$
$$+\left(P^{n+1}_{j,k+1}-P^{n+1}_{j,k}\right)\Delta x=0$$

At any intermediate stage of the SIMPLE iterative procedure the solution is to advance from the (n)th time level to the (n+1)th. The velocity solution is advanced in two stage .First the momentum equation (27) and (35) are solved to obtain an approximation $u^*$and v^* of u^{n+1} and v^{n+1} that does not satisfy continuity, hence we must modify the pressure and velocities.

Modify Pressure and Velocity [10] [11]

The solution of equations (27) and (35) is an approximation solution $u^*$and v^* of u^{n+1} and v^{n+1} respectively. This velocity components (u^*, v^*) will not satisfy the continuity equation (17). Hence using the approximate velocity u^*, a pressure correction δP is sought which will both give $P^{n+1} = P^n + \delta P$ and provide a velocity correction u^c , such that $u^{n+1} = u^n + u^c$ where u^{n+1} satisfies the continuity in the form (17) and similarly for velocity v^*.

The equations (27) and (35) can be written as:

$$\left(\frac{\Delta x\Delta y}{\Delta\tau}+a^u_{j,k}\right)u^{n+1}_{j,k}+\sum a^u_{nb}u^{n+1}_{nb}=-b^u-\left(P^{n+1}_{j+1,k}-P^{n+1}_{j,k}\right)\Delta y$$

..(36)

$$\left(\frac{\Delta x\Delta y}{\Delta\tau}+a^v_{j,k}\right)v^{n+1}_{j,k}+\sum a^v_{nb}v^{n+1}_{nb}=-b^v-\left(P^{n+1}_{j,k+1}-P^{n+1}_{j,k}\right)\Delta x$$

... (37)

To obtain u^* and v^* equations (36) and (37) are approximated as

$$\left(\frac{\Delta x\Delta y}{\Delta\tau}+a^u_{j,k}\right)u^*_{j,k}+\sum a^u_{nb}u^*_{nb}=-b^u-\left(P^n_{j+1,k}-P^n_{j,k}\right)\Delta y$$

... (38)

$$\left(\frac{\Delta x\Delta y}{\Delta\tau}+a^v_{j,k}\right)v^*_{j,k}+\sum a^v_{nb}v^*_{nb}=-b^v-\left(P^n_{j,k+1}-P^n_{j,k}\right)\Delta x$$

... (39)

Subsequently equations (36) and (37) are written as scalar tridiagonal systems along each y grid line (j constant) and solved using the Thomas algorithm.

To obtain equation for subsequent velocity correction u^c equation (36) is subtracted from (27) to give

$$\left(\frac{\Delta x\Delta y}{\Delta\tau}+a^u_{j,k}\right)u^c_{j,k}=-\sum a^u_{nb}u^c_{nb}-\left(\delta P_{j+1,k}-\delta P_{j,k}\right)\Delta y$$

.... (40)

And equation (37) is subtracted from (35) to give a corresponding equation for v^c which is;

$$\left(\frac{\Delta x\Delta y}{\Delta\tau}+a^v_{j,k}\right)v^c_{j,k}=-\sum a^v_{nb}v^c_{nb}-\left(\delta P_{j,k+1}-\delta P_{j,k}\right)\Delta x$$

... (41)

However to make the link between u^c and δP and v^c and δP as $-\sum a^v_{nb}v^c_{nb}$, $-\sum a^u_{nb}u^c_{nb}$ explicit as possible we will drop the quantity for closer approximation.

The SIMPLE algorithm approximates equations (40) and (41) as

$$u^c_{j,k}=d^{(x)}_{j,k}\left(\delta P_{j,k}-\delta P_{j+1,k}\right) \quad ..... (42)$$

Where

$$E=\frac{\Delta\tau\, a^u_{j,k}}{\Delta x\Delta y} \quad \text{and} \quad d^{(x)}_{j,k}=\frac{E\Delta y}{\{(1+E)a^u_{j,k}\}}$$

An equivalent expression can be obtained to link $v_{j,k}^c$ with $(\delta P_{j,k} - \delta P_{j,k+1})$ which is

$$v^c_{j,k} = d^{(y)}_{j,k}\left(\delta P_{j,k} - \delta P_{j,k+1}\right) \quad \text{.......... (43)}$$

Where $d^{(y)}_{j,k} = \frac{E\Delta y}{\{(1+E)a^v_{j,k}\}}$ and

$$E = \frac{\Delta\tau\, a^v_{j,k}}{\Delta x\Delta y} \quad \text{.............................. (44)}$$

To obtain an equation that link δP with the velocity u^* and v^*, introducing the velocity correction in the form

$$u^{n+1}_{j,k} = u^*_{j,k} + u^c_{j,k} \quad \text{.......................... (45)}$$

