COMPARISON OF REDUNDANCY LEVELS USING STOCHASTIC ORDERINGS

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Abstract

In this research we concern with the comparison between the two ways of providing redundant units for a system:-

1. Component redundancy.
2. System redundancy.

The comparison between these ways is carried out by comparing the random variable representing the lifetime of the system resulted from applying component redundancy with the random variable representing the lifetime of that resulted from applying system redundancy using some types of stochastic orderings, namely: (1) usual stochastic ordering, (2) failure rate ordering, (3) likelihood ratio ordering, (4) reversed failure rate ordering and (5) mean residual life ordering.

مقارنة مستويات المجانبة باستخدام ترتيبات تصادفية

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الخلاصة

يهتم هذا البحث بالمقارنة بين طريقتين لتجهيز وحدات اضافية لنظام معين:

١-تجهيز الوحدات الاضافية على مستوى المركب (المجانبة على مستوى المركب)

٢- تجهيز الوحدات الاضافية على مستوى المنظومة (المجانبة على مستوى المنظومة)

ان المقارنة بين هاتين الطريقتين تتم عن طريق مقارنة المتغير العشوائي الذي يمثل فترة الحياة للنظام الناتج عن تجهيز الوحدات الاضافية على مستوى المركب مع المتغير العشوائي الذي يمثل فترة الحياة لذلك النظام الناتج عن تجهيز الوحدات الاضافية على مستوى المنظومة باستخدام بعض الترتيبات التصادفية منها: ١-الترتيب التصادفي الاعتيادي، ٢-ترتيب نسبة الفشل، ٣-ترتيب نسبة الامكانية، ٤-ترتيب نسبة الفشل المعكوسة، ٥-ترتيب متوسط الحياة المتبقية.

Introduction

Reliability of a unit (component or system of components) is defined as its ability to perform its intended function for a specified interval of time and in stated environmental conditions.
Reliability subject deals with improving the performance of devices.

In this research, we present the method of increasing reliability known redundancy; which is a technique for improving reliability via connecting one or more additional units in parallel with the original ones. These additional units, which may be identical or not to the original ones, are called, spare (redundant) units.

When the original unit fails, the spare unit replaces it and carries out its function.
We consider three basic types of redundancy:

(1) Active redundancy (hot or parallel redundancy) [1].
(2) Cold-standby redundancy (or passive redundancy) [1].
(3) Partially energized standby redundancy [2].

Two ways of providing each type of redundancy are studied: (1) component redundancy, that is, providing component(s) as spare(s) to each individual component in the system and (2) system redundancy, that is, providing system(s) as spare(s) to the whole system.
In this research, we have compared between component and system redundancy using some of the stochastic orderings. In some of these results, we have assumed the redundant units to be identical with the original unit. In the results concerning cold-standby redundancy and partially energized standby redundancy, we also take account of the case of perfect switching and that of imperfect switching.

Notation

iid independent, identically distributed.

$T_1 \wedge T_2 \wedge \cdots \wedge T_n$ $\min\{T_1, T_2, \cdots, T_n\}$

$T_1 \vee T_2 \vee \cdots \vee T_n$ $\max\{T_1, T_2, \cdots, T_n\}$

$T_{i:n}$ the ith order statistic, $T_{i:n}$=ith min $\{T_1, T_2, \ldots, T_n\}$

τ_{k-n}:G the lifetime random variable of the k-out-of-n: G system of n components whose independent lifetime random variables are $T_1, T_2, \ldots, T_n$

τ_{k-n}:F the lifetime random variable of the k-out-of-n:F system of n components whose independent lifetime random variables are $T_1, T_2, \ldots, T_n$

$\tau_{1-n:F}(T \vee U)$ $\min\{T_1 \vee U_1, T_2 \vee U_2, \cdots, T_n \vee U_n\}$

$\tau_{1-n:G}(T+U)$ $\max\{T_1+U_1, T_2+U_2, \cdots, T_n+U_n\}$

$\tau_{1-n:F}(T+U)$ $\min\{T_1+U_1, T_2+U_2, \cdots, T_n+U_n\}$

$\tau_{1-n:F}(T)+\tau_{1-n:F}(U)$
$\min\{T_1, T_2, \cdots, T_n\}+\min\{U_1, U_2, \cdots, U_n\}$

