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Some Geometric Properties for an Extended Class Involving Holomorphic Functions Defined by Fractional Calculus

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Abstract
The main objective of this paper is to study a subclass of holomrphic and univalent functions with negative coefficients in the open unit disk U=$\{w \in \mathbb{C}: |w < 1|\}$ defined by Hadamard Product. We obtain coefficients estimates, distortion theorem , fractional derivatives, fractional integrals, and some results.

Keywords: Univalent Function , Hadamard Product , Distortion theorm, farctional derivatives and fractional integrals.

تعميم في بعض الخصائص الهندسية لصنف من الدوال احادية التكافؤ المعرفة بواسطة حساب التفاضل والتكامل الكسري

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الخلاصة
الموضوع الرئيسي من هذا البحث هودراسة دوال لفئة جزئية والتي تتضمن التحليلية واحادية التكافؤمع المعاملات السالبة في قرص الوحدة المعرفة بواسطة هادامارد برودكت. وحصلنا على معاملات تقديرية,نظرية التشوه,المشتقات الكسرية وتكاملات كسرية وبعض النتائج.

Introduction

Let A denotes the class of functions of the form:

$$f(w) = w - \sum_{n=2}^{\infty} a_n w^n, \qquad (a_n \geq 0) \tag{1}$$

which are univalent and holomrphic in the unit disk $U = \{w \in \mathbb{C} : |w| < 1\}$. We define a subclass K of A consisting of the functions by

$$f(w) = w - \sum_{n=2}^{\infty} a_n w^n . \quad (a_n \geq 0) \tag{2}$$

The function $f(w)$ belongs to the class $H(\alpha, \beta, \theta, \lambda)$ if it satisfies

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$$Re\left\{\frac{w(G(f*g)(w))' + \lambda w^2(G(f*g)(w))''}{(1-\lambda)\big(G(f*g)(w)\big) + \lambda w(G(f*g)(w))'} + \emptyset\right\}$$
$$\geq \beta\left\{\frac{w(G(f*g)(w))' + \lambda w^2(G(f*g)(w))''}{(1-\lambda)\big(G(f*g)(w)\big) + \lambda w(G(f*g)(w))'} + \emptyset - 1\right\} + \theta, \qquad (3)$$

where $(0 \leq \theta < 1), (0 \leq \lambda \leq 1), \beta \geq 0, w \in U, (0 < \alpha < 1), (0 \leq \emptyset < 1)$, and

$$g(w) = w - \sum_{n=2}^{\infty} b_n w^n, \quad (b_n \geq 0) \qquad (4)$$

then the Hadamard product or (convolution)$f * g$ of f and g is defined by

$$(f*g)(w) = w - \sum_{n=2}^{\infty} a_n b_n w^n = (g*f)(w) \qquad (5)$$

and $\in U$.

Definition 1 : [1]

The fractional derivative of order α (0< α <1) is defined by

$$D_w^{\alpha} f(w) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dw} \int_0^w \frac{f(t)}{(w-t)^{\alpha}} dt \quad , \qquad (6)$$

where $f(w)$ is an holomrphic function in a simply – connected region of the w – plane containing the origin, and the multiplicity of $(w-t)^{-\alpha}$is removed by requiring log($z - t$) to be real ,when (w – t> 0).

Definition 2: [2]

The fractional integral of order α(α>0) is defined by

$$D_w^{-\alpha} f(w) = \frac{1}{\Gamma(\alpha)} \int_0^w \frac{f(t)}{(w-t)^{1-\alpha}} dt \quad , \qquad (7)$$

where $f(w)$ is a holomrphic function in a simply – connected region of the w – plane containing the origin , and the multiplicity of $(w-t)^{\alpha-1}$ is removed by requiring log($w - t$) to be real , when ($w - t$) > 0 .

Definition 3: [2]

The fractional derivative of order $n + \alpha$ $(n = 0,1,2,\ldots)$ is defined by

$$D_w^{n+\alpha} f(w) = \frac{d^n}{dw^n} D_w^{\alpha} f(w), \qquad (8)$$

We put the holomrphic function $f(w) = w - \sum_{n=2}^{\infty} a_n w^n$ in U, as follows

$$Mf(w) = \Gamma(2+\alpha) w^{-\alpha} D_w^{-\alpha} f(w)$$
$$= w - \sum_{n=2}^{\infty} \frac{n!\Gamma(2+\alpha)}{\Gamma(n+1+\alpha)} a_n w^n \ , \quad \alpha > 0. \qquad (9)$$

And

$$Gf(w) = \Gamma(2-\alpha) w^{\alpha} D_w^{\alpha} f(w).$$
$$= w - \sum_{n=2}^{\infty} \frac{n!\,\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} a_n w^n \ , \quad 0 < \alpha < 1. \qquad (10)$$

Then , from (10) we get

$$G(f*g)(w) = w - \sum_{n=2}^{\infty} \psi(n,\alpha) a_n b_n w^n \qquad (11)$$

where $\psi(n,\alpha)=\frac{n!\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}$ (12)

Lemma 1: [3]

Let $w = u + iv$. Then Re $(w) \geq \sigma$ if and only if $|w-(1+\alpha)| \leq |w+(1-\alpha)|$.