And $v^{n+1}_{j,k} = v^*_{j,k} + v^c_{j,k}$ .... (46)

Substituting equations (45) and (46) into (17) we obtain;

$$\begin{aligned}&\Delta y\left(u^*_{j,k} + u^c_{j,k} - u^*_{j-1,k} - u^c_{j-1,k}\right) + \\ &\Delta x\left(v^*_{j,k} + v^c_{j,k} - v^*_{j,k-1} - v^c_{j,k-1}\right) = 0\end{aligned} \quad \text{.... (47)}$$

From (42) and (43) we have,

$$v^c_{j,k} = d^{(y)}_{j,k}\left(\delta P_{j,k} - \delta P_{j,k+1}\right)$$

$$u^c_{j,k} = d^{(x)}_{j,k}\left(\delta P_{j,k} - \delta P_{j+1,k}\right)$$

And as a consequence (47) becomes

$$\begin{aligned}&\Delta y\left\{d^{(x)}_{j,k}(\delta P_{j,k} - \delta P_{j+1,k}) - d^{(x)}_{j-1,k}(\delta P_{j-1,k} - \delta P_{j,k})\right\} \\ &+ \Delta x\left\{d^{(y)}_{j,k}(\delta P_{j,k} - \delta P_{j,k+1}) - d^{(y)}_{j,k-1}(\delta P_{j,k-1} - \delta P_{j,k})\right\} \\ &= -\Delta y\left(u^*_{j,k} - u^*_{j-1,k}\right) - \Delta x\left(v^*_{j,k} - v^*_{j,k-1}\right)\end{aligned}$$

. . . (48)

and $b^P = -\Delta y\left(u^*_{j,k} - u^*_{j-1,k}\right) - \Delta x\left(v^*_{j,k} - v^*_{j,k-1}\right)$

Let rearrange the terms in equation (48) we have

$$\begin{aligned}a^P_{j,k}\delta P_{j,k} &= a^P_{j+1,k}\delta P_{j+1,k} + a^P_{j-1,k}\delta P_{j-1,k} \\ &+ a^P_{j,k+1}\delta P_{j,k+1} + a^P_{j,k-1}\delta P_{j,k-1} + b^P\end{aligned}$$

.... (49)

Where

$$a^P_{j,k} = \Delta y\left(d^{(x)}_{j,k} + d^{(x)}_{j-1,k}\right) + \Delta x\left(d^{(y)}_{j,k} + d^{(y)}_{j,k-1}\right)$$

$$a^P_{j+1,k} = \Delta y d^{(x)}_{j,k}, \qquad a^P_{j-1,k} = \Delta y d^{(x)}_{j-1,k},$$

$$a^P_{j,k+1} = \Delta x d^{(y)}_{j,k} \qquad a^P_{j,k-1} = \Delta x d^{(y)}_{j,k-1}$$

Equation (49) can be put in summation form as follows:

$$a^P_{j,k}\delta P_{j,k} = \sum a^P_{nb}\,\delta P_{nb} + b^P \quad \text{.... (50)}$$

Equation (50) is a disguised discrete Poisson equation that can be written symbolically as

$$\nabla^2_d \delta P = \frac{1}{\Delta\tau}\nabla_d \cdot u^* \quad \text{...................... (51)}$$

To explain the last equation; let

$d^{(x)}_{j-1,k} = d^{(x)}_{j,k} = \frac{k}{\Delta x}$ and $d^{(y)}_{j,k-1} = d^{(y)}_{j,k} = \frac{k}{\Delta y}$

Then equation (48) reduces to

$$\begin{aligned}&-k\left(\delta P_{j+1,k} - 2\delta P_{j,k} + \delta P_{j-1,k}\right)/\Delta x^2 - \\ &k\left(\delta P_{j,k+1} - 2\delta P_{j,k} + \delta P_{j,k-1}\right)/\Delta y^2 \\ &= -\left(u^*_{j,k} - u^*_{j-1,k}\right)/\Delta x - \left(v^*_{j,k} - v^*_{j,k-1}\right)/\Delta y\end{aligned}$$

.... (52)

The LHS of (52) is a discredited form of

$$-k\left\{\frac{\partial^2}{\partial x^2}\left(\delta P_{j,k}\right) + \frac{\partial^2}{\partial y^2}\left(\delta P_{j,k}\right)\right\}$$

and this shows that (52) is a discredited form of the Poisson equation.

The SIMLPE Algorithm [8] [11]

The complete SIMPLE algorithm can be summarized as follows:

1. u^* and v^* is obtain from (38) and (39)
2. δP is obtaining from (50)
3. u^c and v^c is obtain from (45) and (46) respectively.
4. P^{n+1} is obtain from $P^{n+1}=P^n+\alpha_P\,\delta P$, where α_P is relaxation parameter.