$\tau_{1-n:G}(T)+\tau_{1-n:G}(U)$
$\max\{T_1, T_2, \cdots, T_n\}+\max\{U_1, U_2, \cdots, U_n\}$

$\tau(T)$ $\{T_1 \wedge T_2\} \vee T_3$

R(t) the reliability of the unit.
r(t) the failure rate function.
f(t) the density function.
$\tilde{r}(t)$ the reversed failure rate function of the life distribution F.
$\mu_F(t)$ the mean residual life function.

Let T_1 and T_2 be random variables representing the lifetimes of two different units, the reliability function, distribution function, density function and mean residual life function of T_i are respectively R_i, F_i, f_i, and μ_{Fi}, i=1,2. Comparison between T_1 and T_2 based on functions such as: reliability function, failure rate function, mean residual life function and other suitable functions of probability distributions usually establish partial orders between them. We call them as stochastic orderings.
A random variable T_1 is said to be stochastically smaller than another random variable T_2 ($T_1 \le_{st} T_2$) if $R_1(t) \le R_2(t)$; for all t. [3]
And T_1 is said to be smaller than T_2 in the failure rate ordering sense ($T1 \le_{fr} T_2$) if $r_1(t) \ge r_2(t)$; for all t. [4]
T_1 is said to be smaller than T2 in the likelihood ratio ordering sense $(T_1 \le_{lr} T_2)$ *iff* $\frac{f_2(t)}{f_1(t)}$ is increasing in t. [5]
T_1 is said to be smaller than T_2 in the reversed failure rate ordering sense $(T_1 \le_{rf} T_2)$ *if* $\tilde{r}_1(t) \le \tilde{r}_2(t)$ for all t. [5]
T_1 is said to be smaller than T_2 according to mean residual life (MRL) ordering $(T_1 \le_{mrl} T_2)$ *if* $\mu_{F_1}(t) \le \mu_{F_2}(t)\ \forall\, t \ge 0$. [6]
The definitions of failure rate ordering and reversed failure rate ordering have equivalent statements as given in the next theorem.

Theorem

Let T_1 and T_2 be two absolutely continuous random variables, [4,5].

(i) $T_1 \le_{fr} T_2$ iff $\frac{R_2(t)}{R_1(t)}$ is an increasing function of t.

(ii) $T_1 \le_{fr} T_2$ iff $\frac{F_2(t)}{F_1(t)}$ is an increasing function of t.

In the next theorem, we present the relations among some of the stochastic orderings.

Theorem

Let T_1 and T_2 be two absolutely continuous random variables, [5,6].

(i) $T_1 \le_{lr} T_2 \Rightarrow T_1 \le_{fr} T_2 \Rightarrow T_1 \le_{st} T_2$
(ii) $T_1 \le_{fr} T_2 \Rightarrow T_1 \le_{mrl} T_2$
(iii) $T_1 \le_{lr} T_2 \Rightarrow T_1 \le_{rf} T_2 \Rightarrow T_1 \le_{st} T_2$

In our study, we assume the following:

1. The lifetime random variables of all the units are independent.
2. The lifetime random variables of the systems in system level redundancy are independent.
3. The reliability of every unit is not affected by the type of redundancy.

Active redundancy

In the case of active redundancy all the units (the original and the spare units) are functioning simultaneously. Assume that the lifetimes of the original and the spare units are denoted by T_1, T_2 with reliability functions R_1 and R_2, respectively. The lifetime of the system is the lifetime of the best of its units. Thus, if T_a denotes the lifetime of the resultant system, then

$$T_a = T_1 \vee T_2$$

and

$$\begin{aligned} R_a(t) &= P(T_a > t) \\ &= 1 - P(T_a > t) \\ &= 1 - P(T_1 > t).\, P(T_2 > t), \end{aligned}$$

since the unit lifetimes are independent.

$R_a(t) = 1 - (1-R_1(t)).\,(1-R_2(t))$

Figure 1 below depicts this type of redundancy.

