PRIME HOLLOW MODULES

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Abstract
A non-zero module M is called hollow, if every proper submodule of M is small. In this work we introduce a generalization of this type of modules; we call it prime hollow modules. Some main properties of this kind of modules are investigated and the relation between these modules with hollow modules and some other modules are studied, such as semihollow, amply supplemented and lifting modules.

المقاسات الاولية المجوفة

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الخلاصه
يقال للمقاس الغير صفري M انه مجوف، اذا كان كل مقاس جزئي فعلي فيه صغيرا.في هذا البحث سنعطي اعماما لهذا النوع من المقاسات نطلق عليه اسم المقاسات الاولية المجوفة. ندرس بعض الخواص الاساسية لهذا الصنف من المقاسات مع دراسة العلاقة بينها وبين المقاسات المجوفة من جهة وعلاقتها بأصناف اخرى من المقاسات من جهة اخرى مثل المقاسات شبه المجوفة ، المقاسات التكميلية الواسعة ومقاسات الرفع.

Introduction

Throughout this paper R represents a commutative ring with identity, and all R-modules are left until. A proper submodule N of an R-module M is called small if N+L≠M for every proper submodule L of M [1]. A non-zero module M is called hollow, if every proper submodule of M is small [2].

In this paper we give a generalization of hollow modules, we call it prime hollow module which is a module in which every prime submodule is small.

In section one we give several properties of this class of modules, and in section two we study the Prime hollow modules under the multiplication property. In section three we investigate some conditions under which hollow modules and Prime hollow modules are equivalent.

Finally in section four we investigate the relation between the Prime hollow modules and some other modules such as amply supplemented, semihollow and lifting modules.

Some properties and remarks

In this section we give the definition of Prime hollow modules as generalization of hollow modules. We investigate the main properties of this class of modules.

Definition (1.1): An R-module M is called prime hollow module (simply Pr-hollow module), if each prime submodule of M is small.

It is clear that every hollow module is Pr-hollow module, but the converse is not true, for example the Z-module Q (where Q is the set of all rational numbers), is Pr-hollow module since the only prime submodule of Q is (0) [3], which is small submodule of Q. But Q is not hollow module see [4].

Remarks and Examples (1.2):

1. Z9 module as Z-module is Pr-hollow module. In fact the only prime submodule of Z9 is I= $\{\bar{0},\bar{3},\bar{6}\}$ which is small submodule of Z9 since there is no proper submodule J of Z9 such that I + J = Z9.
2. The Z-module Z is not Pr-hollow module. Since every prime submodule of Z is of the form (p), where p is prime number, but (p) is not small submodule of Z for every prime number p.
3. If an R-module M is finitely generated then hollow modules and Prime hollow modules are equivalent. In fact if M is finitely generated then every proper submodule N of M is contained in a maximal submodule hence in a prime submodule say L. Since M is P-hollow module, then L<<M, hence N$\ll M$. Thus M is hollow module. In particular, the concepts of hollow modules and pr-hollow modules are equivalent on rings.
4. A submodule of Pr-hollow module need not be Pr-hollow module, for example the Z-module Z is a submodule of the module Q which is Pr-hollow module, but Z itself is not Pr-hollow module.

The following proposition gives some properties of Prime hollow modules.

Proposition (1.3): Epimorphic image of Pr-hollow module is Pr-hollow module.

Proof: Let M be Pr-hollow module and let f: M→M′ be an epimorphism with M′ is an R-module. Let N be a prime submodule of M′ with N+L= M′ where L⊆ M′. Since N is a prime submodule of M and f is epimorphism, then $f^{-1}(N)$ is prime submodule of M [3]. Now we have $f^{-1}(N)$ $^{-1}$(N) is a prime submodule of M and M is Pr-hollow module, therefore $f^{-1}(N) \ll M$<<M and hence f ($(f^{-1}(N))$ $\ll$<<f (M) [1], that is N$\ll$ M′.

As corollaries of (1.3), we have the following.