The SIMPLE algorithm contain two relaxation parameters α_P and E(≡Δτ), in our work we take E=1 and α_P = 0.8 (Patankar 1980)[10] to a chive a stable convergence.

Some important Notes about SIMLPE [10]

1. The words Semi-implicit in the name SIMPLE have used to acknowledge that omission of the term. This term $\sum a^v_{nb}v^c_{nb}$, $\sum a^u_{nb}u^c_{nb}$

represents an indirect or implicit influence of the pressure correction on velocity; pressure corrections at nearby locations can alter the neighboring velocities and thus cause a velocity correction at the point under consideration.

2. The omission of any term would, of course, be unacceptable if it meant that the ultimate solution would not be the true solution of the discretize forms of momentum and continuity equations. It so happens that the converged solution given by SIMPLE does not contain any error resulting from the omission of $\sum a_{nb}^{v} v_{nb}^{c}, \sum a_{nb}^{u} u_{nb}^{c}$.

3. The mass source b^P in the equation (50) that serves a useful indicator of the convergence of the fluid flow solution. The iterations should be continued, until the value of b^P every where becomes sufficiently small.

4. In many problems, the value of the absolute pressure is much larger than the local differences in pressure that are encountered. If the absolute value of pressure were used for p, round-off errors would arise in calculations, hence is best to set P= 0 as a references value at suitable grid point and to calculate all other values of P as pressures relative to the reference value.

Discussion the Results

In this section we will explain the relationship between Reynolds number and time increment (dt) and it's effect on the eddy vortices from grow or decay i.e disappear and vanish the eddy vortex in the cross-section, we study this for Reynolds number takes the various values which is 12, 24, 48, and 96 at time increment 0.01, 0.03 and 0.05.

In the fig (1 – 3) we note that when time increment increase the eddy vortex take varies shapes this mean; when dt = 0.01 we see that there is a single eddy vortex in the center of cross-section while when dt = 0.03 it observed that a new eddy vortex are created and the original eddy is shifting toward boundary with range (0.003 – 0.001) on other hand from fig (25) we see that the new eddy vortex are shifting toward boundary; and the original eddy have the range (0.003 – 0.0015) near the boundary and the corner of cross – section have a new eddy vortex generate with range (0.0005 – 0.001)

The above explanation exhibited that the flow of fluid is unsteady and flow field influence by time, but if we set time equal to zero we certainly get the steady flow.

Fig (4 – 6) shows that the eddy vortex shifting toward boundary from the left side and we also observed that when RE=24 and dt = 0.01 the range of eddy vortex have (0.005 – 0.001) while at dt = 0.03 it have the range (0.003 – 0.0005) and last figure(6) at dt = 0.05 we will see that the range is (0.0012 – 0.0002).At RE = 48 and dt = 0.01 ,see fig (7), is observed that there is a single eddy vortex in the middle of cross-section and parallel to y axis, it see that at the corner an eddy become generate. Fig (8) explain that this corner becomes a new eddy vortex with range (0.0035 – 0.0005) and the original eddy have (0.003 – 0.0005) when dt = 0.03 but in the case dt = 0.05 fig (9) we will see that the eddy vortex are shifting toward boundary from right side and it note that the two vortex have the range (0.002 – 0.005) and (0.003 – 0.0005) respectively Finally, fig (10 – 12) shows that there are two eddy vortex near boundary with range (0.0014 – 0.0004) and (0.001 – 0.0004) at d t= 0.01 while at dt = 0.03 and dt = 0.05 we observe that these vortexes are shifting up towered boundary, the stream line in the corner will be vanish.

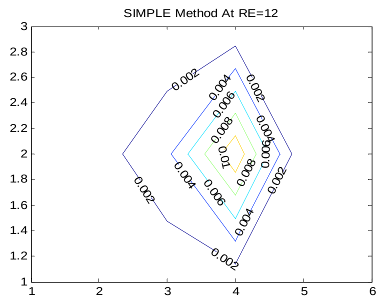

Figure (1)

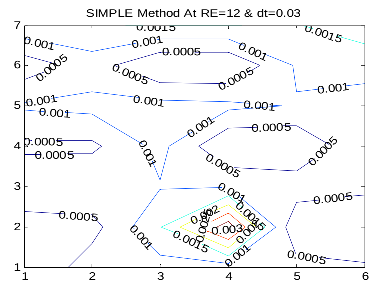

Figure (2)