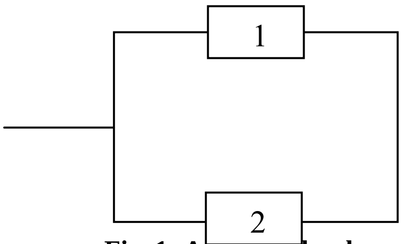


Fig. 1: Active Redundancy

Below we compare between component and system level in the case of active redundancy. Under certain conditions and considering 1-out-of-n:F system Boland and EL-Neweihi, 1995 [4] proved that if $T_1, T_2, \ldots, T_n$ are the lifetime random variables of the original units and $U_1, U_2, \ldots, U_n$ are the lifetime random variables of the spare units such that T_i and U_i are iid, $1 \le i \le n$, then

$$\tau_{1-n:F}(T)\vee\tau_{1-n:F}(U) \le_{fr} \tau_{1-n:F}(T\vee U)$$

Since, failure rate ordering implies mean residual life ordering, then the following claim is valid.

Claim

Under the same conditions of Boland and EL-Neweihi and considering 1-out-of-n:F system if $T_1, T_2, \ldots, T_n$ are the lifetimes of the original units and $U_1, U_2, \ldots, U_n$ are the lifetimes of the spare units such that T_i and U_i are iid. Then

$$\tau_{1-n:F}(T)\vee\tau_{1-n:F}(U) \le_{mrl} \tau_{1-n:F}(T\vee U)$$

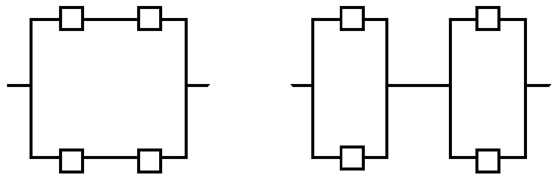
If all units in the redundant group (the original and the spare units) have iid lifetimes then we prove, in the next result, that active redundancy on component level is better than active redundancy on system level in the likelihood ratio ordering sense when the original system is 1-out-of-2:F system.

Result-1- (Use assumptions 1-3)

Consider 1-out-of-2: F system. Let T_1, T_2, U_1, U_2 be iid lifetimes with common reliability function R, where T_1, T_2 are the lifetimes of the two original units and U_1, U_2 are the lifetimes of the two spare units. Then

$$\tau_{1-2:F}(T)\vee\tau_{1-2:F}(U) \le_{lr} \tau_{1-2:F}(T\vee U)$$

Figure 2 below represents active redundancy on system and component levels).



(a) (b)

Fig. 2: a) System redundancy, b) component redundancy

Proof:

It suffices to show that $\frac{f_c(t)}{f_s(t)}$ is an increasing function of t.

Where

$$\begin{aligned} f_c(t) &= -\frac{d}{dt}\{p(\tau_{1-2:F}(T\vee U) > t)\} \\ &= 4f(t).(1-r(t)).\left(1-(1-r(t))^2\right), \end{aligned}$$

where $f(t) = -\frac{d}{dt}R(t)$

and

$$f_s(t) = -\frac{d}{dt}\{P(\tau_{1-2:F}(T)\vee\tau_{1-2:F}(U) > t)\}$$

$$= 4f(t).R(t).\left(1-R^2(t)\right)$$

Letting

$$g(t) = \frac{f_c(t)}{f_s(t)} = \frac{4f(t)(1-R(t))(1-(1-R(t))^2)}{4f(t)R(t)(1-R^2(t))}$$

$$= \frac{2-R(t)}{1+R(t)}$$

We see that,

$$\frac{d}{dt}g(t) = \frac{3f(t)}{(1+R(t))^2} > 0$$

This completes the proof.

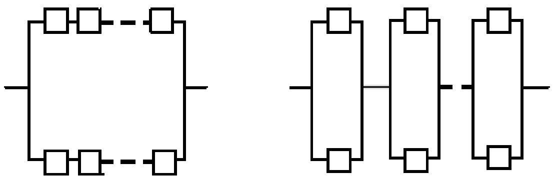
For the general case when the original system is 1-out-of-n: F system, we put condition under which the above result is still true as shown below.