Corollary (1.4): Let M be a Pr-hollow module, and let N be a proper sub module of M, then $\frac{M}{N}$ is Pr-hollow module.

Corollary (1.5): A direct summand of Pr-hollow module is Pr-hollow module.

Note that, the converse of (1.4) is not true in general, for example $\frac{Z_{12}}{(\bar{2})} \cong Z_2$ is Pr-hollow module. But Z_{12} itself is not Pr-hollow module, since there exists a prime submodule $(\bar{3})$ of Z_{12} which is not small submodule of Z_{12}.

Proposition (1.6): Let M be Pr-hollow module and let N be a prime submodule of M such that $\frac{M}{N}$ is finitely generated, then M is hollow module.

Proof: Since $\frac{M}{N}$ is finitely generated then $\frac{M}{N}$ = $R(x_1+N)+R(x_2+N)+\ldots\ldots\ldots\ldots.+R(x_n+N)$, where $x_i \in M$ for all i=1,2,……n. We claim that $M=Rx_1+Rx_2+\ldots\ldots\ldots+Rx_n$. Let m∈M, then m+N∈ $\frac{M}{N}$, thus $(m+N) = r_1(x_1+N)+ r_2(x_2+N)+ \ldots\ldots r_n(x_n+N) = r_1x_1+ r_2x_2+\ldots. r_nx_n +N \in \frac{M}{N}$. This implies that $m= r_1x_1+ r_2x_2+\ldots. r_nx_n +n$, for some n∈N. Thus $M = Rx_1+ Rx_2+\ldots.Rx_n +N$. But M is Pr-hollow module, therefore $M = Rx_1+ Rx_2+\ldots.Rx_n$. That is M is finitely generated and by (1.2) (3), M is hollow module.

Proposition (1.7): Every finitely generated Pr-hollow module is cyclic.

Proof: Since M is finitely generated Pr-hollow module then by (1.2) (3), M is hollow module and hence M is cyclic [4, P.33].

From proposition (1.7) we conclude the following result, before that an R-module M is called projective, if for every epimorphism f: B → A where A and B be any R-modules, and for every homomorphism g: M→ A there exists a homomorphism h: M → B such that f h = g [1]. And an R-module M is called C.P module if every cyclic submodule of M is projective [5].

Corollary (1.8): Let M be a finitely generated C.P module. If M is Pr-hollow module then M is projective.

Proof: Since M is finitely generated Pr-hollow module, then by (1.7), M is cyclic. But M is C.P module thus M is projective module.

Recall that a submodule N of a module M is called coclosed in M if whenever $\frac{N}{K} \ll \frac{M}{K}$ then N = K [6].

We saw in (1.2) (4), that a submodule of Pr-hollow module need not be Pr-hollow module. This statement is true in some cases as in the following proposition.

Proposition (1.9): Let M be a Pr-hollow module in which every proper submodule of M is prime submodule, then every non-zero coclosed submodule of M is Pr-hollow module.

Proof: Let N be a non-zero coclosed submodule of M, and let L be a proper submodule of N. then $L \subset M$. By assumption L is prime submodule of M. Since M is Pr-hollow module, so $L \ll M$. But N is coclosed submodule of M, thus $L \ll N$ [4, P.27].

A module M is called a small cover for a module N, if there exists a small epimorphism $\phi : M \longrightarrow N$ [7].

Proposition (1.10): Let Q be a small cover of M. If Q is Pr-hollow module then M is pr-hollow module.

Proof: Let f: Q→M be a small cover of M and assume that Q is a Pr-hollow module. By the first isomorphism theorem $\frac{Q}{\ker f} \cong M$. Since Q is Pr-hollow module, then by (1.4), M is Pr-hollow module.

We think that the converse of (1.10) is true but we cannot prove it.

Multiplication Prime hollow modules

In this section we study the multiplication Pr-hollow module. An R-module M is called multiplication if each submodule N of M can be written as the form N=IM for some ideal I of R [8].

