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Dynamical Behavior Effect of Environmental Toxin on the Prey-Predator Model

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Abstract:
In this research, we investigated the impact of toxins in a predator -prey paradigm where several externally harmful chemicals directly infect the prey. During feeding, the predator indirectly inherits the detrimental health. To investigate the impact of toxin-producing prey on the predator population, a mathematical model is made public. It is believed that the poison emitted by the prey population has a deleterious effect on the quantity of predators. In this paper the stability and Bifurcation, be modeled. As demonstrated by the numerical aspect addressed, the numerical solution showed that changing the parameters has an effect on the model. The results are explained by numerical simulations. The system is shown to have an extensive array of behaviours, including chaos. The effect of toxin is to show the stability of the system, or it causes the predator to go extinct. Increasing toxins have the potential to deplete symbiosis and wipe out entire populations.

Keywords: Bifurcation analysis (BA); Equilibrium point (Ep); predator-prey; Toxin effect; Stability.

السلوك الديناميكي لتاثير السموم البيئية على نموذج الفريسة والمفترس

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الخلاصة :
في هذا البحث قمنا بدراسة تاثير السموم في نموذج الفريسة والمفترس حيث تصيب العديد من المواد الكيمائية الضارة الخارجية الفريسه مباشرة اثناء التغذية , وتنتقل هذه السموم بشكل غير مباشر الى المفترس, وتم اقتراح نموذج رياضي لدراسة تاثير الفرائس المنتجة للسموم على اعداد الحيوانات المفترسة ويعتقد ان السم الذي تطلقه الفرائس له تاثير ضار . في هذا البحث تمت مناقشة الاستقرارية والتشعب ودراسة سلوكيات النظام الديناميكي باستخدام المحاكاة بما في ذلك تاثير السم على النموذج المقترح, وكشف الحل العددي أن تغيير المعلمات له تأثير على النموذج، كما يتضح من الجانب العددي. تُفسَّر النتائج من خلال عمليات محاكاة رقمية. وقد تبيّن أن النظام يتميز بمجموعة واسعة من السلوكيات، بما في ذلك الفوضى. ويُظهر تأثير السم

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استقرار النظام، أو أنه يتسبب في انقراض المفترس. وقد تُؤدي زيادة السموم إلى استنزاف التعايش والقضاء على مجموعات بأكملها.

1. Introduction

The study of system dynamics changes in prey-predator model or environmental is essential to ecological research because it provides interactions future vision or organismal. Traditional models frequently presume a perfect ecology free of outside effects. Becoming increasingly vulnerable to a number of concrete effects, such as the introduction of pollutants through human activity. These environmental toxins can have a major influence on both prey and predator populations. They can emit environmental pollutants, land runoff or other beginnings. The Toxins can have a direct impact on dynamical system. These dynamics are further complicated by indirect impacts, such as shifts in the availability of prey or the efficiency of predators.

The complex interactions knowledge is necessary for predicting how resilient ecosystems will be to pollution and for developing strategies to mitigate the negative impacts of environmental pollutants on biodiversity. The acceleration of global environmental changes is creating new challenges for the stability and sustainability of natural ecosystems, making this field of study more and more important. By including toxin dynamics into prey-predator models, ecologists and environmental scientists may more accurately forecast the potential long-term impacts of pollution and help develop effective conservation and management methods.

Prey species in predator-prey food systems are impacted by environmental toxins, which can change their mental and behavioral patterns and negatively affect their growth. Prey species' aversiveness to predators is influenced by their toxicity; the more toxic they are, the easier it is for them to evade predators and the fewer attacks they endure. Environmental pollutants have an impact on the populations and behaviors of prey and predator species, as well as the dynamics of the prey-predator interaction. Numerous predator-prey models, notably those involving amphibians and woodfrog tadpoles, have shown these effects. A mathematical model was developed and analyzed by Hallam et al., [1] and [2] to investigate how a toxicant affects the rates at which biological organisms grow. A model was presented by Shukla et al., [3] to investigate the combined effects of two distinct toxicants that are released from various external sources, etc. These effects have been demonstrated in a number of predator-prey models. Zhu et al., developed a model that considered the impact of toxins on interactions between three distinct predator-prey species, [4-9].

They discovered that parameters pertaining to the release and consumption of toxins had an impact on the stability of the system. Several of the most important studies are those by Huang and others [10], who focused on how toxicity affects a population's first-order kinetics. They suggested a terminal organismal concentration threshold value that, even in the presence of limiting toxicants, can regulate the population's pace of extinction. Zhang, Tian and others [11], [12] conducted more research on the existence of toxicity and adjusted the model to account for toxicity from both environmental and food chain pathways. Some researchers studied type of model helps in understanding how environmental factors like toxins and human activities like fishing can influence the long-term sustainability of fish populations and their ecosystems [13], [14]. Smith et al., [15] and [16], modified the prey-predator model to take into consideration the impact of ambient toxicants, exposing both the predator and the prey to the toxicants concurrently. However, most models pertaining to toxicants are generic in nature, and they do not account for specific aquatic ecosystems. For the purpose of assessing ecological risk and managing ecosystems responsibly, it is essential to comprehend how poisons affect ecological systems, as in [17], [18], and [19].

The sustainability of biodiversity depends on the establishment of fascinating links between toxicology and ecology. Toxinology should therefore only be viewed as a barrier when it is first discovered. In similar investigations, other researchers represented process in tritrophic food chain and food web models using different forms of function response (Holling types II, III, IV), among others [20 - 27].

In general, toxins have an enormous effect on how ecological systems behave and how they function. According to the toxinology notes, toxic animals can be researched like substances, and toxicity needs to be looked at from all angles that are important for survival, such as population control and parasites [28 - 33]. Tritrophic food systems have been the subject of some research, with an emphasis on how toxicants affect the system's ability to survive or go extinct [34 - 40].

In view of the previously mentioned, a tri-trophic food-web system is developed in this study. Effects of environmental toxins on multiple species: unlike earlier models that consider toxins affecting only prey or a single predator, this model incorporates toxin-induced mortality across all levels. This allows a more accurate perspective on the impacts of bioaccumulation in food chains.

The paper's outline can be summarized as follows: A detailed explanation of the model and its dimensionlessness is provided in section 2. Section 3 describes the equilibrium point and status. In section 4, the subject of local stability is covered. While the persistence of the model is discussed in Section 5. Furthermore, the basin of attractions for equilibrium sites is specified in Section 6. The local bifurcation (LB) is covered in Section 7. Section 8 deals with the model's simulation. Lastly, conclusions provided in the concluding section.

2. The mathematical formulation

The ecological model in this section, which consists of a food chain system with three species, can be mathematically described by the differential equations system that follows, which is formulated mathematically as follows: A poisonous substance produced by another outside source infects every species.

$$\begin{aligned}
\frac{dX}{dT} &= rX\left(1-\frac{X}{K}\right)-\frac{\alpha_1 XY}{1+m_1X}-\gamma_1X^3,\\
\frac{dY}{dT} &= \frac{e_1\alpha_1 XY}{1+m_1X}-\frac{\alpha_2 YZ}{1+m_2Y}-\gamma_2Y^2,\\
\frac{dZ}{dT} &= \frac{e_2\alpha_2 YZ}{1+m_2Y}-\gamma_3Z^2-dZ\,.
\end{aligned} \tag{1}$$

Where, $X(t), Y(t)$ and $Z(t)$ determine the size of the populations of the following at time t: prey, predator, and top predator. The parameters are considered to be all positive and are described by the following hypotheses based on the most recent Holling type-II functional response, where $X(0) \geq 0, Y(0) \geq 0$, and $Z(t) \geq 0$.

- The juvenile prey depends logistically on the mature prey for all sustenance; in the absence of the predator, it needs the mature prey's carrying capacity and necessary growth rate (r).
- In the meantime, the predator $Y(t)$, uses maximal attack rates of $\alpha_1 > 0$ and $\alpha_2 > 0$ to consume both adults and juvenile prey in accordance with Holling type II. As a result, both adult and juvenile prey with conversion rates of $0 < e_1 < 1$ and $0 < e_2 < 1$.
- m_1, m_2 are prey and predator preference rates, respectively.
- d is natural rates of top predator death.

- Finally, $\gamma_i, i = 1,2,3$ is thought to represent the toxicity of the top predator, predator, and mature and immature prey, respectively. The predator and top predator are indirectly exposed to the dangerous compounds through their diet, but the detrimental substances in the environment directly affect the prey species. As a result, poison has less of an impact on predator species than on prey species.

System (1) involves 12 parameters, this makes analysis challenging. Consequently, the set of parameters is whittled down to 8 using the non-dimensional variables and parameters listed below, yielding the model (2).

$$x = \frac{X}{K}, y = \frac{\alpha_1 Y}{r}, z = \frac{\alpha_2 Z}{r}, t = rT, \quad \xi_1 = km_1, \quad \xi_2 = \frac{\gamma_1 K^2}{r},$$

$$\xi_3 = \frac{e_1\alpha_1 K}{r}, \xi_4 = \frac{m_2 r}{\alpha_1}, \quad \xi_5 = \frac{\gamma_2}{\alpha_1}, \quad \xi_6 = \frac{e_2\alpha_2}{\alpha_1}, \xi_7 = \frac{\gamma_3}{\alpha_2}, \quad \xi_8 = \frac{d}{r}.$$

$$a_{11} = -x + \frac{\xi_1 xy}{(1+\xi_1 x)^2} - 2\xi_2 x^2 + (1-x) - \frac{y}{1+\xi_1 x} - \xi_2 x^2.$$

Consequently, the following is an outline of the non-dimensional system related to system (2):

$$\begin{aligned}
\frac{dx}{dt} &= x\left[(1-x) - \frac{y}{1+\xi_1 x} - \xi_2 x^2\right] = xf_1(x,y,z),\\
\frac{dy}{dt} &= y\left[\frac{\xi_3 x}{1+\xi_1 x} - \frac{z}{1+\xi_4 y} - \xi_5 y\right] = yf_2(x,y,z),\\
\frac{dz}{dt} &= z\left[\frac{\xi_6 y}{1+\xi_4 y} - \xi_7 z - \xi_8\right] = zf_3(x,y,z).
\end{aligned} \tag{2}$$

The interactivity functions are defined on $\mathbb{R}_+^3 = \{(x,y,z): x(t) \geq 0, y(t) \geq 0, z(t) \geq 0\}$. Additionally, because the interactive functions in system (2) have continuous partial derivatives, they are Lipschitzian functions. Consequently, a unique system (2) solution exists.

Theorem 2.1: The solutions to system (2) beginning in $\mathbb{R}_+^3$ are uniformly bounded**.**

Proof. From the first equation of the system (2), it is obtaining

$\frac{dx}{dt} \leq x - x^2$

Therefore, by resolving this differential inequality, the result is $x \leq 1$ as $t \to \infty$.

In a comparable way, it can be seen from second equation of system (2) that: $\frac{dy}{dt} \leq y - \xi_5 y^2$, which gives that $y \leq \frac{1}{\xi_5}$ as $t \to \infty$.