Result -2- (Use assumptions 1-3)

Consider 1-out-of-n: F system. Let $T_1, T_2,\ldots, T_n$, $U_1, U_2,\ldots, U_n$ be iid lifetimes with common reliability function R, where $T_1, T_2,\ldots,T_n$ are the lifetimes of the n original units and $U_1, U_2,\ldots,U_n$ are the lifetimes of the n spare units. Then

$$\tau_{1-n:F}(T)\backslash/\tau_{1-n:F}(U) \leq_{lr} \tau_{1-n:F}(T\backslash/U)$$

Figure 3 below represents active redundancy on system and component levels).



(a) **(b)**

Fig. 3: a) System redundancy, b) component redundancy

Proof:

It suffices to show that $\frac{f_c(t)}{f_s(t)}$ is an increasing function of t.

Where

$$f_c(t) = -\frac{d}{dt}\{P(\tau_{1-n:F}(T\backslash/U) > t)\}$$

$$= -\frac{d}{dt}\left\{\left(2R(t) - R^2(t)\right)^n\right\}$$

$$= n\left(2R(t) - R^2(t)\right)^{n-1}.(2f(t) - R(t).f(t))$$

where $f(t) = \frac{d}{dt}R(t)$

$$=2nf(t)\cdot\left(R(t)\right)^{n-1}\cdot\left(2-R(t)\right)^{n-1}\cdot\left(1-R(t)\right)$$

and

$$f_s(t) = -\frac{d}{dt}\{P(\tau_{1-n:F}(T)\backslash/\tau_{1-n:F}(U) > t)\}$$

$$=-\frac{d}{dt}\{2R^n(t) - R^{2n}(t)\}$$

$$=2nf(t)\cdot\left(R(t)\right)^{n-1}\cdot(1-R^n(t))$$

Letting

$$g(t) = \frac{f_c(t)}{f_s(t)}$$

$$= \frac{2nf(t)\cdot\left(R(t)\right)^{n-1}\cdot\left(2-R(t)\right)^{n-1}\cdot\left(1-R(t)\right)}{2nf(t)\cdot\left(R(t)\right)^{n-1}\cdot(1-R^n(t))}$$

$$= \frac{(2-R(t))^{n-1}\cdot(1-R(t))}{(1-R^n(t))}$$

To show that g(t) is an increasing function of t, we must show that $\frac{d}{dt}g(t) > 0$

But

$$\frac{d}{dt}g(t)$$

$$\frac{\{1-R^n(t)\}\cdot\left\{\begin{matrix}(n-1)(2-R(t))^{n-2}f(t)(1-R(t)) + \\ f(t)(2-R(t))^{n-1}\end{matrix}\right\} - \left\{(2-R(t))^{n-1}(1-R(t))\right\}.\{nf(t)R^{n-1}(t)\}}{(1-R^n(t))^2}$$

i.e., we must show that

$$nf(t)(1-R(t))(1-R^n(t))(2-R(t))^{n-2} - f(t)(1-R(t))(1-R^n(t))(2-R(t))^{n-2} + f(t)(1-R^n(t))(2-R(t))^{n-1} - nf(t)R^{n-1}(t)(1-R(t))(2-R(t))^{n-1} > 0$$

Notice that,

$$f(t)(1-R^n(t))(2-R(t))^{n-1} f(t)(1-R(t))(1-R^n(t))(2-R(t))^{n-2}$$

$$= f(t)(1-R^n(t))(2-R(t))^{n-2} > 0$$

Since, $f(t), (1-R^n(t))\ and\ (2-R(t))^{n-2} > 0$ and that,

$$nf(t)(1-R(t))(1-R^n(t))(2-R(t))^{n-2} - nf(t)R^{n-1}(t)(1-R(t))(2-R(t))^{n-1}$$

$$= nf(t)(1-R(t))(2-R(t))^{n-2}(1-2R^{n-1}(t)) > 0$$

under the condition that

$$(1-2R^{n-1}(t)) > 0\ i.e.,\ R^{n-1}(t) < \frac{1}{2}$$

This completes the proof.