Lemma 2: [3]

Let $w = u + iv$ and α , β are real numbers. Then

Re $w > \alpha\,|w-1| + \beta$ if and only if Re$\{w(1+\alpha e^{i\emptyset})-\alpha e^{i\emptyset}\} > \beta$.

2. Coefficient Estimates

In the Theorem(1), we get the sufficient condition for the function $f(w)$ in the class $H(\alpha,\beta,\theta,\lambda)$

Theorem 1:

A function $f(w)$ defined by (2) is in the class $H(\alpha,\beta,\theta,\lambda)$ if and only if

$$\sum_{n=2}^{\infty}(1-\lambda+n\lambda)[n(1+\beta)+\emptyset(1+\beta)-(\theta+\beta)]\psi(n,\alpha)a_nb_n \leq 1+\emptyset-\theta \tag{13}$$

where $(0 \leq \theta < 1), (0 \leq \lambda \leq 1), \beta \geq 0, (0 < \alpha < 1), (0 \leq \emptyset < 1)$.

Proof:

By using Definition "3" , we get

$$Re\left\{\frac{w(G(f*g)(w))'+\lambda w^2(G(f*g)(w))''}{(1-\lambda)(G(f*g)(w))+\lambda w(G(f*g)(w))'}+\emptyset\right\} \geq$$

$$\beta\left\{\frac{w(G(f*g)(w))'+\lambda w^2(G(f*g)(w))''}{(1-\lambda)(G(f*g)(w))+\lambda w(G(f*g)(w))'}+\emptyset-1\right\}+\theta,$$

By Lemma (2) , we have

$$Re\left\{\left(\frac{w(G(f*g)(w))'+\lambda w^2(G(f*g)(w))''}{(1-\lambda)(G(f*g)(w))+\lambda w(G(f*g)(w))'}+\emptyset\right)(1+\beta e^{i\gamma})-\beta e^{i\gamma}\right\} \geq \theta,$$
$$-\pi < \gamma \leq \pi$$

or equivalently

$$Re\left\{\frac{w(G(f*g)(w))'(1+\beta e^{i\gamma})+\lambda w^2(G(f*g)(w))''(1+\beta e^{i\gamma})}{(1-\lambda)(G(f*g)(w))+\lambda w(G(f*g)(w))'}\right.$$
$$+\frac{\emptyset(1+\beta e^{i\gamma})\left[(1-\lambda)(G(f*g)(w))+\lambda w(G(f*g)(w))'\right]}{(1-\lambda)(G(f*g)(w))+\lambda w(G(f*g)(w))'}$$
$$\left.-\frac{\beta e^{i\gamma}\left[(1-\lambda)(G(f*g)(w))+\lambda w(G(f*g)(w))'\right]}{(1-\lambda)(G(f*g)(w))+\lambda w(G(f*g)(w))'}\right\} \geq \theta. \tag{14}$$

Let

$$A(w)=\left[w(G(f*g)(w))'(1+\beta e^{i\gamma})+\lambda w^2(G(f*g)(w))''(1+\beta e^{i\gamma})\right]$$
$$+\emptyset(1+\beta e^{i\gamma})\left[(1-\lambda)(G(f*g)(w))+\lambda w(G(f*g)(w))'\right]$$
$$-\beta e^{i\gamma}\left[(1-\lambda)(G(f*g)(w))+\lambda w(G(f*g)(w))'\right]$$

And $B(w)=\left[(1-\lambda)(G(f*g)(w))+\lambda w(G(f*g)(w))'\right]$

By using (11) , we have

$$(1-\lambda)(G(f*g)(w)) = w-\sum_{n=2}^{\infty}\psi(n,\alpha)a_nb_nw^n-\lambda w+\sum_{n=2}^{\infty}\lambda\psi(n,\alpha)a_nb_nw^n,$$

$$\lambda w(G(f*g)(w))' = \lambda w - \sum_{n=2}^{\infty} \lambda n\psi(n,\alpha)a_n b_n w^n .$$