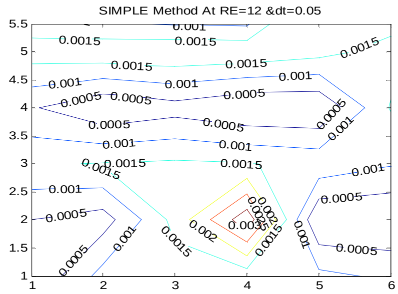

Figure (3)

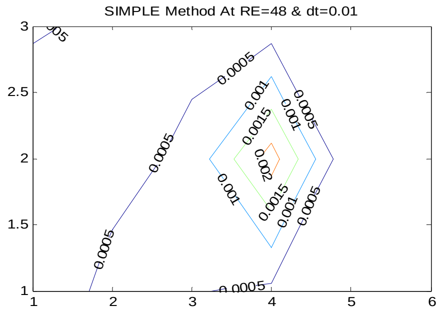

Figure (7)

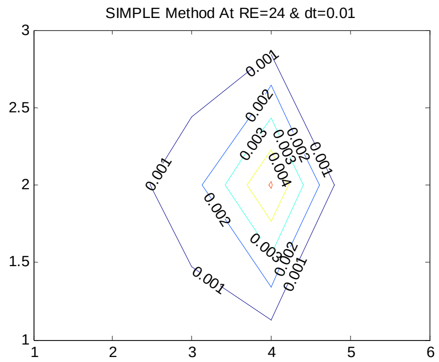

Figure (4)

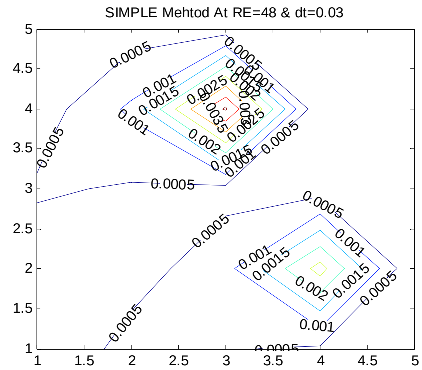

Figure (8)

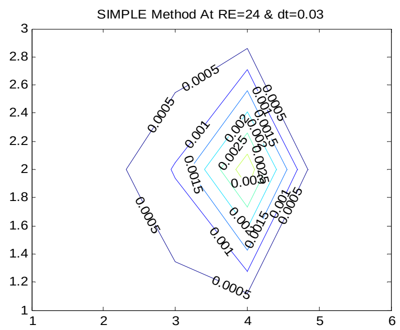

Figure (5)

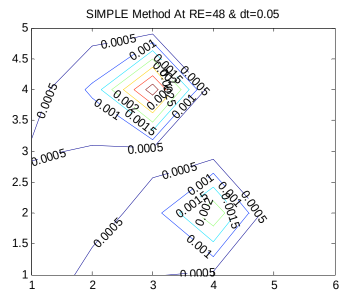

Figure (9)

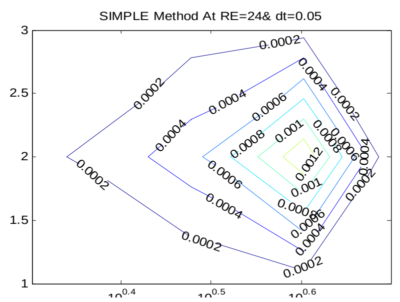

Figure (6)

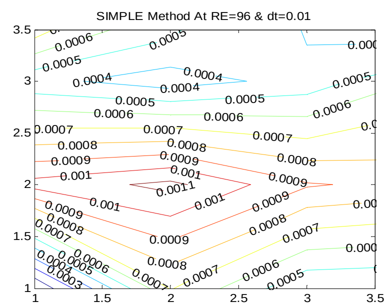

Figure (10)

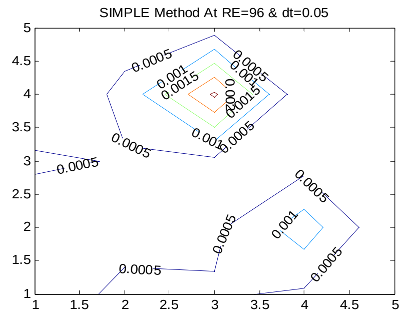

Figure (11)

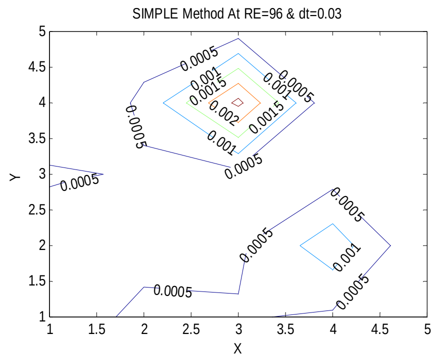

Figure (12)

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