Note

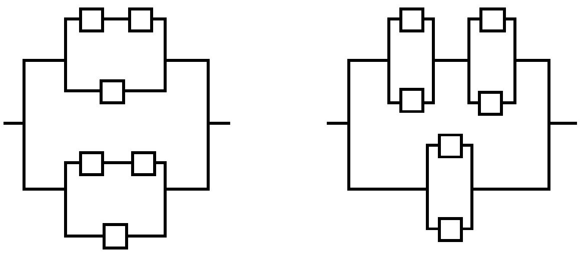
Since likelihood ratio ordering implies mean residual life ordering and reversed failure rate ordering, then we can say that the previous results are also true under the mean residual life ordering and the reversed failure rate ordering.

Now, consider the parallel-series system of three units whose lifetime is defined by $\{T_1 \wedge T_2\} \vee T_3$, where T_1, T_2 and T_3 are its components lifetimes, T_1, T_2 and T_3 are iid exponentially distributed random variables with parameter r, then active redundancy on component level is better than that on system level in the usual stochastic ordering sense as proved in the next result.

Result -3- (Use assumptions 1-3)
Consider the parallel-series system of three units. If T_1, T_2, T_3, U_1, U_2, U_3 are all iid lifetimes whose common distribution is exponential with parameter r. T_i and U_i are the lifetimes of the ith original and the ith spare unit, respectively. Then

$$\tau(T)\vee\tau(U) \leq_{st} \tau(T\vee U)$$

Figure 4 below represents active redundancy on system and component levels).



(a) **(b)**

Fig. 4: a) System redundancy, b) component redundancy

<u>Proof:</u>

We have,

$R_s(t) = P(\tau(T)\vee\tau(U) > t) = 1-(1-e^{-2rt})^2.(1-$ $e-rt)2$

$R_c(t) = P(\tau(T\vee U) > t)$

$$= 1-(1-e^{-rt})^4.(2-(1-e^{-rt})^2)$$

The wanted result can be obtained by proving that $R_s(t) \leq R_c(t)$; i.e., by showing that

$$1-(1-e^{-2rt})^2.(1-e^{-rt})^2$$

$$\leq 1-(1-e^{-rt})^4.(2-(1-e^{-rt})^2)\ ;\ \forall t.$$

Simplifying this, we get

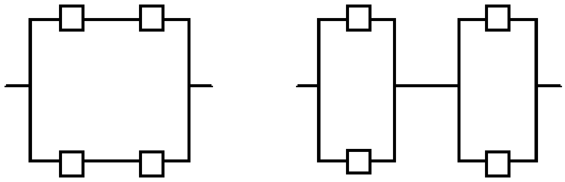
$$R_s(t) \leq R_c(t)\ \ if\ \ e^{2rt}+1 \geq 2e^{rt}$$

Since the last inequality is always true, the proof is completed.
In the previous results we have assumed that the spares and the original units have iid lifetimes.
Now, we consider the case when the spare units are not identical with the original ones.

Result -4- (Use assumptions 1-3)
Consider 1-out-of-2: F system. Let T_1, T_2 be iid lifetimes of the original units where T_i is Weibull random variable with shape parameter α and scale parameter r. Let U_1, U_2 be iid lifetimes of the spare units where U_i is Weibull random variable with shape parameter α and scale parameter r^*. Then

$$\tau_{1-2:F}(T)\vee\tau_{1-2:F}(U) \leq_{st} \tau_{1-2:F}(T\vee U)$$

Figure 5 below represents active redundancy on system and component levels).



(a) **(b)**

Fig. 5: a) System redundancy, b) component redundancy

Proof:

We have,

$R_s(t) = P(\tau_{1-2:F}(T)\vee\tau_{1-2:F}(U) > t)$

$= e^{-2((rt)^{\alpha}+(r^*t)^{\alpha})}.(e^{2(r^*t)^{\alpha}}+e^{2(rt)^{\alpha}}-1)$

$R_c(t) = P(\tau_{1-2:F}(T\vee U) > t)$

$$= e^{-2((rt)^{\alpha}+(r^*t)^{\alpha})}.(e^{(r^*t)^{\alpha}}+e^{(rt)^{\alpha}}-1)^2$$