By Lemma (1), we have that (14) is equivalent to
$|A(w) + (1-\theta)B(w)| \geq |A(w) - (1+\theta)B(w)| \quad for\ 0 \leq \theta < 1$

But $|A(w) + (1-\theta)B(w)|$

$$= \left|\left[\left(w - \sum_{n=2}^{\infty} n\psi(n,\alpha)a_n b_n w^n\right)(1+\beta e^{i\gamma}) - \sum_{n=2}^{\infty}(\lambda n(n-1)\psi(n,\alpha)a_n b_n w^n)(1+\beta e^{i\gamma})\right.\right.$$

$$+\emptyset(1+\beta e^{i\gamma})\left(w - \sum_{n=2}^{\infty}\psi(n,\alpha)a_n b_n w^n - \lambda w\right.$$

$$\left.+\sum_{n=2}^{\infty}\lambda\psi(n,\alpha)a_n b_n w^n + \lambda w - \sum_{n=2}^{\infty}\lambda n\psi(n,\alpha)a_n b_n w^n\right)$$

$$-\beta e^{i\gamma}\left(w - \sum_{n=2}^{\infty}\psi(n,\alpha)a_n b_n w^n - \lambda w + \sum_{n=2}^{\infty}\lambda\psi(n,\alpha)a_n b_n w^n + \lambda w - \sum_{n=2}^{\infty}\lambda n\psi(n,\alpha)a_n b_n w^n\right)$$

$$\left.\left.+(1-\theta)\left(w - \sum_{n=2}^{\infty}(1-\lambda+n\lambda)\psi(n,\alpha)a_n b_n w^n\right)\right]\right|$$

$$= |w - \sum_{n=2}^{\infty} n\psi(n,\alpha)a_n b_n w^n + \beta e^{i\gamma} w$$

$$-\beta e^{i\gamma}\sum_{n=2}^{\infty} n\psi(n,\alpha)a_n b_n w^n - \sum_{n=2}^{\infty}(\lambda n(n-1))\psi(n,\alpha)a_n b_n w^n$$

$$-\beta e^{i\gamma}\sum_{n=2}^{\infty}(\lambda n(n-1))\psi(n,\alpha)a_n b_n w^n + \emptyset w - \emptyset\sum_{n=2}^{\infty}\psi(n,\alpha)a_n b_n w^n + \emptyset\sum_{n=2}^{\infty}\lambda\psi(n,\alpha)a_n b_n w^n$$

$$-\emptyset\sum_{n=2}^{\infty}\lambda n\psi(n,\alpha)a_n b_n w^n + \emptyset\beta e^{i\gamma} w - \emptyset\beta e^{i\gamma}\sum_{n=2}^{\infty}\psi(n,\alpha)a_n b_n w^n + \emptyset\beta e^{i\gamma}\sum_{n=2}^{\infty}\lambda\psi(n,\alpha)a_n b_n w^n$$

$$-\emptyset\beta e^{i\gamma}\sum_{n=2}^{\infty}\lambda n\psi(n,\alpha)a_n b_n w^n - \beta e^{i\gamma} w + \beta e^{i\gamma}\sum_{n=2}^{\infty}\psi(n,\alpha)a_n b_n w^n - \beta e^{i\gamma}\sum_{n=2}^{\infty}\lambda\psi(n,\alpha)a_n b_n w^n$$

$$+\beta e^{i\gamma}\sum_{n=2}^{\infty}\lambda n\psi(n,\alpha)a_n b_n w^n + w - \sum_{n=2}^{\infty}(1-\lambda+n\lambda)\psi(n,\alpha)a_n b_n w^n - \theta w$$

$$\left.+\theta\sum_{n=2}^{\infty}(1-\lambda+n\lambda)\psi(n,\alpha)a_n b_n w^n\right|$$

$$= \left|(2+\emptyset-\theta)w\right.$$

$$-\sum_{n=2}^{\infty}\left[(n+\lambda n(n-1)) + \emptyset(1-\lambda+n\lambda) + (1-\theta)(1-\lambda+n\lambda)\right]\psi(n,\alpha)a_n b_n w^n$$

$$\left.-\beta e^{i\gamma}\sum_{n=2}^{\infty}[(n+\lambda n(n-1)) + \emptyset(1-\lambda+n\lambda) - (1-\lambda+n\lambda)]\psi(n,\alpha)a_n b_n w^n\right|$$

$$\geq (2+\emptyset-\theta)|w| - \sum_{n=2}^{\infty}[(n+\lambda n(n-1))+\emptyset(1-\lambda+n\lambda)+(1-\theta)(1-\lambda+n\lambda)]\psi(n,\alpha)a_n b_n|w|^n$$

$$-\beta\sum_{n=2}^{\infty}[(n+\lambda n(n-1))+\emptyset(1-\lambda+n\lambda)-(1-\lambda+n\lambda)]\psi(n,\alpha)a_n b_n|w|^n$$