Theorem (2.1): Let R be a Pr-hollow ring. If M is multiplication finitely generated and faithful module over R, then M is Pr-hollow module.

Proof: Let N be a prime submodule of M. Since M is multiplication module, there exists a prime ideal I of R such that N=IM [9]. But R is Pr-hollow ring, thus I is small ideal of R Since M is finitely generated, faithful and multiplication module, so N is small submodule of M [10].

Recall that an R-module M is called cancelation module, if whenever AM=BM where A & B be two ideals of R then A = B. And M is called weak cancelation, if whenever AM = BM, where A & B be two ideals of R then A+ ann M = B+ ann M. Also M is called quasi-cancelation module, if whenever AM = BM where A & B be two finitely generated ideals of R then A = B [11].

In [11], Mijbass proved that if M is multiplication and cancelation module then M is finitely generated and faithful, so we have the following proposition.

Proposition (2.2): Let M be a multiplication module. If M is cancelation module over a Pr-hollow ring then M is Pr-hollow module.

Proof: Since M is a multiplication and cancelation module, then M is faithful and finitely generated [11, P.52], and by theorem (2.1), M is Pr-hollow module.

Corollary (2.3): Let M be a multiplication module over Pr-hollow ring such that M contains a cancelation submodule. Then M is Pr-hollow module.

Proof: Suppose that N is a cancelation submodule of M. Since M is a multiplication module, then M is cancelation module [11, P.61]. And by (2.2), M is Pr- hollow module.

Prime hollow modules and hollow modules

In section 1 we said that every hollow module is Pr-hollow module, and we give an example shows that the converse is not true. In this section we investigate conditions under which Pr-hollow modules can be hollow modules.

Proposition (3.1): Let M be a multiplication module containing a finitely generated prime submodule. If M is Pr-hollow module then M is hollow module.

Proof: Since M is multiplication module containing a finitely generated prime submodule, then M is finitely generated [11, P.58], and by (1.2) (3), we get the result.

Note that the Z-module Q contains only prime submodule which is zero but Q is not hollow module, in fact Q is not multiplication module.

Corollary (3.2): Let M be a multiplication module with prime annihilator. If M is Pr-hollow module then M is hollow module.

Proof: Since M is a multiplication module with prime annihilator, then M is finitely generated [11, P.56], hence M is hollow module.

Corollary (3.3): Let M be a multiplication and weak cancelation module. If M is Pr-hollow module then M is hollow module.

Proof: Since M be a multiplication and weak cancelation module, then M is finitely generated [11, p.60], and by (1.2) (3), M is hollow module.

Corollary (3.4): Let M be a multiplication and weak cancelation module. If M is Pr-hollow module, then M is cyclic module (hence M is hollow module).

Proof: Since M be a multiplication module weak cancelation module, then M is finitely generated [11, P.60]. But M is Pr-hollow module so by (1.8), M is cyclic. .

Recall that a submodule N of an R-module M is pure in M, if IM $\cap$ N = IN [3]. So we can give the following corollary.

Corollary (3.5): Let M be a multiplication R-module such that M contains a pure weak cancelation submodule N with ann (M) = ann (N). If M is Pr-hollow module then M is hollow module.

Proof: By assumption M is weak cancelation [11, p.62], and by proposition (3.3), we get the result.

Corollary (3.6): Let M be a multiplication R-module such that M contains a pure cancelation submodule. If M is Pr-hollow module, then M is hollow module.

Proposition (3.7): Let M be a multiplication faithful over integral domain R. If M is Pr-hollow module then M is hollow module.

Proof: Since M is a multiplication faithful over integral domain R, then M is finitely generated [11, p.54], and by (1.2) (3) we are done.

Proposition (3.8): Let M be a multiplication module which has a finitely generated faithful submodule N of M. If M is Pr-hollow module, then M is hollow module.

Proof: Since M is a multiplication module and the submodule N of M is finitely generated faithful, then M is finitely generated [11, p.56], hence M is hollow module.