Hence,

$$\tau_{1-2:F}(T)\vee\tau_{1-2:F}(U) \leq_{st} \tau_{1-2:F}(T\vee U)\ \text{iff}\ R_s(t) \leq R_c(t)\ ;\ \forall t.$$

After simplifications we get:

$$R_s(t) \leq R_c(t)\ iff\ e^{(rt)^{\alpha}+(r^*t)^{\alpha}} - e^{(rt)^{\alpha}} - e^{(r^*t)^{\alpha}} + 1 \geq 0$$

Which holds iff $(e^{rt})^{\alpha} \geq 1$.

This is always true.

If we consider 1-out-of-2:F system whose original unit lifetimes are iid exponentially distributed with parameter α and whose spare unit lifetimes are iid exponentially distributed with parameter 0.5 α, the previous result still valid using the reversed failure rate ordering as the following result shows:

Result -5- (Use assumptions 1-3)
Consider 1-out-of-2: F system. Let T_1, T_2 be iid lifetimes of the original units where T_i is an exponential random variable with parameter α. Let U_1, U_2 be iid lifetimes of the spare units where U_i is an exponential random variable with parameter 0.5 α. Then

$$\tau_{1-2:F}(T)\vee\tau_{1-2:F}(U) \leq_{rf} \tau_{1-2:F}(T\vee U)$$

Proof:

We have,

$F_c(t) = P(\tau_{1-2:F}(T\vee U) \leq t)$

$= 1-(1-(1-e^{-\alpha t}).(1-e^{-0.5\alpha t}))^2$

$$F_s(t) = P(\tau_{1-2:F}(T) \vee \tau_{1-2:F}(U) \le t)$$
$$=(1-e^{-2\propto t}).(1-e^{-\propto t})$$
$$g(t) = \frac{F_c(t)}{F_s(t)} = \frac{1-(1-(1-e^{-\propto t}).(1-e^{-0.5\propto t}))^2}{(1-e^{-2\propto t}).(1-e^{-\propto t})}$$
$$= \frac{1-2e^{-1.5\propto t}+e^{-2\propto t}}{1-e^{-2\propto t}}$$

We need to show that $g(t)$ is increasing in t.

The numerator of

$$\frac{d}{dt}(g(t)) is\ (-4 \propto e^{-2\propto t} + 3 \propto e^{-1.5\propto t} + \propto e^{-3.5\propto t}), which\ is\ nonnegative\ if$$

$$3e^{0.5\propto t} + e^{-1.5\propto t} \ge 4$$

Note that, at t=0 the left hand side is 4 and the inequality above is true.
Now if we show that the expression in the left hand side is an increasing function we are through.
The derivative of the left hand side expression is $(1.5\ \alpha\ (e^{0.5\alpha t}-e^{-1.5\alpha t}))$ which is positive always. Thus, the left hand side expression is an increasing function of t, and since it is equal to 4 at t=0, then it is always greater than 4 as required

Cold- Standby redundancy

In the case of cold-standby redundancy, one unit is always in operation (the original unit) and the other units (the spare units) are not operating they are in the standby position. It is assumed that the standby redundant units neither degrade nor fail while in standby state. When the standby redundant unit takes the place of the failed unit in the redundant group (the original unit and the spare unit), its state is new.
To replace the failed unit by the standby one we need a switching device as shown in figure 6.

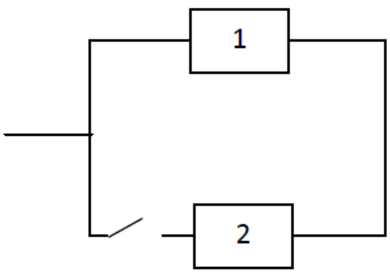


Fig. 6: Cold-Standby Redundancy

Here, we discuss cold-standby redundancy on component and system levels using an automatic switching device, which instantaneously inserts the next unit when the original unit has failed.