Also

$$|A(w)-(1+\theta)B(w)|$$

$$=\left|\left[\left(w-\sum_{n=2}^{\infty}n\psi(n,\alpha)a_n b_n w^n\right)(1+\beta e^{i\gamma})-\sum_{n=2}^{\infty}(\lambda n(n-1)\psi(n,\alpha)a_n b_n w^n)(1+\beta e^{i\gamma})\right.\right.$$

$$+\emptyset(1+\beta e^{i\gamma})\left(w-\sum_{n=2}^{\infty}\psi(n,\alpha)a_n b_n w^n+\sum_{n=2}^{\infty}\lambda\psi(n,\alpha)a_n b_n w^n-\sum_{n=2}^{\infty}\lambda n\psi(n,\alpha)a_n b_n w^n\right)$$

$$-\beta e^{i\gamma}\left((1-\lambda)\left(w-\sum_{n=2}^{\infty}\psi(n,\alpha)a_n b_n w^n\right)+\lambda w-\lambda\sum_{n=2}^{\infty}n\psi(n,\alpha)a_n b_n w^n\right)$$

$$\left.\left.-(1+\theta)\left(w-\sum_{n=2}^{\infty}(1-\lambda+n\lambda)\psi(n,\alpha)a_n b_n w^n\right)\right]\right|$$

$$=\left|(\emptyset-\theta)w-\sum_{n=2}^{\infty}[(n+\lambda n(n-1))+\emptyset(1-\lambda+n\lambda)-(1+\theta)(1-\lambda+n\lambda)]\,\psi(n,\alpha)a_n b_n w^n\right.$$

$$\left.-\beta e^{i\gamma}\sum_{n=2}^{\infty}[(n+\lambda n(n-1))+\emptyset(1-\lambda+n\lambda)-(1-\lambda+n\lambda)]\psi(n,\alpha)a_n b_n w^n\right|$$

$$\leq(\emptyset-\theta)|w|+\sum_{n=2}^{\infty}\left[\left(n+\lambda n(n-1)\right)+\emptyset(1-\lambda+n\lambda)-(1+\theta)(1-\lambda+n\lambda)\right]\psi(n,\alpha)a_n b_n|w|^n$$

$$+\beta\sum_{n=2}^{\infty}[(n+\lambda n(n-1))+\emptyset(1-\lambda+n\lambda)-(1-\lambda+n\lambda)]\psi(n,\alpha)a_n b_n|w|^n$$

and, so, $|A(w)+(1-\alpha)B(w)|-|A(w)-(1+\alpha)B(w)|$

$$\geq 2(1+\emptyset-\theta)|w|$$

$$-\sum_{n=2}^{\infty}\left[\left(2n+2\lambda n(n-1)\right)+2\emptyset(1-\lambda+n\lambda)-2\theta(1-\lambda+n\lambda)\right]$$

$$-\beta\left[\left(2n+2\lambda n(n-1)\right)+2\emptyset(1-\lambda+n\lambda)-2(1-\lambda+n\lambda)\right]\psi(n,\alpha)a_n b_n|w|^n$$

$$\geq 0$$

$$\sum_{n=2}^{\infty}\left[\left(n+\lambda n(n-1)\right)+\emptyset(1-\lambda+n\lambda)-\theta(1-\lambda+n\lambda)\right.$$

$$\left.+\beta[n+\lambda n(n-1)+\emptyset(1-\lambda+n\lambda)-(1-\lambda+n\lambda)]\right]\psi(n,\alpha)a_n b_n|w|^n\geq 0$$

$$\sum_{n=2}^{\infty}[n(1+\beta)+\lambda n(n-1)(1+\beta)+\emptyset(1+\beta)(1-\lambda+n\lambda)-(\theta+\beta)(1-\lambda+n\lambda)]\psi(n,\alpha)a_n b_n$$

$$\leq 1+\emptyset-\theta.$$

This is equivalent to

$$\sum_{n=2}^{\infty}(1-\lambda+n\lambda)[n(1+\beta)+\emptyset(1+\beta)-(\theta+\beta)]\psi(n,\alpha)a_nb_n \leq 1+\emptyset-\theta.$$

By putting $\emptyset=0$ in the above theorem, we have the result achieved by Abdul Hussein and Buti[4].