Now, $Q(t) = \xi_3 x(t) + y(t) + \frac{1}{\xi_6} z(t)$, then

$$\frac{dQ}{dt} \leq 2\xi_3 + \frac{1}{\xi_5} - \delta Q,$$

where $\delta = \{1, \xi_8\}$.

Consequently, for $t \to \infty$, straightforward computation yields that:

$$Q(t) \leq \frac{(2\xi_5\xi_3 + 1)}{\delta\xi_5}.$$

Therefore, every solution in the following domain is uniformly bounded.

3. The existence of eqilibiurm points (EPs)

There are four non-negative EPs; the points' forms and the conditions under which they exist are given below.

1. The point of trivial EP that is shown by $E_0 = (0,0,0)$ always exists.

2. The axial EP that is as shown by $E_1 = (\bar{x}, 0,0)$ exists when

$\bar{x} = \frac{-1+\sqrt{1+4\xi_2}}{2\xi_2}$.

3. The top predator free EP, is shown by $E_2 = (\bar{\bar{x}}, \bar{\bar{y}}, 0)$, represented by where $\bar{\bar{y}} = \frac{\xi_3 x}{\xi_5(1+\xi_1 x)}$, and $\bar{\bar{x}}$ is obtain by the fourth order polynomial equation where $A_o\lambda^4 + A_1\lambda^3 + A_2\lambda^2 + A_3\lambda + A_4 = 0$, exists if and only if the condition is met:

$2\xi_1 + \xi_2 > \xi_1^2$.

(3)

Where

$A_o = -\xi_2\xi_5\xi_1^2, A_1 = -[\xi_1^2\xi_5 + 2\xi_2\xi_1\xi_5], A_2 = -[-\xi_1^2\xi_5 + 2\xi_1\xi_5 + \xi_2\xi_5]$,
$A_3 = -[2\xi_1\xi_5 + \xi_3 + \xi_5], A_4 = \xi_5$.

4. EP $E_3 = (x^*, y^*, z^*)$, arises uniquely in the interior of $\mathbb{R}_+^3$.

$f_i(x, y, z) = 0$, where f_i , $i = 1,2,3$ are written in system (2) from $f_3(x, y, z) = 0$ we get

$$z^* = \frac{\xi_6 y - \xi_8(1+\xi_4 y)}{\xi_7(1+\xi_4 y)}, \text{ where } \xi_6 y > \xi_8(1 + \xi_4 y). \quad (4a)$$

However, in the interior of the first quadrant of the the $xy -$ plane, the point (x^*, y^*) denotes the specific place at which the following two isoclines cross:

$$v_1(x, y) = (1 - x) - \frac{y}{1+\xi_1 x} - \xi_2 x^2 \quad (4b)$$

$$v_2(x, y) = \frac{\xi_3 x}{1+\xi_1 x} - \frac{z}{1+\xi_4 y} - \xi_5 y \quad (4c)$$

Clearly, from Equation (4b) we get $y^* = (1 - x - \xi_2 x^2)(1 + \xi_1 x)$ as $x = 0$, the two isoclines become:

$$v_1(0, y) = 1, \quad (5a)$$

$$v_2(0, y) = \rho_1 y^3 + \rho_2 y^2 + \rho_3 y + \rho_4 = 0, \quad (5b)$$

where:

$\rho_1 = \xi_4^2\xi_5\xi_7$,
$\rho_2 = 2\,\xi_4\xi_5\xi_7$,
$\rho_3 = \xi_5\xi_7 + \xi_6 - \xi_4\xi_8$,
$\rho_4 = \xi_8$.

The following sufficient requirements have to be satisfied for each polynomial Equation (5a) and (5b) to be given a unique positive root:

$\xi_5\xi_7 + \xi_6 - \xi_4\xi_8 > 0$ or,
$\xi_5\xi_7 + \xi_6 - \xi_4\xi_8 < 0$. (6)

The point exists uniquely if, in addition to conditions (6). These conditions are satisfied: $\frac{dy}{dx} = -\left(\frac{\partial v_2(x,y)}{\partial x}\right) / \left(\frac{\partial v_2(x,y)}{\partial y}\right) < 0$.

4. Local stability analysis

In this part, the system's local stability study is examined using the linearization technique around the previously indicated equilibrium locations. At the position (*x, y, z*), the Jacobin matrix (Jm) of system (2) can be expressed as follows:

$$J=\begin{pmatrix} x\frac{\partial f_1}{\partial x}+f_1 & x\frac{\partial f_1}{\partial y} & x\frac{\partial f_1}{\partial z} \\ y\frac{\partial f_2}{\partial x} & y\frac{\partial f_2}{\partial y}+f_2 & y\frac{\partial f_2}{\partial z} \\ z\frac{\partial f_3}{\partial x} & z\frac{\partial f_3}{\partial y} & z\frac{\partial f_3}{\partial z}+f_3 \end{pmatrix}=\left(a_{ij}\right)_{3\times 3}, \quad (7)$$

where

$a_{11}=-x+\frac{\xi_1 xy}{(1+\xi_1 x)^2}-2\xi_2 x^2+(1-x)-\frac{y}{1+\xi_1 x}-\xi_2 x^2$, $a_{12}=\frac{-x}{1+\xi_1 x}$, $a_{13}=0$, $a_{21}=\frac{\xi_3 y}{(1+\xi_1 x)^2}$, $a_{22}=\frac{\xi_4 zy}{(1+\xi_4 y)^2}-\xi_5 y+\frac{\xi_3 x}{1+\xi_1 x}-\frac{z}{1+\xi_4 y}-\xi_5 y$, $a_{23}=\frac{-y}{1+\xi_4 y}$,

$a_{31}=0$, $a_{32}=\frac{\xi_6 z}{(1+\xi_4 y)^2}$, $a_{33}=-2\xi_7 z+\frac{\xi_6 y}{1+\xi_4 y}-\xi_8$.

Therefore,

- The Jm at E_0 has eigenvalue $\lambda_{01}=1$, $\lambda_{02}=0$, $\lambda_{03}=-\xi_8$, since there is a positive Eq hence E_0 is unstable non-hyperbolic point.
- The Jm at E_1 has the eigenvalues $\lambda_{11}=-(1+2\xi_2 x)x$, $\lambda_{12}=\frac{\xi_3 x}{1+\xi_1 x}$, and $\lambda_{13}=-\xi_8$, hence is a saddle point.
- The Jm at E_2 is reduced to:

$$J_{e_2}=\left(b_{ij}\right)_{3x3}, \quad (8a)$$

where

$b_{11}=-x+\frac{\xi_1 xy}{(1+\xi_1 x)^2}-2\xi_2 x^2$, $b_{12}=\frac{-x}{1+\xi_1 x}$, $b_{13}=0$,

$b_{21}=\frac{\xi_3 y}{(1+\xi_1 x)^2}$, $b_{22}=-\xi_5 y$, $b_{23}=\frac{-y}{1+\xi_4 y}$,

$b_{31}=0$, $b_{32}=0$, $b_{33}=\frac{\xi_6 y}{1+\xi_4 y}-\xi_8$.

Then, the characteristic formula of J_{e_2} can written as;

$$[\lambda^2-(b_{11}+b_{22})\lambda+(b_{11}b_{22}-b_{12}b_{21}](b_{33}-\lambda)=0. \quad (8b)$$

Clearly, Eigen values of the Equation (8b) can written as:

$$\lambda_{21}=\frac{(b_{11}+b_{22})+\sqrt{(b_{11}+b_{22})^2-4(b_{11}b_{22}-b_{12}b_{21})}}{2},\ \lambda_{22}=\frac{(b_{11}+b_{22})-\sqrt{(b_{11}+b_{22})^2-4(b_{11}b_{22}-b_{12}b_{21})}}{2},$$

If the following conditions are met, it is simple to confirm that J_{e_2} has all negative eigenvalues and that E_2 is thus locally asymptotically stable.

$$\frac{\xi_6 y}{1+\xi_4 y}<\xi_8. \quad (9a)$$

$$1+2\xi_2 x>\frac{\xi_1 y}{(1+\xi_1 x)^2} \quad (9b)$$

$$(\xi_5+2\xi_2\xi_5 x)(1+\xi_1 x)^2+\frac{\xi_3}{(1+\xi_1 x)}<\xi_1\xi_5 y \quad (9c)$$

- The Jm of the system (2) at E_3 the can be written as:

$$J_{E_3}=\left(c_{ij}\right)_{3x3}, \quad (10)$$

where

$c_{11}=-x^*+\frac{\xi_1 x^* y^*}{(1+\xi_1 x^*)^2}-2\xi_2 x^{*2}$, $c_{12}=\frac{-x^*}{1+\xi_1 x^*}$, $c_{13}=0$, $c_{21}=\frac{\xi_3 y^*}{(1+\xi_1 x^*)^2}$,

$c_{22}=\frac{\xi_4 z^* y^*}{(1+\xi_4 y^*)^2}-\xi_5 y^*$, $c_{23}=\frac{-y^*}{1+\xi_4 y^*}$, $c_{31}=0$, $c_{32}=\frac{\xi_6 z^* y^*}{(1+\xi_4 y^*)^2}$, $c_{33}=-\xi_7 z^*$.

Thus, the characteristic equation of J_{E_3} can be expressed as:

$$\lambda^3+A_1\lambda^2+A_2\lambda+A_3=0, \quad (11)$$

where
$A_1 = -(c_{11} + c_{22} + c_{33})$, $A_2 = c_{11}c_{22} + c_{11}c_{33} + c_{22}c_{33} - c_{12}c_{21} - c_{23}c_{32}$, and $A_3 = [c_{32}c_{11}c_{23} + c_{12}\, c_{21}c_{33} - c_{11}c_{22}c_{33}]$,with $\Delta = A_1A_2 - A_3 = -(c_{11} + c_{22} + c_{33})(c_{11}c_{22} + c_{11}c_{33} + c_{22}c_{33} - c_{23}c_{32} - c_{12}c_{21}) - (c_{11}c_{32}c_{23} + c_{12}\, c_{21}c_{33} - c_{11}c_{22}c_{33})$.

As a result, the E_3 becomes locally asymptotically stable When the real parts of each of the eigenvalues in the characteristic Equation (11) are negative, in accordance with the Routh Hurwitz Criterion, $A_1 > 0$, $A_3 > 0$, and $\Delta > 0$. Accordingly, the following condition

$$1 + 2\xi_2 x^* > \frac{\xi_1 y^*}{(1+\xi_1 x^*)^2}. \tag{12}$$

5. Persistence

The system (2) persistence is examined in this section. Since it is commonly accepted that a system can only be said to persist if none of its species are extinct, system (2) can only continue to exist if its trajectory begins on boundary planes where the omega limit is not reached.

One subsystem of system (2) is located in a positive of the xy −plane and is expressed as follows:

$$\begin{aligned} \frac{dx}{dt} &= x\left[(1-x) - \frac{y}{1+\xi_1 y} - \xi_2 x^2\right] = \mu_1\,(x,y), \\ \frac{dy}{dt} &= y\left[\frac{\xi_3 x}{1+\xi_1 x} - \xi_5 y\right] = \mu_2\,(x,y). \end{aligned} \tag{13}$$

It is simple to confirm that the subsystem has positive EP inside the positive quadrant of the xy −plane, which coincides with system (2). The Dulac function approach is now applied to determine whether periodic dynamics could exist around interior positive point of (13).