In the first step, we assume that the switching device is absolutely reliable (has reliability equal to one) this switching is described as perfect switching.
To derive the reliability function of the resultant system, where the switching is absolutely reliable, assume that T_1 and T_2 are independent lifetimes, where T_1 represents the operating unit lifetime and T_2 represents the lifetime of the standby unit. Let F_i be the distribution function of T_i, i=1,2 and $T_c= T_1+ T_2$ be the lifetime of the resultant system, that the lifetime of the original unit plus that of the cold-standby unit.

$$R_c(t) = P(T_c > t) = 1 - P(T_c \le t)$$
$$=1 - P(T_1 + T_2 \le t)$$

But, $P(T_1+T_2\le t)$ is the distribution function of the sum of the two independent random variables T_1 and T_2, so it is given by the formula

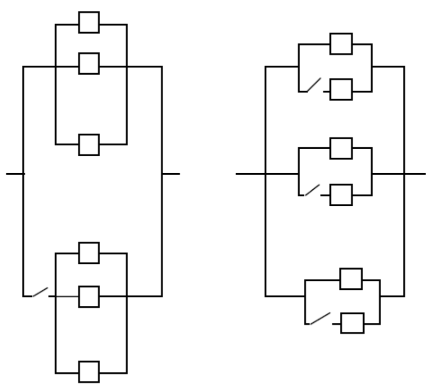
$$P(T_1 + T_2 \le t) = \int_0^t F_2(t-x)dF_1(x)$$

Thus,

$$R_c(t) = 1 - \int_0^t F_2(t-x)dF_1(x)$$

In the next result, we prove that if the spare units do not match the original units and the original system is 1-out-of-n: G system, then cold-standby redundancy on system level is better than that on component level in the usual stochastic ordering sense.

Result -6- (Use assumptions 1-3)
Consider 1-out-of-n: G system. Let T_1, T_2,…, T_3, be the lifetimes of the original units and let U_1, U_2,…, U_n be the lifetimes of the spare units. Then
$\tau_{1-n:G}(T+U) \le_{st} \tau_{1-n:G}(T) + \tau_{1-n:G}(U)$
Figure 7 below represents cold-standby redundancy on system and component levels).



(a) **(b)**

Fig. 7: a) System redundancy, b) component redundancy

Proof:

Obviously, we have to show that the inequality

$$\tau_{1-n:G}(T+U) \le \tau_{1-n:G}(T)+\tau_{1-n:G}(U)$$

holds.

As it is obvious that for any two finite sets of real numbers A and B,

$$\max\{A+B\} \le \max\{A\}+\max\{B\}$$

and since,

$$\tau_{1-n:G}(T) = \max\{T_1, T_2,\dots, T_3\}$$

and

$$\tau_{1-n:G}(U) = \max\{U_1, U_2,\dots, U_3\}$$

One may conclude that

$$\tau_{1-n:G}(T+U) \le \tau_{1-n:G}(T) + \tau_{1-n:G}(U)$$

This means that the reliability function of the system whose lifetime is $\tau_{1-n:G}(T+U)$ is not greater than that of the system whose lifetime is $\tau_{1-n:G}(T) + \tau_{1-n:G}(U)$.

Consider the case of 1-out-of-2: F system whose original unit lifetimes are iid exponentially distributed with parameter α and whose spare unit lifetimes are iid exponentially distributed with parameter 0.5α. In the next result we show that cold-standby redundancy on component level is better than that on system level in the likelihood ratio ordering sense.

Result -7- (Use assumptions 1-3)

Consider 1-out-of-2:F system. Let T_1, T_2 be iid lifetimes of the original units. T_i is an exponential random variable with parameter α. Let U_1, U_2 be iid lifetimes of the spare units. U_i is an exponential random variable with parameter 0.5α. Then

$$\tau_{1-2:F}(T)+\tau_{1-2:F}(U) \le_{lr} \tau_{1-2:F}(T+U)$$

(Figure 8) below represents cold-standby redundancy on system and component levels).