Conversely, assume that (2.1) holds , then we show that

$$Re\left\{\frac{w\big(G(f*g)(w)\big)'(1+\beta e^{i\gamma})+\lambda w^2\big(G(f*g)(w)\big)''(1+\beta e^{i\gamma})}{(1-\lambda)\big(G(f*g)(w)\big)+\lambda w\big(G(f*g)(w)\big)'} + \frac{\emptyset(1+\beta e^{i\gamma})[(1-\lambda)(G(f*g)(w))+\lambda w\big(G(f*g)(w)\big)'}{(1-\lambda)(G(f*g)(w))+\lambda w\big(G(f*g)(w)\big)'} - \frac{(\theta+\beta e^{i\gamma})\left[(1-\lambda)(G(f*g)(w))+\lambda w\big(G(f*g)(w)\big)'\right]}{(1-\lambda)(G(f*g)(w))+\lambda w\big(G(f*g)(w)\big)'}\right\}\geq 0.$$

Upon choosing the values of z on the positive real axis where $0\leq w=r<1$, the above inequality reduces to

$$Re\left\{\frac{\left(w-\sum_{n=2}^{\infty}(n\psi(n,\alpha)a_nb_nw^n\right)(1+\beta e^{i\gamma})-\left(\sum_{n=2}^{\infty}\lambda n(n-1)\psi(n,\alpha)a_nb_nw^n\right)(1+\beta e^{i\gamma})}{1-\sum_{n=2}^{\infty}(1-\lambda+n\lambda)\psi(n,\alpha)a_nb_nw^{n-1}} + \frac{\emptyset(1+\beta e^{i\gamma})[w-\sum_{n=2}^{\infty}(1-\lambda+n\lambda)\psi(n,\alpha)a_nb_nw^n]}{1-\sum_{n=2}^{\infty}(1-\lambda+n\lambda)\psi(n,\alpha)a_nb_nw^{n-1}} - \frac{(\theta+\beta e^{i\gamma})[w-\sum_{n=2}^{\infty}(1-\lambda+n\lambda)\psi(n,\alpha)a_nb_nw^n]}{1-\sum_{n=2}^{\infty}(1-\lambda+n\lambda)\psi(n,\alpha)a_nb_nw^{n-1}}\right\}\geq 0.$$

$$Re\left\{\frac{(1+\emptyset-\theta)-\sum_{n=2}^{\infty}[(n(1+\beta e^{i\gamma})+(\lambda n(n-1))(1+\beta e^{i\gamma})}{1-\sum_{n=2}^{\infty}(1-\lambda+n\lambda)\psi(n,\alpha)a_nb_nw^{n-1}} + \frac{\emptyset(1+\beta e^{i\gamma})(1-\lambda+n\lambda)-(\theta+\beta e^{i\gamma})(1-\lambda+n\lambda)]\psi(n,\alpha)a_nb_nw^n}{1-\sum_{n=2}^{\infty}(1-\lambda+n\lambda)\psi(n,\alpha)a_nb_nw^{n-1}}\right\}\geq 0.$$

Since $(-e^{i\gamma})\geq -|e^{i\gamma}|=-1$, the above inequality reduces to

$$Re\left\{\frac{(1+\emptyset-\theta)-\sum_{n=2}^{\infty}[(n(1+\beta)+(\lambda n(n-1))(1+\beta)}{1-\sum_{n=2}^{\infty}(1-\lambda+n\lambda)\psi(n,\alpha)a_nb_nr^{n-1}} + \frac{\emptyset(1+\beta)(1-\lambda+n\lambda)-(\theta+\beta)(1-\lambda+n\lambda)]\psi(n,\alpha)a_nb_nr^{n-1}}{1-\sum_{n=2}^{\infty}(1-\lambda+n\lambda)\psi(n,\alpha)a_nb_nr^{n-1}}\right\}\geq 0.$$

Letting $r\to 1^-$, we get the desired conclusion.

Corollary 1 :

Let $f(w)\in H(\alpha,\beta,\theta,\lambda)$. Then ,

$$a_n\leq \frac{1+\emptyset-\theta}{(1-\lambda+n\lambda)[n(1+\beta)+\emptyset(1+\beta)-(\theta+\beta)]\psi(n,\alpha)b_n}$$

3. Distortion Theorem

In the Theorem(2) , we obtain the distortion theorem of $f(w)\in H(\alpha,\beta,\theta,\lambda)$.