Define $\vartheta(x,y) = \frac{1}{xy}$. This is obviously continuously differentiable within the positive quadrant of the xy − plane. $\vartheta(x,y) > 0$, for all $(x,y) \in \mathbb{R}_+^2$.
Furthermore, direct computation gives that
$\Delta(x,y) = \frac{\partial}{\partial x}(\vartheta \cdot \mu_1) + \frac{\partial}{\partial y}(\vartheta \cdot \mu_2) = -\frac{1}{y} + \frac{\xi_1}{(1+\xi_1 x)^2} - \frac{2\xi_2 x}{y} - \frac{\xi_5}{x}$.
Therefore, $\Delta(x,y)$ is does not change sign, not identically to zero under this condition:

$$\frac{\xi_1}{(1+\xi_1 x)^2} < \frac{2\xi_2 x}{(1+\xi_1 x)^2} + \frac{\xi_5}{x} + \frac{1}{y}. \tag{14}$$

Observe that anytime the subsystem has locally asymptotically stable in interior of $\mathbb{R}_+^2$, it indicates that subsystem (14) is globally asymptotically stable in the interior of the positive quadrant of the xy −plane. As a result, the Dulac technique states that the subsystem's positive quadrant of the xy −plane has no periodic dynamics.

Theorem 5.1: Assuming that the planes of boundaries do not exhibit Periodic dynamics, then the system (2) is uniformly persistent as long as the criteria are met.

$$\frac{\xi_6 y}{1+\xi_4 y} > \xi_8. \tag{15a}$$

Proof: Define $\phi(x,y,z) = x^{p_1}y^{p_2}z^{p_3}$, where p_1, p_2, p_3 are positive constants, and $\phi(x,y,z) > 0$ for $(x,y,z) \in Int\,\mathbb{R}_+^3$ with $\phi(x,y,z) = 0$ if one of $x, y,$ and z approaches zero. Thus, direct calculation yields:

$\Omega(x,y,z)=\frac{\phi'(x,y,z)}{\phi(x,y,z)}=p_1f_1+p_2f_2+p_3f_3,$

where the system (2) contains the functions $f_i; i=1,2,3$. This is now the proof in terms of the average Lyapunov method, assuming that $\Omega(x,y,z)>0$ for all border equilibrium sites. Therefore,

$$\Omega(x,y,z)=p_1\left[(1-x)-\frac{y}{(1+\xi_1x)}-\xi_2x^2\right]$$
$$+p_2\left[\frac{\xi_3x}{1+\xi_1x}-\frac{z}{1+\xi_4x}-\xi_5y\right]+p_3\left[\frac{\xi_6y}{1+\xi_4y}-\xi_7z-\xi_8\right].$$

We have that

$\Omega(E_0)=p_1[1]+p_2[0]+p_3[-\xi_8].$

It is determined that $\Omega(E_0)>0$ by selecting an arbitrary positive value for p_1 that is sufficiently large in relation to p_3.

$\Omega(E_1)=p_1[0]+p_2\left[\frac{\xi_3x}{1+\xi_1x}\right]+p_3[-\xi_8].$

By suitable chose of the parameters p_3, so that p_2 is suitable to be large in comparison to p_3, then obtain that $\Omega(E_1)>0$.

And, we have:

$\Omega(E_2)=p_3\left[\frac{\xi_6y}{1+\xi_4y}-\xi_8\right].$

Clearly, the condition (15a) assurances that $\Omega(E_2)>0$.

Hence, the system (2) uniformly persistent.

6. Globally stability

This part delves deeper into the dynamics of system (2) by utilizing the Lyapunov function. Determining the basin of attraction for the non-hyperbolic point and the locally asymptotically stable EPs is the goal.

Theorem 6.1: The first axial EP $E_1=(\bar{x},0,0)$ of system (2) is globally asymptotically stable if the condition is hold:

$$1>\frac{\xi_3\bar{x}}{1+\xi_1\bar{x}}. \quad (16)$$

Proof: Consider the following scalar function $N_1=(x-\bar{x}-\bar{x}\ln\frac{x}{\bar{x}})+\frac{1}{\xi_3}y+\frac{1}{\xi_3\xi_6}z.$

It is clear that $N_1:\mathbb{R}^3_+\to\mathbb{R}$, so that $N_1(E_1)=0$ and $E_1(x.y.z)>0$ for $\{(x,y,z)\in\mathbb{R}^3_+: x>0, y\geq 0, z\geq 0, (x,y,z)\neq E_1\}$. Hence, N_1 is a positive definite function. Now, by differential N_1 with respect to time and simplify the result it is obtain that

$$\frac{dN_1}{dt}=-[1+\xi_2(x+\bar{x})](x-\bar{x})^2+\frac{\bar{x}y}{1+\xi_1\bar{x}}-\frac{\xi_5}{\xi_3}y^2-\frac{\xi_7}{\xi_3\xi_6}z^2-\frac{\xi_8}{\xi_3\xi_6}z.$$

Hence,

$$\frac{dN_1}{dt}\leq-[1+\xi_2(x+\bar{x})](x-\bar{x})^2-\frac{1}{\xi_3\xi_5}\left[1-\frac{\xi_3\bar{x}}{1+\xi_1x}\right]-\frac{\xi_7}{\xi_3\xi_6}z^2-\frac{\xi_8}{\xi_3\xi_6}z.$$

Therefore, using the conditions (16) leads to $\frac{dN_1}{dt}$ is a negative definite. Accordingly, the function N_1 is strong Lyapunov function, hence the E_1 is globally asymptotically stable.

Theorem 6.2: In the interior of the $\mathbb{R}^3_+$ sub-region, there is $E_2=(\bar{\bar{x}},\bar{\bar{y}},0)$ that is asymptotically stable and meets the following requirements:

$$1+\ \xi_2(x+\ \bar{\bar{x}})>\ \frac{\xi_1\,\bar{\bar{y}}}{(1+\xi_1 x)(1+\xi_1\bar{\bar{x}})}\,, \tag{17a}$$

$$\frac{\xi_8}{\xi_6}>\ \frac{\bar{\bar{y}}}{(1+\xi_4 y)}, \tag{17b}$$

$$(\alpha_{12})^2<4\ \alpha_{11}\alpha_{22}, \tag{17c}$$

where all the symbols $\alpha_{ij};\ \ i\,,j=1{,}2{,}3$ is given in the proof.

Proof: Let the scalar function $N_2=\left(x-\ \bar{\bar{x}}-\bar{\bar{x}}\ln\frac{x}{\bar{\bar{x}}}\right)+\left(y-\ \bar{\bar{y}}-\bar{\bar{y}}\ln\frac{y}{\bar{\bar{y}}}\right)+\frac{1}{\xi_6}\ z.$

It is clear that $N_2\colon\mathbb{R}_+{}^3\to\mathbb{R}$, so that $N_2(E_2)=0$, and $N_2(x,y,z)>0$ for any $\{(x,y,z)\in\mathbb{R}^3_+\colon \mathrm{x}>0,\ \mathrm{y}>0,\ \mathrm{z}\geq 0,\ (\mathrm{x,y,z})\neq E_2\}$. Hence, the function N_2 is positive function.

Now, by differentiate N_2 with respect to time, and then simplify the result, it is obtained that

$$\frac{dN_2}{dt}=-\alpha_{11}(x-\ \bar{\bar{x}})^2+\alpha_{12}(x-\ \bar{\bar{x}})(y-\ \bar{\bar{y}})-\xi_5\,(y-\ \bar{\bar{y}})^2$$
$$-\left(\frac{\xi_8}{\xi_6}-\frac{\bar{\bar{y}}}{(1+\xi_4 y)}\right)z-\frac{\xi_7}{\xi_6}z^2,$$

where $\alpha_{11}=\left(1+\ \xi_2(x+\ \bar{\bar{x}})-\ \frac{\xi_1\,\bar{\bar{y}}}{(1+\xi_1 x)(1+\xi_1\bar{\bar{x}})}\right),\alpha_{12}=\frac{1+\ \xi_1\,\bar{\bar{x}}-\xi_3}{(1+\xi_1 x)(1+\xi_1\bar{\bar{x}})}$.

Hence, using the above conditions (17a) - (17c) gives

$$\frac{dN_2}{dt}\leq-\left[\sqrt{\alpha_{11}}\,(x-\ \bar{\bar{x}})+\sqrt{\xi_5}\ (y-\ \bar{\bar{y}})\right]^2-\left(\frac{\xi_8}{\xi_6}-\frac{\bar{\bar{y}}}{(1+\xi_4 y)}\right)z-\frac{\xi_7}{\xi_6}z^2.$$

N_2 is a strong Lyapunov function since it is evident that $\frac{dN_2}{dt}$ is negative definite. The E_2 is therefore asymptotically stable in the area that meets the aforementioned set of require.

Theorem 6.3: Let us assume that $E_3=(x^*,y^*,z^*)$ is a locally asymptotically stable EP, then it has a basin of attraction in the interior of $R_+{}^3$ that satisfy the conditions:

$$1+\ \xi_2(x+\ x^*)>\ \frac{\xi_1\,y^*}{(1+\xi_1 x)(1+\xi_1 x^*)}\,, \tag{18a}$$

$$\xi_5>\frac{\xi_4 z^*}{(1+\xi_4 y^*)\,(1+\xi_4 y)}, \tag{18b}$$

$$(\mu_{12})^2<2\mu_{11}\mu_{22}, \tag{18c}$$

$$(\mu_{23})^2<2\mu_{22}\mu_{33}, \tag{18d}$$

where all the symbols $\mu_{ij}; i,j=1{,}2{,}3$ are given in proof.

Proof: Let the following scalar function

$$N_3=\left(x-\ x^*-x^*\ln\frac{x}{x^*}\right)+\left(y-\ y^*-y^*\ln\frac{y}{y^*}\right)+\left(z-\ z^*-z^*\ln\frac{z}{z^*}\right).$$

It is clear that $N_3\colon\mathbb{R}_+{}^3\to\mathbb{R}$, so that $N_3(E_3)=0$, and $N_3(x,y,z)>0$ for any $\{(x,y,z)\in\mathbb{R}^3_+\colon \mathrm{x,y,z}>0,\ (\mathrm{x,y,z})\neq E_3\}$. Hence, the function N_3 is positive definite function. Now by differentiate N_3 with respect to time, and then simplify the result, it is obtaining that:

$$\frac{dN_3}{dt}\leq-\mu_{11}(x-\ x^*)^2-\mu_{12}(x\ -\ x^*)(y\ -\ y^*)-\mu_{22}\frac{(y\ -\ y^*)^2}{2}$$
$$-\mu_{22}\frac{(y\ -\ y^*)^2}{2}-\mu_{23}(z-\ z^*)(y\ -\ y^*)-\mu_{33}(z-\ z^*)^2,$$

where $\mu_{11}=1+\ \xi_2(x+\ x^*)-\ \frac{\xi_1\,y^*}{(1+\xi_1 x)(1+\xi_1 x^*)},\mu_{22}=\xi_5-\frac{\xi_4 z^*}{(1+\xi_4 y^*)\,(1+\xi_4 y)}$,

$\mu_{12}=\left(\frac{1}{(1+\xi_1 x)}-\frac{\xi_3}{(1+\xi_1 x)(1+\xi_1 x^*)}\right),\mu_{23}=\left(\frac{1}{(1+\xi_4 y)}-\frac{\xi_6}{(1+\xi_4 y)(1+\xi_4 y^*)}\right),\mu_{33}=\xi_7.$

Hence, using the above conditions (18a) - (18d) gives

$$\frac{dN_3}{dt}\leq-\left[\sqrt{\mu_{11}}(x-\ x^*)+\sqrt{\frac{\mu_{22}}{2}}(y\ -\ y^*)\right]^2-\left[\sqrt{\frac{\mu_{22}}{2}}(y\ -\ y^*)+\sqrt{\mu_{33}}(z-\ z^*)\right]^2.$$

N_3 is a strong Lyapunov function in the Sub region of $\mathbb{R}^3_+$: that satisfies the requirements (18a)–(18d) since it is evident that $\frac{dN_3}{dt}$ is negative definite. As a result, for any trajectory that begins at a point within the region and satisfies the aforementioned set of requirements, E_3 is asymptotically stable.