Prime hollow modules and some other modules

We study in this section the relation between Pr-hollow module and other modules such as semihollow modules, amply supplemented modules and lifting modules.

An R-module M is called semihollow if every proper finitely generated submodule of M is small [12].

It is clear that every hollow module is semihollow module, but we cannot find a direct relation between Pr- hollow module and semihollow module. However semihollow module is Pr- hollow module when M is finitely generated Noetherian module, where a Noetherian module is a module in which every submodule is finitely generated [13].

Proposition (4.1): Let M be finitely generated Noetherian module. If M is semihollow module then M is Pr- hollow module.

Proof: Let N be a prime submodule of M. Since M is finitely generated Noetherian module, then N is finitely generated [3, p.33]. But M is semihollow module, thus N is small submodule of M.

Let A and B be submodules of a module M. Then A is a supplement of B in M if M = A + B and A $\cap$ B << A, and A module M is called supplemented, if every submodule of M, has a supplement in M. A module M is called amply supplemented, if for every two submodules U, V of M, such that M = U + V, there exists a supplement V1 of U in M, such that V1 $\leq$ V [14]. In the following proposition we show that Prime hollow modules are subclass of amply supplemented modules.

Proposition (4.2): Every Pr-hollow module is amply supplemented module.

Proof: Let M be a Pr-hollow module and let U be a prime submodule of M. Since M is Pr-hollow module, then we have U + M = M and U$\ll$ M. Therefore M is amply supplemented.

We introduce the following definition.

Definition (4.3): An R-module M is called Pr-supplemented, if every prime submodule of M has a supplement in M.

Proposition (4.4): Every Pr-hollow module is Pr- supplemented module.

Proof: Let M be a Pr-hollow module and let N be a prime submodule of M. Now M +N =M and N$\cap$ M=N. But N is prime submodule of M and M is Pr-hollow module, therefore N$\ll$M, that is M is Pr-supplemented module.

Recall that an R-module M is called lifting if for every submodule N of M there are submodules K and K′ of M such that M = K$\oplus$ K′, K $\subseteq$ N and N$\cap$ K′ $\ll$ M [15].

It is known that every hollow module is lifting module. This property is not satisfying in Prime hollow modules as we see in the following example.

Examples (4.5): Consider the Z-module Q, where Q is the set of all rational numbers. This module is Pr-hollow module (1.2) (4), but it is not lifting module [16].

If M is finitely generated then Pr-hollow module can be lifting module, as the following proposition shows.

Proposition (4.6): Every finitely generated Pr-hollow module is lifting module.

Proof: Clear.

The converse of (4.6) is not true in general for example the Z-module $M = Z_2 \oplus Z_4$ is lifting module [16], but it is not Pr-hollow module, since there exists a prime submodule $N = Z_2 \oplus (0)$ of M which is not small submodule of M. However the converse is true in the following case.

Proposition (4.7): Every indecomposable lifting module is Pr-hollow module.

Proof: Let M be a indecomposable lifting module, and let N be a prime submodule of M. then $M = K \oplus K'$, $K \subseteq N$ and $N \cap K' \ll M$. But M is indecomposable, thus $K' = 0$ and hence $K = M$ which implies that $N \cap M = N \ll M$.

As a corollary of (4.6), we have the following.

Proposition (4.8): Let M be a faithful multiplication over integral domain R, If M is Pr-hollow module then M is lifting module.

Proof: Since M be a faithful multiplication module over integral domain, then M is finitely generated [11, P.54], but M is Pr-hollow module so by (4.6), M is lifting module.

Also by using (4.6) and [11, p.56] we can easily prove the following.

Corollary (4.9): Let M be a non-zero multiplication module with prime annihilator. If M is Pr-hollow module then M is lifting module.

We end this work by the following proposition.

Proposition (4.10): If an R-module M is finitely generated faithful and multiplication over a Pr-hollow ring, then M is lifting module.

Proof: Since M is a finitely generated faithful and multiplication module over integral domain, then by (2.1), M is Pr-hollow module, and by (4.6), M is lifting module.

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