Theorem 2:

If $f(w)\in H(\alpha,\beta,\theta,\lambda)$, then

$$|w|-|w|^2\frac{(1+\emptyset-\theta)(3-\alpha)}{2(\lambda+1)[2(\beta+1)+\emptyset(\beta+1)-(\theta+\beta)b_2]}\leq |f(w)| \leq |w|+|w|^2\frac{(1+\emptyset-\theta)(3-\alpha)}{2(\lambda+1)[2(1+\beta)+\emptyset(1+\beta)-(\theta+\beta)b_2]}$$

Proof :
Since $|f(w)| \leq |w| + |w|^2 \sum_{n=2}^{\infty} a_n$
from (13) , we get

$$\sum_{n=2}^{\infty} a_n \leq \frac{(1+\emptyset-\theta)(3-\alpha)}{2(\lambda+1)[2(1+\beta)+\emptyset(1+\beta)-(\theta+\beta)b_2]}, \quad (15)$$

hence

$$|f(w)| \leq |w| + |w|^2 \frac{(1+\emptyset-\theta)(3-\alpha)}{2(1+\lambda)[2(\beta+1)+\emptyset(\beta+1)-(\theta+\beta)b_2]}.$$

Similarly , we get

$$|f(w)| \geq |w| - |w|^2 \frac{(1+\emptyset-\theta)(3-\alpha)}{2(1+\lambda)[2(1+\beta)+\emptyset(1+\beta)-(\theta+\beta)b_2]}.$$

Theorem(3) proves that the class $H(\alpha,\beta,\theta,\lambda)$ is closed under arithmetic mean and closed under convex linear combinations .
The function $f_k(w)$ is defined by

$$f_k(w) = w - \sum_{n=2}^{\infty} a_{n,k} w^n, (a_{n,k} \geq 0, n \in \mathbb{N}) \quad (16)$$

Theorem 3:
A function $f_k(w)$ in equation (16) is in the class $H(\alpha,\beta,\theta,\lambda)$ for (k = 1,2,...,m). Then the function

$$\Phi(w) = w - \sum_{n=2}^{\infty} c_n w^n, (c_n \geq 0, n \in \mathbb{N}) \quad (17)$$

is also in the class $H(\alpha,\beta,\theta,\lambda)$, where

$$c_n = \frac{1}{m}\sum_{k=1}^{m} a_{n,k} .$$

Proof :
A function $f_k(w) \in H(\alpha,\beta,\theta,\lambda)$, then from Theorem (1) , we get

$$\sum_{n=2}^{\infty} (1-\lambda+n\lambda)[n(1+\beta)+\emptyset(1+\beta)-(\theta+\beta)]\psi(n,\alpha)a_{n,k}b_n \leq 1+\emptyset-\theta.$$

Hence

$$\sum_{n=2}^{\infty} (1-\lambda+n\lambda)[n(1+\beta)+\emptyset(1+\beta)-(\theta+\beta)]\psi(n,\alpha)c_n b_n \leq 1+\emptyset-\theta.$$

$$\sum_{n=2}^{\infty} (1-\lambda+n\lambda)[n(1+\beta)+\emptyset(1+\beta)-(\theta+\beta)]\psi(n,\alpha)b_n \left[\frac{1}{m}\sum_{k=1}^{m} a_{n,k}\right] \leq 1+\emptyset-\theta.$$

The $\Phi(w) \in H(\alpha,\beta,\theta,\lambda)$.

Theorem 4:
The class $H(\alpha,\beta,\theta,\lambda)$ is closed under linear combinations .

Proof :
Let the function $f_k(w)(k = 1,2)$, defined by (16), be in the class $H(\alpha,\beta,\theta,\lambda)$. We show that the function $E(w) = \ell f_1(w) + (1-\ell) f_2(w), (0 \leq \ell \leq 1)$ is also in the class $H(\alpha,\beta,\theta,\lambda)$. Since $f_1(w) \in H(\alpha,\beta,\theta,\lambda)$, then from (13), we get

$$\sum_{n=2}^{\infty} (1-\lambda+n\lambda)[n(1+\beta)+\emptyset(1+\beta)-(\theta+\beta)]\psi(n,\alpha)a_{n,1}b_n \leq 1+\emptyset-\theta.$$

And, so, $f_2(w) \in H(\alpha,\beta,\theta,\lambda)$. Then from (13) we get

$$\sum_{n=2}^{\infty}(1-\lambda+n\lambda)[n(1+\beta)+\emptyset(1+\beta)-(\theta+\beta)]\psi(n,\alpha)a_{n,2}b_n \leq 1+\emptyset-\theta.$$

Then

$$E(w)=w-\sum_{n=2}^{\infty}[\ell a_{n,1}+(1-\ell)a_{n,2}]w^n.$$

Therefore, by Theorem 1 we have

$$\sum_{n=2}^{\infty}(1-\lambda+n\lambda)[n(1+\beta)+\emptyset(1+\beta)-(\theta+\beta)]\psi(n,\alpha)b_n[\ell a_{n,1}+(1-\ell)a_{n,2}] \leq 1+\emptyset-\theta.$$

Hence, by Theorem (1) we have $E(w) \in H(\alpha,\beta,\theta,\lambda)$.