7. Local bifurcation analysis (LBA)

In this section, the Sotomayor's theorem for LB is used to study the sensitivity of the dynamical behavior around the Locally asymptotically stable of the (2) to changes in a particular parameter. It is not a sufficient requirement because LB can only occur when a non-hyperbolic EP exists. This parameter is chosen to ensure that, for a given value of the suggested Bifurcation parameter, the EP is non-hyperbolic.

Next, rewrite system (2) as follows:

$$\frac{dH}{dt} = F(X), X = (x, y, z)^T, \text{ and } F = (xf_1, yf_2, zf_3)^T. \quad (19)$$

Then the second derivative of F with respect to Y can be written as:

$$D^2F(X)(\wp, \wp) = [\mathfrak{H}_{i1}]_{3\times1}, \quad (20)$$

where $\wp = (o_1, o_2, o_3)^T$ be anon zero real vector, with

$$\mathfrak{H}_{11} = \left[-2 - 6\xi_2 x + \frac{2\xi_1 y}{(1+\xi_1 x)^3}\right] {o_1}^2 - 2\left[\frac{1}{(1+\xi_1 x)^2}\right] o_1 o_2,$$

$$\mathfrak{H}_{21} = \left[\frac{-2\xi_1\xi_3 y}{(1+\xi_1 x)^3}\right] {o_1}^2 + 2\left[\frac{\xi_3}{(1+\xi_1 x)^2}\right] o_1 o_2 + \left[\frac{2\xi_4 z}{(1+\xi_4 y)^3} - 2\xi_5\right] {o_2}^2 - 2\left[\frac{1}{(1+\xi_4 y)^2}\right] o_2 o_3,$$

$$\mathfrak{H}_{31} = \left[-\frac{2\xi_4\xi_6 z}{(1+\xi_4 y)^2}\right] {o_2}^2 + \left[\frac{2\xi_6}{(1+\xi_4 y)^2}\right] o_2 o_3 - 2\xi_7 {o_3}^2.$$

The resulting theorems examine the possibility of LB within the system by utilizing the computation (2) mentioned earlier.

Theorem 7.1: A saddle node bifurcation of the system (2) at EP E_1 happens whenever ξ_2 passes over the value $\xi_2^* = \frac{1-2x}{3x^2}$.

Proof: At first axial equiliburm point with ξ_2^*, the Jm of the (2) is expressed as:

$$J_1 = J(E_1, \xi_2^*) = \begin{pmatrix} -2x + 1 - 3\xi_2 x^2 & \frac{-x}{1+\xi_1 x} & 0 \\ 0 & \frac{-\xi_3 y}{1+\xi_1 x} & 0 \\ 0 & 0 & -\xi_8 \end{pmatrix}.$$

In this matrix, we have two eigenvalues have negative real part, and the third eigenvalue is zero, $\lambda_{11}^* = 0$.Thus, first axial EP is a non-hyperbolic point at ξ_2^*.

Let $\aleph_1 = (n_{11}, n_{12}, n_{13})^T$ is eigenvector conjugate to the eigenvalue $\lambda_{11}^* = 0$.

So, $J_1\aleph_1 = 0$, gives that $\aleph_1 = (n_{11}, 0,0)^T$, and $n_{11} \neq 0$ is any real number.

Now, let $\Theta_1 = (\vartheta_{11}, \vartheta_{12}, \vartheta_{13})^T$ represents the eigenvector conjugate with the eigenvalue $\lambda_{11}^* = 0$, of the matrix ${J_1}^T$.

Thus, ${J_1}^T\Theta_1 = 0$ we have $\Theta_1 = (-\frac{a_{22}}{a_{11}}\vartheta_{12}, \vartheta_{12}, 0)^T$, with $\vartheta_{12} \neq 0$ is any real number.

Then by Sotomayor theorem, given that

$\frac{\partial F}{\partial \xi_2} = F_{\xi_2}(X, \xi_2) = (-x^3, 0, 0)^T$.

Therefore, ${\Theta_1}^T F_{\xi_2^*}(E_1, \xi_2^*) = \frac{a_{22}}{a_{12}} x^3 \vartheta_{12} \neq 0$, as a result, the first condition for the occurrence of saddle node bifurcation is met. Moreover, since

$$DF_{\xi_2}(X,\xi_2)=\begin{pmatrix}-3x^2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\Longrightarrow DF_{\xi_2}(E_1,{\xi_2}^*)\aleph_1=(-3x^2\, n_{11},0,0)^T.$$

Then, ${\Theta_1}^T DF_{\xi_2}(E,{\xi_2}^*)\aleph_1 = 3x^2\frac{a_{22}}{a_{11}}\,\vartheta_{12}n_{11}\;\neq 0.$

Also, by using Equation (20), it is obtained that

$$D^2F(E_1,{\xi_2}^*)(\aleph_1,\aleph_1)=\begin{pmatrix}(-2-6\xi_2 x)\,{\vartheta_{11}}^3\\ 0\\ 0\end{pmatrix}.$$

Accordingly, the following is obtained:

${\Theta_1}^T D^2F(E_1,{\xi_2}^*)(\aleph_1,\aleph_1)=\frac{a_{22}}{a_{12}}\,\vartheta_{12}(2+6\xi_2 x)\,{\vartheta_{11}}^3\;\;\neq 0.$

Then, a sable node Bifurcation take place.

Theorem 7.2: The saddle node bifurcation of the system (2) at the EP E_2 happens if ξ_8 processes the amount ${\xi_8}^{**}=\frac{\xi_6\, y}{1+\xi_4 y}$, provided that the following condition holds.

$$\left(\frac{2\xi_6}{(1+\xi_4 y)^2}-2\xi_7\left(\frac{a_{12}\,a_{21}}{a_{11}\,a_{23}}-\frac{a_{22}}{a_{23}}\right)\right)\left(\frac{a_{12}\,a_{21}}{a_{11}\,a_{23}}-\frac{a_{22}}{a_{23}}\right)(n_{22})^2\vartheta_{23}\neq 0. \qquad (21)$$

Proof: The Jm at $(E_2,{\xi_8}^{**})$ is determined by:

$J_2=J(E_2,{\xi_8}^{**})=a_{ij}=$

$$\begin{pmatrix}1-2x+\frac{\xi_1 xy}{(1+\xi_1 x)^2}-2\xi_2 x^2-\frac{y}{1+\xi_1 x} & \frac{-x}{1+\xi_1 x} & 0\\ \frac{\xi_3 y}{(1+\xi_1 x)^2} & -2\xi_5 y+\frac{\xi_3\, x}{1+\xi_1 x} & \frac{-y}{1+\xi_4 y}\\ 0 & 0 & -\xi_8+\frac{\xi_6 y}{1+\xi_4 y}\end{pmatrix}.$$

Two of the eigenvalues clearly have negative real parts under condition (9a), whereas the third eigenvalue, designated ${\lambda_{33}}^{**}=0$, is zero. The non-hyperbolic point at ${\xi_8}^{**}$ is hence the top predator.

Let $\aleph_2=(n_{21},n_{22},n_{23})^T$ be an eigenvector conjugate to the eigenvalue ${\lambda_{33}}^{**}=0$.

Hence, $J_2\,\aleph_2=0$, gives that $\aleph_2=\left(-\frac{a_{12}}{a_{11}}n_{22},n_{22},\left(\frac{a_{12}\,a_{21}}{a_{11}\,a_{23}}-\frac{a_{22}}{a_{23}}\right)n_{22}\right)^T$, with $n_{22}\neq 0$ is any real number.

Now, let $\Theta_2=(\vartheta_{21},\vartheta_{22},\vartheta_{23})^T$ be an eigenvector conjugate to the eigenvalue ${\lambda_{33}}^{**}=0$ of the matrix ${J_2}^T$.

Thus, ${J_2}^T\Theta_2=0$ we have $\Theta_2=(0,0,\vartheta_{23})^T$, with $\vartheta_{23}\neq 0$ is any real number.

Now, since:

$\frac{\partial F}{\partial\,\xi_8}=F_{\xi_8}(X,\xi_8)=(0,0\,,-\mathrm{z})^T\Rightarrow\frac{\partial F}{\partial\xi_8}=F_{\xi_8}(E_2,{\xi_8}^{**})=(0,0\,,0)^T.$

Therefore, ${\Theta_2}^T F_{\xi_8}(E_2,{\xi_8}^{**})=0$, then have the firs condition occurrence of saddle node bifurcation is met. Moreover, since

$$DF_{\xi_8}(X,\xi_8)=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}\Rightarrow DF_{\xi_8}(\varepsilon_2,{\xi_8}^{**})\mathcal{L}_2=\left(0,0,-\left(\frac{a_{12}\,a_{21}}{a_{11}\,a_{23}}-\frac{a_{22}}{a_{23}}\right)\ell_{22}\right)^T.$$

Then, ${\Theta_2}^T DF_{\xi_8}(E_2,{\xi_8}^{**})\aleph_2=-\left(\frac{a_{12}\,a_{21}}{a_{11}\,a_{23}}-\frac{a_{22}}{a_{23}}\right)\vartheta_{23}n_{22}\neq 0.$

Also, by using Equation (46), it is obtained that:

$D^2F(\varepsilon_2,{\xi_8}^{**})(\aleph_2,\aleph_2) =$

$$\begin{pmatrix} \left[-2-6\xi_2\bar{x}+\frac{2\xi_1\bar{y}}{(1+\xi_1\bar{y})^3}\right]\left(\frac{a_{12}}{a_{11}}\,n_{22}\right)^2-2\left[\frac{1}{(1+\xi_1\bar{x})^2}\right]\left(\frac{a_{12}}{a_{11}}n_{22}\right)^2 \\ \left[\frac{-2\xi_1\xi_3\bar{y}}{(1+\xi_1\bar{x})^3}\right]\left(\frac{a_{12}}{a_{11}}n_{22}\right)^2+2\left[\frac{\xi_3}{(1+\xi_1\bar{x})^2}\right]\left(\frac{a_{12}}{a_{11}}n_{22}\right)^2 \\ +[-2\,\xi_5]{n_{22}}^2\ -2\left[\frac{1}{(1+\xi_4\bar{y})^2}\right]\left(\left(\frac{a_{12}\,a_{21}}{a_{11}\,a_{23}}-\frac{a_{22}}{a_{23}}\right)n_{22}\right)^2 \\ \left[\frac{2\xi_6}{(1+\xi_4\bar{y})^2}\right]\left(\left(\frac{a_{12}\,a_{21}}{a_{11}\,a_{23}}-\frac{a_{22}}{a_{23}}\right)n_{22}\right)^2-2\xi_7\left(\left(\frac{a_{12}\,a_{21}}{a_{11}\,a_{23}}-\frac{a_{22}}{a_{23}}\right)n_{22}\right)^2 \end{pmatrix}.$$

Accordingly, the following is obtained:

$${\Theta_2}^T D^2F(E_2,{\xi_8}^{**})(\aleph_2,\aleph_2)=\left(\frac{2\xi_6}{(1+\xi_4 y)^2}-2\xi_7\left(\frac{a_{12}\,a_{21}}{a_{11}\,a_{23}}-\frac{a_{22}}{a_{23}}\right)\right)\left(\frac{a_{12}\,a_{21}}{a_{11}\,a_{23}}-\frac{a_{22}}{a_{23}}\right)(n_{22})^2\vartheta_{23}\neq 0.$$

Then, a saddle node bifurcation take place.