Theo rem 5:

A function $f_k(w)$ of the from (16) is in the class $H(\alpha,\beta_k,\theta_k,\lambda_k)$, where $(0 \leq \theta_k < 1, \beta_k \geq 0, 0 < \alpha < 1, 0 \leq \lambda_k \leq 1, 0 \leq \emptyset < 1, n \geq 2)$, for each ($k$ = 1,2,...,m) , then the function

$$s(w)=w-\frac{1}{m}\sum_{n=2}^{\infty}\left[\sum_{k=1}^{m}a_{n,k}\right]w^n$$

is also in the class $H(\alpha,\beta,\theta,\lambda)$, where

$$\beta=\min_{1\leq k\leq m}\{\beta_k\},\quad \theta=\min_{1\leq k\leq m}\{\theta_k\},\quad \lambda=\min_{1\leq k\leq m}\{\lambda_k\} \text{ and } \emptyset=\min_{1\leq k\leq m}\{\emptyset_k\}$$

Proof :

Let the functions $f_k(w)$ $H(\alpha,\beta_k,\theta_k,\lambda_k)$, then from Theorem (1) we get

$$\sum_{n=2}^{\infty}(1-\lambda_k+n\lambda_k)[n(1+\beta_k)+\emptyset_k(1+\beta_k)-(\theta_k+\beta_k)]\psi(n,\alpha)a_{n,k}b_n \leq 1+\emptyset_k-\theta_k,$$

hence

$$\sum_{n=2}^{\infty}(1-\lambda_k+n\lambda_k)[n(1+\beta_k)+\emptyset_k(1+\beta_k)-(\theta_k+\beta_k)]\psi(n,\alpha)b_n\left[\frac{1}{m}\sum_{k=1}^{m}a_{n,k}\right]$$

$$\leq \frac{1}{m}\sum_{k=1}^{m}(1+\emptyset_k-\theta_k),$$

Therefore, $(w) \in H(\alpha,\beta,\theta,\lambda)$.

In the next two theorems we want to show the fractional integral and fractional derivative introduced by Srivastava[5- 10].

Theorem 6: Let the function $f(w)$ be in the class $H(\alpha,\beta,\theta,\lambda)$.

Then

$$|D_w^{-\alpha}f(w)| \leq \frac{1}{\Gamma(\alpha+2)}|w|^{\alpha+1}\left|1+\frac{2(1+\emptyset-\theta)}{(\alpha+2)(\lambda+1)[2(\beta+1)+\emptyset(\beta+1)-(\theta+\beta)]}|w|\right| \quad (18)$$

and

$$|D_w^{-\alpha}f(w)| \leq \frac{1}{\Gamma(2+\alpha)}|w|^{\alpha+1}\left|1-\frac{2(1+\emptyset-\theta)}{(2+\alpha)(1+\lambda)[2(1+\beta)+\emptyset(\beta+1)-(\theta+\beta)]}|w|\right| \quad (19)$$

$$\left|(1+\emptyset-\theta) \leq \frac{(2+\alpha)(\lambda+1)[2(\beta+1)+\emptyset(\beta+1)-(\theta+\beta)]}{2\Gamma(2+\alpha)}\right|$$

The last equalities in (18) and (19) are accomplished for the function

$$f(w) = w - \frac{(1+\emptyset-\theta)}{(1+\lambda)[2(1+\beta)+\emptyset(1+\beta)-(\theta+\beta)]}w^2.$$

Proof: By using the Theorem (1) , we have

$$\sum_{n=2}^{\infty} a_n \leq \frac{(1+\emptyset-\theta)}{(1+\lambda)[2(1+\beta)+\emptyset(1+\beta)-(\theta+\beta)]}. \quad (20)$$

From Definition (3), we get

$$D_w^{-\alpha} f(w) = \frac{1}{\Gamma(2+\alpha)} w^{1+\alpha} - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} a_n w^{n+\alpha}$$

and

$$\Gamma(2+\alpha) w^{-\alpha} D_w^{-\alpha} f(w) = w - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(\alpha+2)}{\Gamma(n+\alpha+1)} a_n w^n$$

$$= w - \sum_{n=2}^{\infty} \psi(n) a_n w^n \quad (21)$$

Where

$$\psi(n) = \frac{\Gamma(n+1)\Gamma(\alpha+2)}{\Gamma(n+\alpha+1)}.$$

We get $\psi(n)$ is a decreasing univalent function of n and $0 < \psi(n) \leq \psi(2) = \frac{2}{2+\alpha}$.