Theorem 7.3: The saddle node Bifurcation of (2) at E_3 is happens when ξ_8 passes over the value ${\xi_8}^*=-2\xi_7 z^*+\frac{\xi_6\,y^*}{1+\xi_4 y^*}-\frac{c_{11}c_{23}c_{32}}{(c_{11}c_{22}-c_{12}c_{21})}$, iff the resulting condition is fulfilled.

$$(\rho_1\varphi_{11}^*+\rho_2\varphi_{21}^*+\varphi_{31}^*)\neq 0, \tag{22}$$

where the symbols of (22) are computed in proof.

Proof: The Jm at E_3 and $\xi_8={\xi_8}^*$ can be written as:

$$J_3=J(E_3,{\xi_8}^*)=\begin{pmatrix} d_{11} & d_{12} & 0 \\ d_{21} & d_{22} & d_{23} \\ 0 & d_{32} & {d_{33}}^* \end{pmatrix}.$$

The Jm (10) has the entries $d_{ij}, i,j=1,2,3$, where ${d_{33}}^*=d_{33}({\xi_8}^*)$, the J_3 determinant equals 0, as demonstrated by direct computation. Thus, ${\lambda_{33}}^*=0$ gives the zero eigenvalue of the matrix J_3, and the two real parts eigenvalues are negative.

Let $\aleph_3=(n_{31},n_{32},n_{33})^T$ be an eigenvector conjugate to the eigenvalue $\lambda^*=0$. Thus, $J_3\aleph_3=0$, gives that $\aleph_3=(H_2n_{33},H_1n_{33},n_{33})^T$, where $n_{33}\neq 0$ be any real number, and $H_1=\frac{-d_{33}}{d_{32}}$, while $H_2=\frac{d_{22}d_{33}-d_{23}d_{32}}{d_{21}d_{32}}$.

Now, let $\Theta_3=(\vartheta_{31},\vartheta_{32},\vartheta_{33})^T$ be an eigenvector conjugate to the eigenvalue $\lambda^*=0$ of the matrix ${J_3}^T$. Thus, ${J_3}^T\Theta_3=0$, we have $\Theta_3=(\rho_1\vartheta_{33},\rho_2\vartheta_{33},\vartheta_{33})^T$, where $\epsilon_{33}\neq 0$ be any real number , and $\rho_1=\frac{d_{22}d_{33}-d_{23}d_{32}}{d_{12}d_{23}}$, while $\rho_2=\frac{-d_{33}}{d_{23}}$.

Then by using he Sotomoyar's theorem, we obtain

$$\frac{\partial F}{\partial\xi_8}=F_{\xi_8}(X,\xi_8)=(0,0,-\mathrm{z})^T,$$

where F_{ξ_8} is given in Equation (19). Therefore, it is obtained that:

$${\Theta_3}^*F_{\xi_8}(E_3,{\xi_8}^*)=\ \vartheta_{33}\neq 0. \tag{23}$$

Now by using Equation (20), it is obtained that:

$$D^2F(E_3,{\xi_8}^*)(\aleph_3,\aleph_3)=[\varphi_{i1}^*]_{3\times 1},$$

where

$$\left[-2-6\xi_2\bar{x}+\frac{2\xi_1\bar{y}}{(1+\xi_1\bar{x})^3}\right](H_2\,n_{33})^2-2\left[\frac{1}{(1+\xi_1\bar{x})^2}\right]H_1\,H_2(n_{33})^2$$
$$\left[\frac{-2\xi_1\xi_3\bar{y}}{(1+\xi_1\bar{x})^3}\right](H_2 n_{33})^2+2\left[\frac{\xi_3}{(1+\xi_1\bar{x})^2}\right]H_1\,H_2(n_{33})^2$$
$$+\left[\frac{2\xi_4\,z}{(1+\xi_4\bar{y})^3}-2\,\xi_5\right](H_1\,n_{33})^2\;-2\left[\frac{1}{(1+\xi_4\bar{y})^2}\right]H_1\,(n_{33})^2$$
$$\left[-\frac{2\xi_4\xi_6\,z}{(1+\xi_4\bar{y})^3}\right](H_1\,n_{33})^2+\frac{2\xi_6}{(1+\xi_4\bar{y})^2}H_1\,(n_{33})^2-2\xi_7(n_{33})^2.$$

Consequently, it is easy to confirm that:
$\Theta_3{}^T D^2F(E_3,\xi_8{}^*)(\mathrm{n}_3,\mathrm{n}_3)=\epsilon_{33}(\rho_1\varphi_{11}^*+\rho_2\varphi_{21}^*+\varphi_{31}^*)$.
Clearly, $\Theta_3{}^T D^2F(E_3,\xi_8{}^*)(\aleph_3,\aleph_3)\neq 0$, under the (23), Then, a saddle node bifurcation take place.

8. Numerical results

The advantage of analyzing numerical simulations is that they help us recognize the effects of changing the system's parameter values. System (2) has a globally asymptotically stable to E_3 for the values in Equation (24).

$\xi_1=0.1, \xi_2=0.05, \xi_3=0.5, \xi_4=0.1, \xi_5=0.03, \xi_6=0.25, \xi_7=0.01, \xi_8=0.1$ (24)

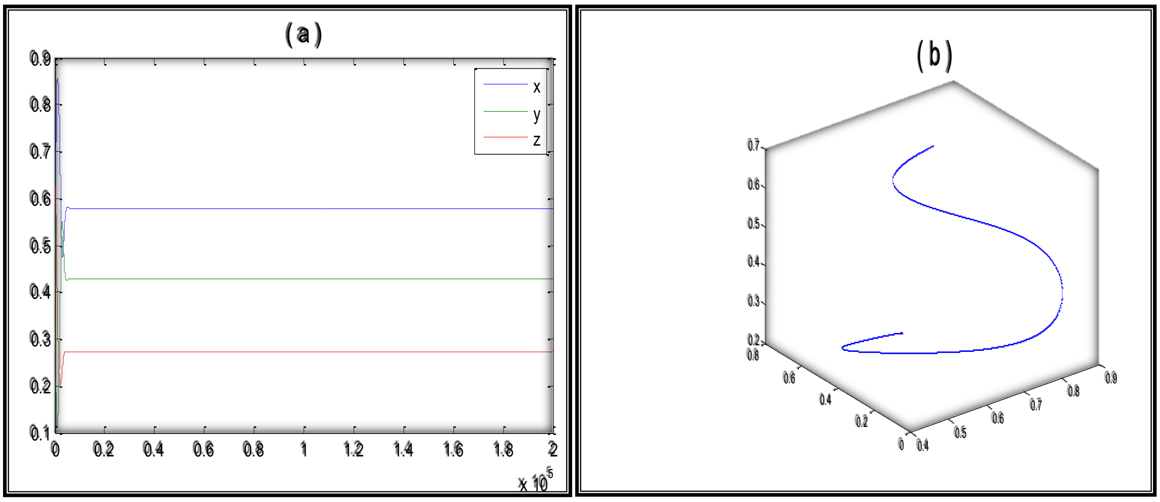

Figure 1: For the date given in Equation (24), the trajectory of system (2) with $\xi_1=0.1$, approaches asymptotically to the E_3, This trajectory's time series, as (a) and (b).

By changing the values of the parameters in Equation (24) we start with point ξ_1. We notice that the solution is still approaching the EP, when $\xi_1=0.5$ and as the system continues to increase $\xi_1\geq 0.6$, the system still exists, as in the Figure 2.

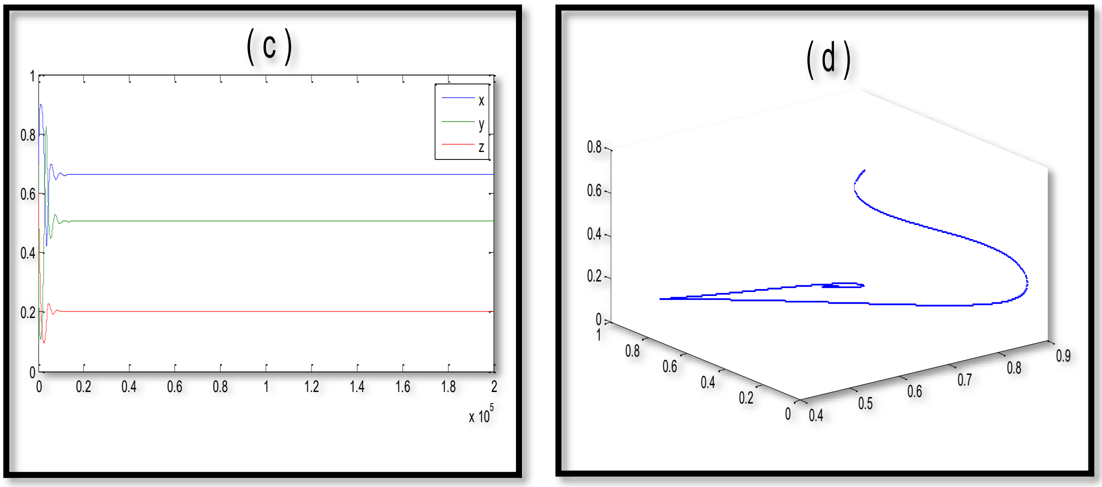

Figure 2: When $\xi_1 \geq 0.6$, the system (2)'s trajectory asymptotically approaches the positive point E_3, according to the trajectory's time series as (c) and (d).

Change the rate of the toxin parameter of the prey as the value of $0.5 \leq \xi_2 \leq 3$, and by increasing $\xi_2 \geq 9$ approaches asymptotically to the E_3.

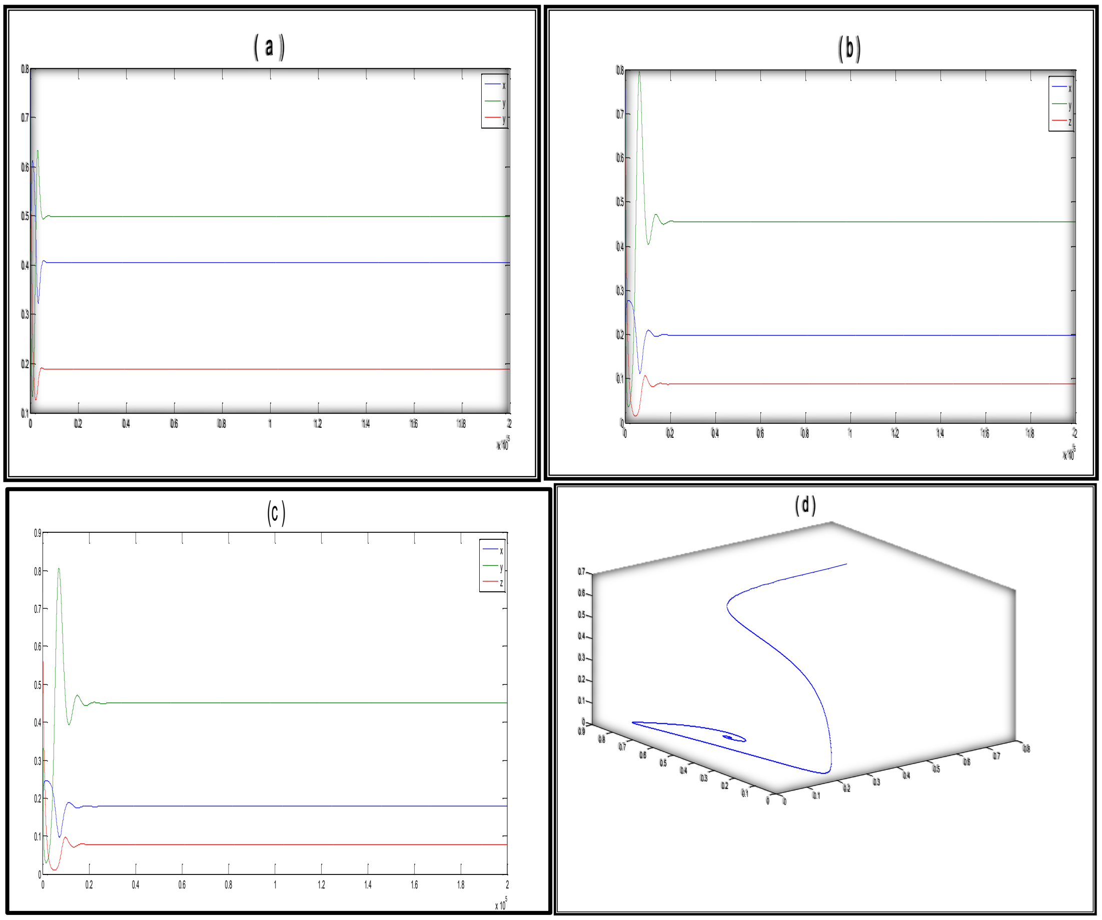

Figure 3: (a) The time series of system (2) when $\xi_2 = 0.5$, approaches asymptotically to the positive point, (b) The trajectory of system(2) when $0.5 < \xi_2 \leq 3$, (c) The time series of system (2) when $\xi_2 \geq 9_2$, (d) The trajectory of the system.

Change the predator's toxin parameter rate ξ_5, and observe that the trajectory of system (2) approaches the E_3 asymptotically when $\xi_5 = 0.3, 0.5, 0.6$, as shown in Figure. (b). However, when $\xi_5 \geq 0.7$, the system (2) loses persistence, and the solution of system (2) approaches E_2, as shown in Figure (c).

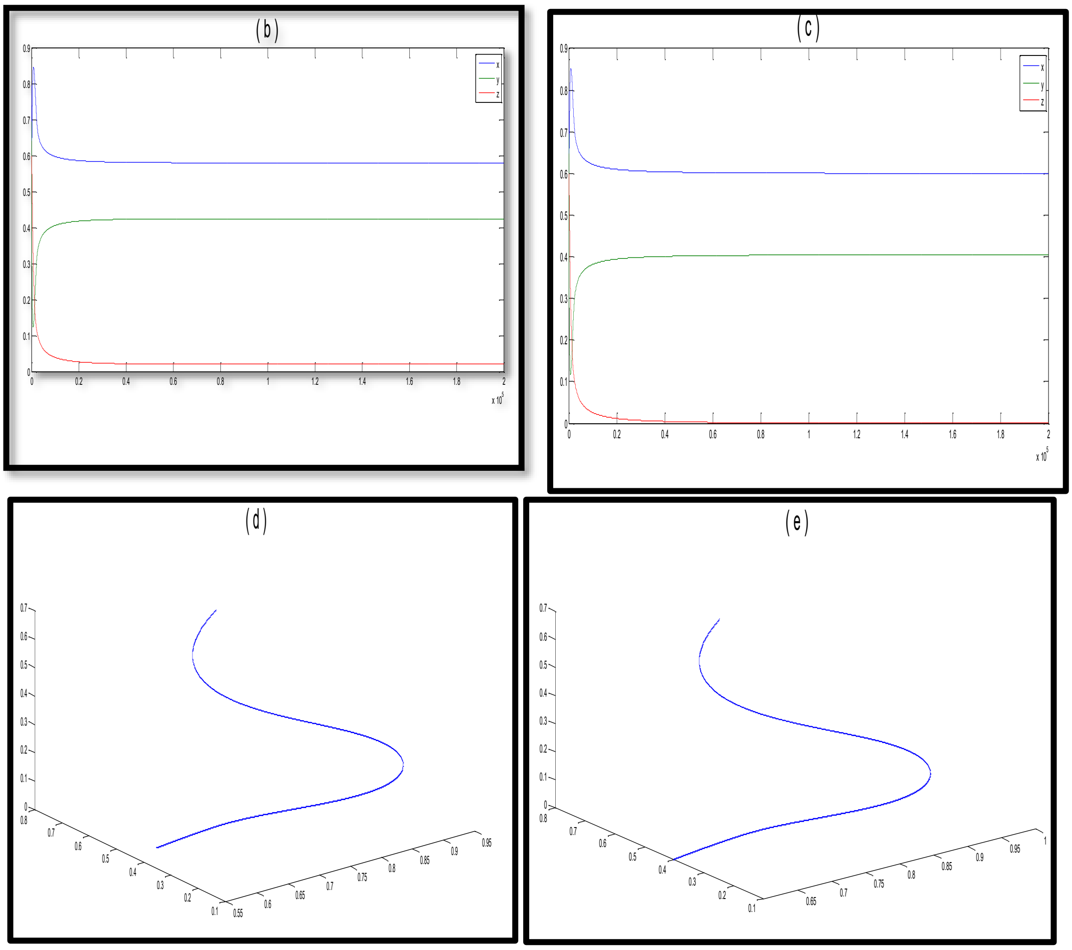

Figure 5. The time series of system (2) when $\xi_5 = 0.3, 0.5, 0.6$ approaches asymptotically to the points (E_3), while $\xi_5 = 0.7$, the time series appere the equiburum point (E_2) of system (2), (d-e) the trajectory of the system (2).

Change the rate of the toxin parameter of the top predator ξ_7 , when $0.3 \leq \xi_7 \leq 0.7$. It is obvious trajectory of system (2) approaches asymptotically to the EP E_3, as Figures (a-d), but when $\xi_7 > 0.7$, the system (2) losses the persistence and the solution of system (2) approaches asymptotically to E_2, as Figures (e-f).

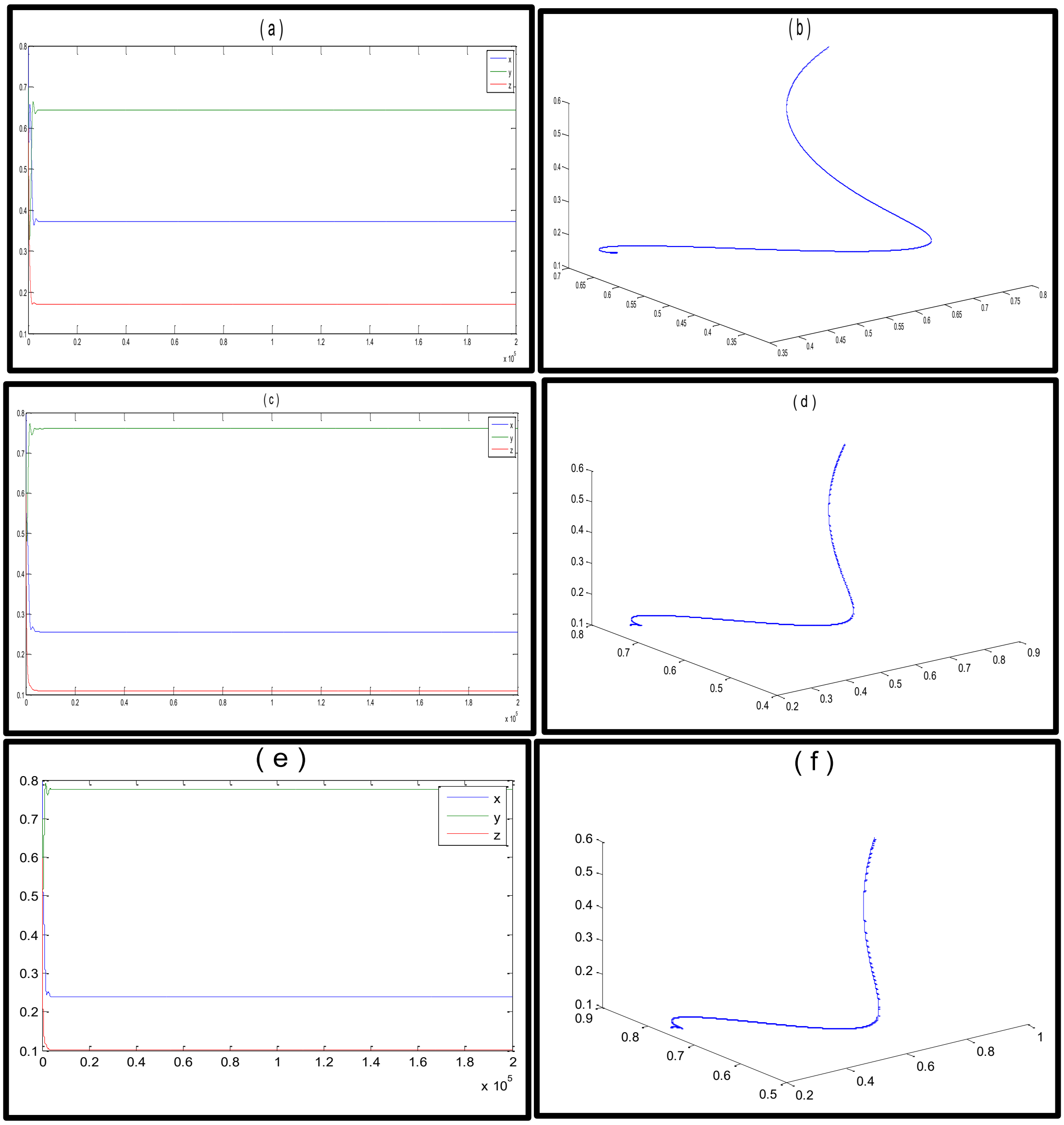

Figure 6: The trajectory of system (2) when $0.3 \le \xi_7 \le 0.7$, and time series of this trajectory respectively, when $\xi_7 > 0.7$ the trajectory of system (2), this trajectory's time series.