By using (20) and (21) , we get

$$|\Gamma(2+\alpha) w^{-\alpha} D_w^{-\alpha} f(w)| \leq |w| + \psi(2)|w|^2 \sum_{n=2}^{\infty} a_n$$

$$\leq |w| + \frac{2(1+\emptyset-\theta)}{(2+\alpha)(\lambda+1)[2(\beta+1)+\emptyset(\beta+1)-(\theta+\beta)]}|w|^2.$$

and

$$|\Gamma(2+\alpha) w^{-\alpha} D_w^{-\alpha} f(w)| \geq |w| - \psi(2)|w|^2 \sum_{n=2}^{\infty} a_n$$

$$\geq |w| - \frac{2(1+\emptyset-\theta)}{(2+\alpha)(1+\lambda)[2(1+\beta)+\emptyset(1+\beta)-(\theta+\beta)]}|w|^2.$$

The proof is complete.

Theorem 7: A function *f*(w) is in the c lass $H(\alpha, \beta, \theta, \lambda)$.

Then

$$|D_w^{\alpha} f(w)| \leq \frac{1}{\Gamma(2-\alpha)} |w|^{1-\alpha} \left[1 + \frac{2(1+\emptyset-\theta)}{(2-\alpha)(1+\lambda)[2(1+\beta)+\emptyset(\beta+1)-(\theta+\beta)]}|w|\right] \quad (22)$$

and

$$|D_w^{\alpha} f(w)| \geq \frac{1}{\Gamma(2-\alpha)} |w|^{1-\alpha} \left[1 - \frac{2(1+\emptyset-\theta)}{(2-\alpha)(1+\lambda)[2(1+\beta)+\emptyset(\beta+1)-(\theta+\beta)]}|w|\right] \quad (23)$$

$$\left[(1+\emptyset-\theta) \leq \frac{(2-\alpha)(1-\lambda)[2(\beta+1)+\emptyset(1+\beta)-(\theta+\beta)]}{2\Gamma(2-\alpha)}\right].$$

The equalities in (22) and (23) are accomplished for a univalent function

$$f(w) = w - \frac{(1+\emptyset-\theta)}{(1+\lambda)[2(1+\beta)+\emptyset(1+\beta)-(\theta+\beta)]}w^2.$$

Proof: By using Theorem (1) , we have

$$\sum_{n=2}^{\infty} n a_n \leq \frac{2(1+\emptyset-\theta)}{(\lambda+1)[2(\beta+1)+\emptyset(\beta+1)-(\theta+\beta)]}. \quad (24)$$

From Definition (2) , we obtain

$$D_w^\alpha f(w) = \frac{1}{\Gamma(2-\alpha)} w^{1-\alpha} - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} a_n w^{n-\alpha}$$

and

$$\Gamma(2-\alpha) w^\alpha D_w^\alpha f(w) = w - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n-\alpha+1)} a_n w^n$$

$$= w - \sum_{n=2}^{\infty} \frac{\Gamma(n)\Gamma(2-\alpha)}{\Gamma(n-\alpha+1)} n a_n w^n = w - \sum_{n=2}^{\infty} n\, \Phi(n) a_n w^n, \qquad (25)$$

since $\Phi(n) = \frac{\Gamma(n)\Gamma(2-\alpha)}{\Gamma(n-\alpha+1)}$

for know that $\Phi(n)$ is a decreasing univalent function of n and $0 < \Phi(n) \le \Phi(2) = \frac{2}{2-\alpha}$.

Using (24) and (25), we have

$$|\Gamma(2-\alpha) w^\alpha D_w^\alpha f(w)| \le |w| + \Phi(2)|w|^2 \sum_{n=2}^{\infty} n a_n$$

$$\le |w| + \frac{2(1+\emptyset-\theta)}{(2-\alpha)(\lambda+1)[2(\beta+1)+\emptyset(\beta+1)-(\theta+\beta)]} |w|^2,$$

we also have

$$|\Gamma(2-\alpha) w^\alpha D_w^\alpha f(w)| \le |w| - \Phi(2)|w|^2 \sum_{n=2}^{\infty} n a_n$$

$$\le |w| - \frac{2(1+\emptyset-\theta)}{(2-\alpha)(\lambda+1)[2(\beta+1)+\emptyset(\beta+1)-(\theta+\beta)]} |w|^2,$$

The proof is complete.

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