Change the rate of the toxin parameter of the prey and predator $\xi_2 = 2$, $\xi_5 = 3$ and $\xi_3 = 0.01$, it's seen that trajectory of system (2) approaches to the EP E_1, as Figure (a).

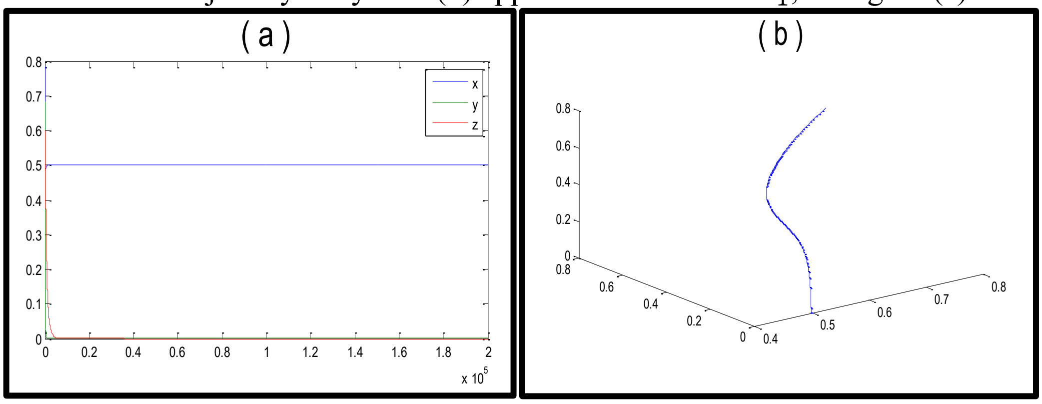

Figure 7. The time serie and trajectory of system (2) when $\xi_2 = 2$, $\xi_5 = 3$ and $\xi_3 = 0.01$.

The collection of data (25) contains the second set of parameter values that cause chaotic dynamics to occur in the system (2):

$\xi_1 = 6\,, \xi_2 = 0.6, \xi_3 = 0.99\,, \xi_4 = 0.002,\ \ \xi_5 = 0.005, \xi_6 = 0.6\,, \xi_7 = 0.002,\ \xi_8 = 0.03$ (25)

The dynamic behavior of the system (2) in the Figures (8) and (9) below.

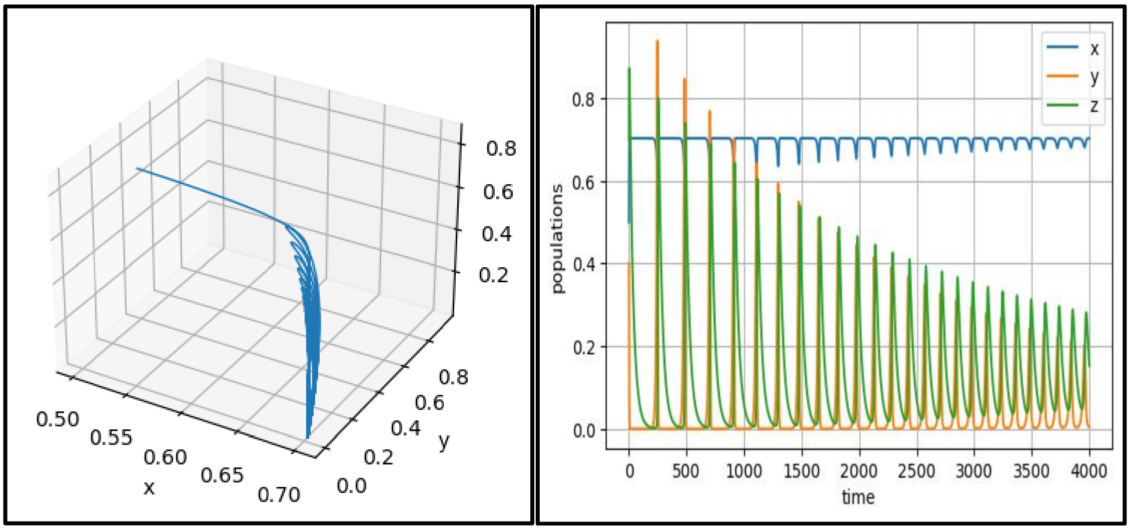

Figure 8: (a) Using the data set provided (25), and the time series of the chaotic attractors provided, (b) The trajectory of the system (2) approaches a chaotic attractor.

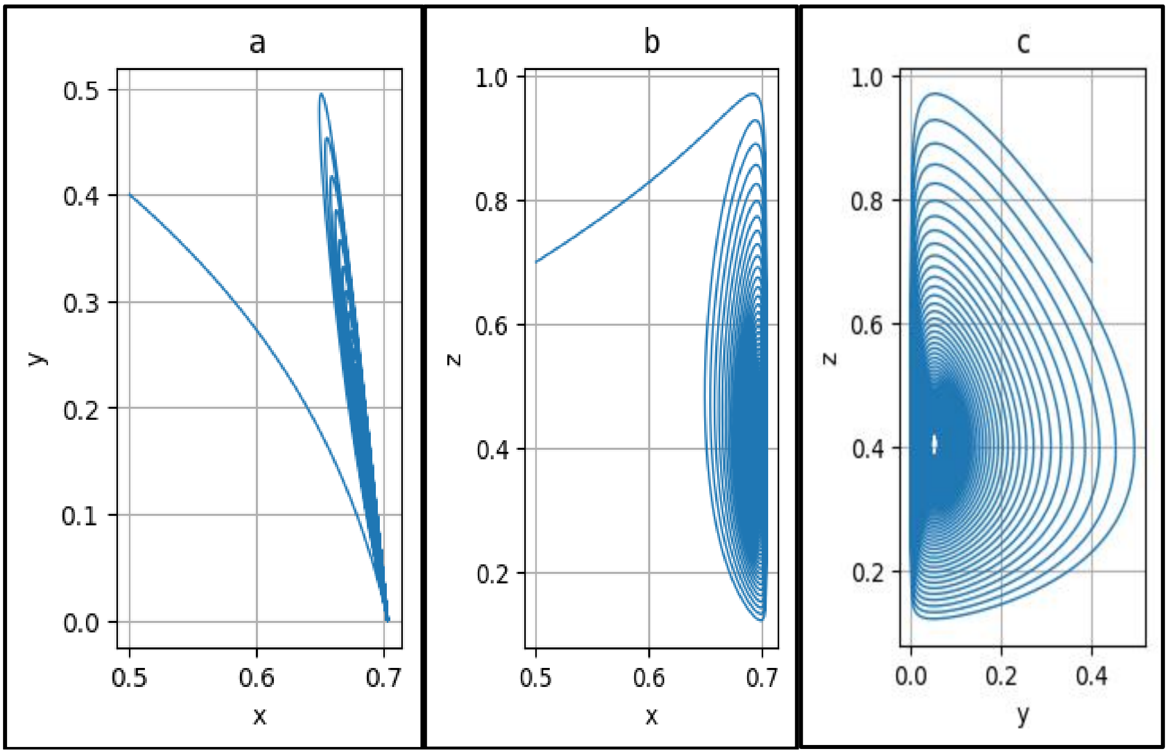

Figure 9: The projection on xy, xz, and yz planes are shown in (a), (b), and (c), respectively.

Figures (8) and (9) show that the dynamics of the system (2) are rich, containing chaos, periodic attractors, and point attractors. As the parameter values are changed, there are numerous bifurcation points. It is commonly recognized that the chaotic attractor's sensitivity to changing starting points is its most defining characteristic. Thus, Figure (10) demonstrates unequivocally how sensitive the odd attractor shown in Figure (8) is to a slight alteration in their starting position.

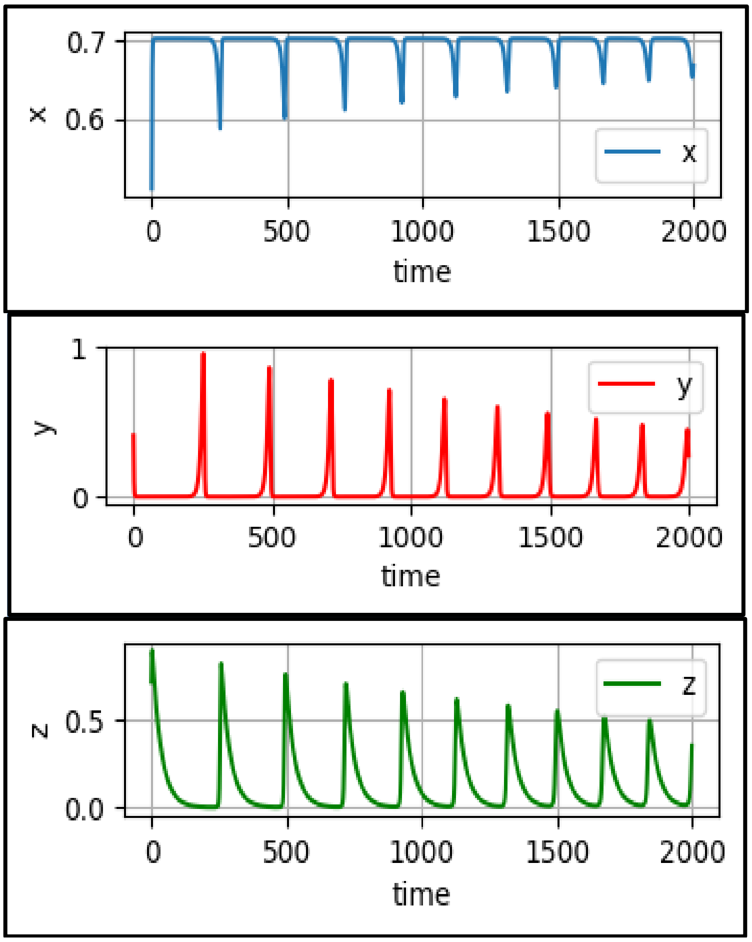

Figure 10: The chaotic attractor's trajectory shown in Figure (8) is sensitive to the starting values. (0.51, 0.61, 0.71). The graphs show the trajectories of x, y, and z depending on the time interval.

Using the bifurcation diagrams, the effects of changing various parameter values on the chaotic regions and the dynamic behavior of the system (2) are now investigated. It is known that altering the parameter values will result in the birth and death of the attractors, as represented by the bifurcation diagram. The bifurcation diagrams as a function of the parameters $\xi_1, \xi_2, \xi_3, \xi_6, \xi_7$ and ξ_8 are shown in Figures (11–16) and (17). Although the bifurcation diagrams clearly show the existence of chaotic zones, it is shown that the solution approaches periodic dynamics in the plane and that raising these parameters leads to the extinction of one predator species.

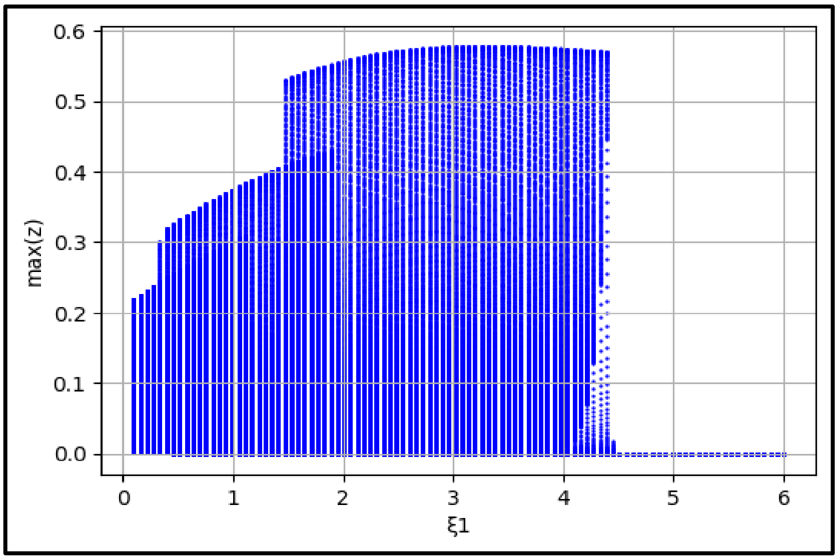

Figure 11: The bifurcation diagrams of a system (2) of ξ_1 in the range $0.1 < \xi_1 < 7$ with the parameter values in (25).

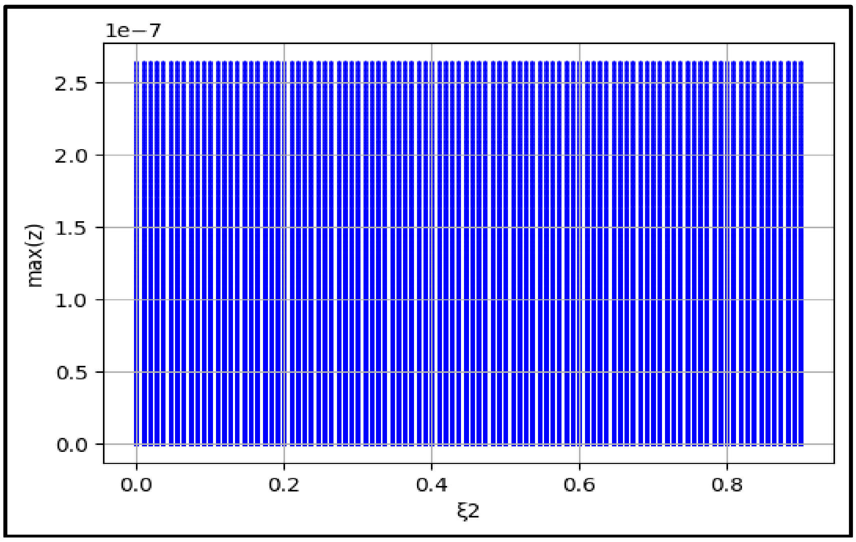

Figure 12: The bifurcation diagrams of a system (2) as ξ_2 in the range $0.1 < \xi_2 < 0.9$ with the parameter values in (25).

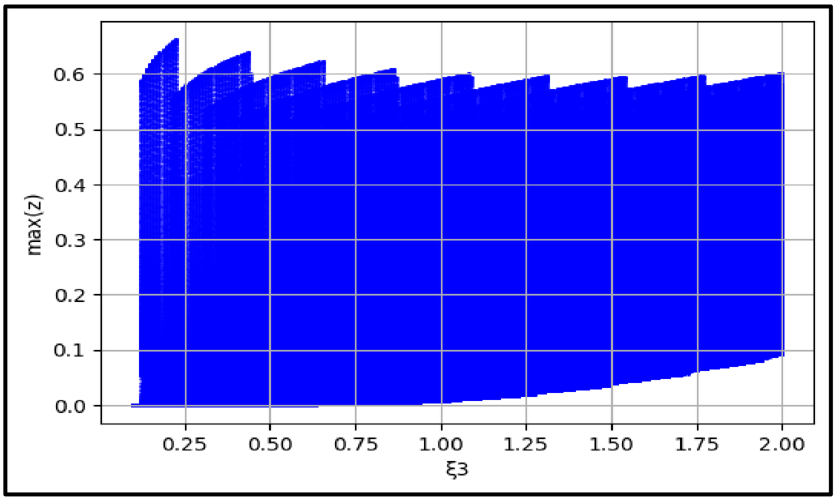

Figure 13: The bifurcation diagrams of a system (2) as ξ_3 in range $0.1 < \xi_3 < 2$ using the parameter values in (25).

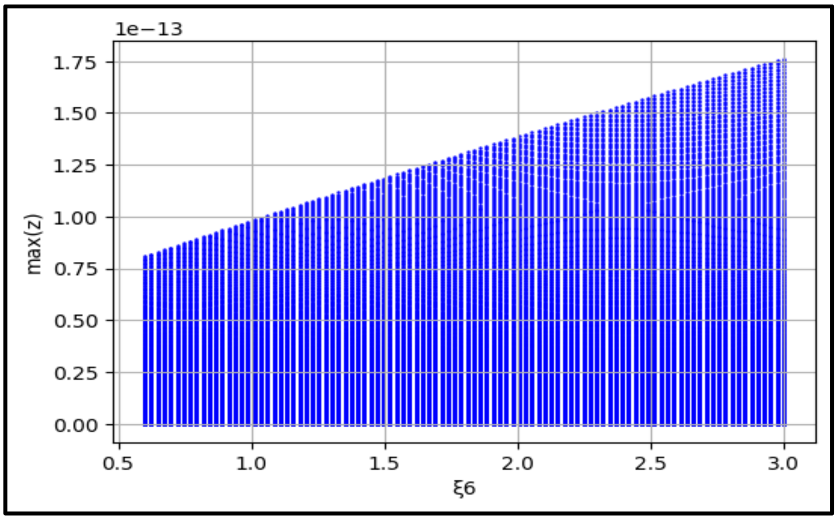

Figure 14: The bifurcation diagrams of a system (2) as a ξ_6 in the range $0.5 < \xi_6 \leq 3$ using the parameter values in (25).

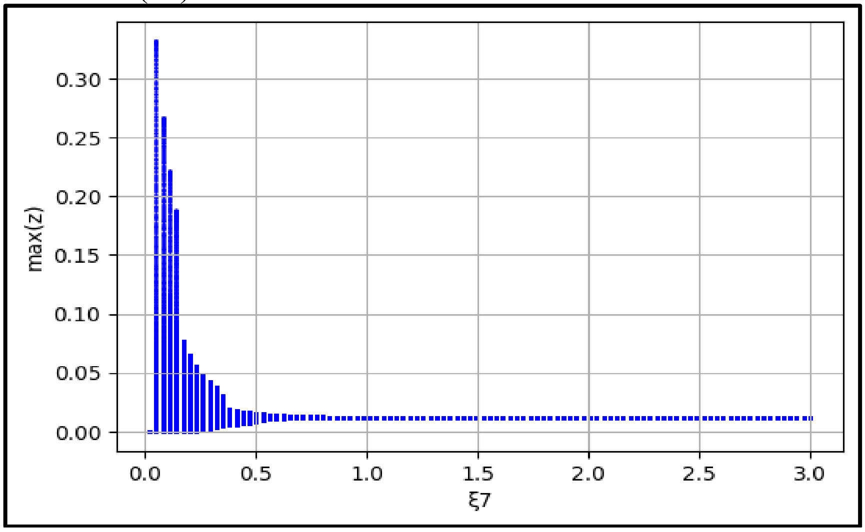

Figure15: The bifurcation diagrams of a system (2) as ξ_7 in the range $0.1 < \xi_7 < 3$ using the parameter values in (25).

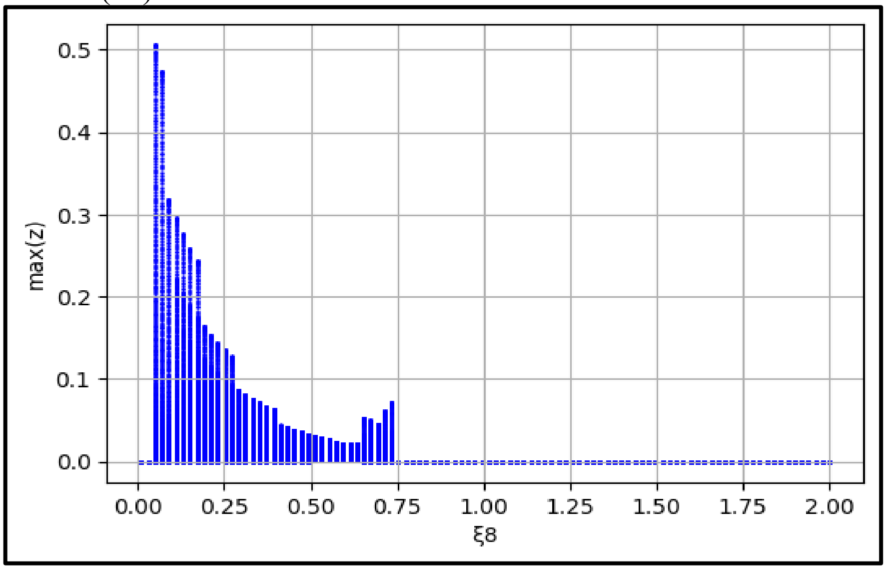

Figure 16: The bifurcation diagrams of a system (2) as ξ_8 in the range $0.1 < \xi_8 < 2$ with the parameter values in (25).

9. Discussion and conclusions

Important new information about the intricate relationships between species can be gained by studying environmental toxins in prey-predator models. Toxin introduction can significantly change a species' long-term survival, stability, and population levels within an environment. Reductions in prey and predator populations brought on by the toxin may result in the extinction of species that are already at risk or changes in the equilibrium of the environment. Toxins can also alter reproduction rates, impair prey defenses, and change the functional response of predators, which further complicates interactions. All of the solution's features are discussed. We examine each possible EP's existence and local stability. The basin of attraction for the equilibrium sites was calculated or global dynamics were examined using the Lyapunov function approach. The conditions for LB near EPs are set. Finally, the possibility of complicated dynamics was explored using a range of techniques, such as the 3D phase portrait and bifurcation diagrams, in combination with numerical simulation. The numerical simulation yielded the following results using a fake set of biologically realistic parameter values. The food chain system is an example of a chaotic system that is subject to a variety of attractors, system (2). A useful tool for comprehending the possible results of toxin exposure and aiding in the prediction of long-term effects is mathematical modeling of these effects,

To summarize up, it is important to incorporate environmental toxin dynamics into prey predator models as a means to correctly represent ecological realities, finally, the effects of environmental pollutants can be lessened by combining cutting-edge technology with scientific studies. In order to develop sustainable solutions, future research should incorporate ecological restoration and AI-based monitoring, depending on the type of environment being examined and the type of toxin or pollution harming it.

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