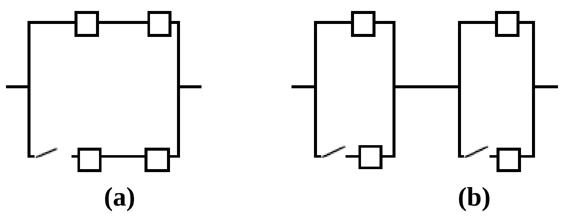


Fig. 8: a) System redundancy, b) component redundancy

Proof:

Define the distribution function of $\tau_{1-2:F}(T)$ by $(1-e^{-2\alpha t})$ and that of $\tau_{1-2:F}(U)$ by $(1-e^{-\alpha t})$ we have,

$$F_s(t) = P(\tau_{1-2:F}(T)+\tau_{1-2:F}(U) \le t)$$

$$=\int_0^t \left(1-e^{-\propto(t-x)}\right) d\left(1-e^{-2\propto x}\right)$$

$$=\int_0^t \left(1-e^{-\propto(t-x)}\right).\left(2\propto e^{-2\propto x}\right)dx$$

$$=1+e^{-2\propto t}-2e^{-\propto t}$$

and

$$f_s(t) = \frac{d}{dt}F_s(t)$$

$$=2\propto\left(e^{-\propto t}-e^{-2\propto t}\right)$$

The distribution function of T_i+U_i is defined by:

$$P(T_i+U_i \le t) = \int_0^t \left(1-e^{-0.5\propto(t-x)}\right) d\left(1-e^{-\propto x}\right)$$

$$=1+e^{-\propto t}-2e^{-0.5\propto t}$$

Thus,

$$F_c(t) = P(\tau_{1-2:F}(T+U) \le t)$$

$$=1-\left(1-P(T_i+U_i \le t)\right)^2$$

$$=1-\left(2e^{-0.5\propto t}-e^{-\propto t}\right)^2$$

and

$$f_c(t) = \frac{d}{dt}F_c(t)$$

$$=2\propto\left(2e^{-\propto t}+e^{-2\propto t}-3e^{-1.5\propto t}\right)$$

let,

$$g(t) = \frac{f_c(t)}{f_s(t)} = \frac{2\propto\left(2e^{-\propto t}+e^{-2\propto t}-3e^{-1.5\propto t}\right)}{2\propto\left(e^{-\propto t}-e^{-2\propto t}\right)}$$

$$=\frac{2e^{-\propto t}+e^{-2\propto t}-3e^{-1.5\propto t}}{e^{-\propto t}-e^{-2\propto t}}$$

We need to show that g(t) is an increasing function of t.

The numerator of $\frac{d}{dt(g(t))}$ is given by:

$1.5e^{-2.5\alpha t}+1.5e^{-3.5\alpha t}-3e^{-3\alpha t}$ and is greater than 0 if $e^{\alpha t}-2e^{0.5\alpha t}+1\ge 0$ which is always true since $e^{\alpha t}-2e^{0.5\alpha t}+1 = (e^{0.5\alpha t}-1)^2 \ge 0$ always.

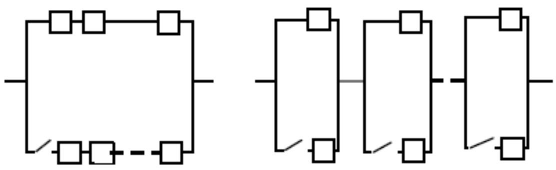
When the original system is 1-out-of-n:F system with unit lifetimes are iid exponentially distributed with parameter α and the spare unit lifetimes are iid exponent-tially distributed with parameter 0.5 α, then the previous result is still true under certain condition as shown below.

Result -8- (Use assumptions 1-3)

Consider 1-out-of-2: F system. Let T_1, T_2,…, T_n be iid lifetimes of the original units. T_i is an exponential random variable with parameter α. Let U_1, U_2,…, U_n be iid lifetimes of the spare units. U_i is an exponential random variable with parameter 0.5 α. Then

$$\tau_{1-n:F}(T) + \tau_{1-n:F}(U) \leq_{lr} \tau_{1-n:F}(T+U)$$

Figure 9 below represents cold-standby redundancy on system and component levels.



(a) **(b)**

Fig. 9: a) System redundancy, b) component redundancy

Proof:

Define the distribution function of $\tau_{1-n:F}(T)$ by $(1-e^{-n\alpha t})$ and that of $\tau_{1-n:F}(U)$ by$(1-e^{-0.5n\alpha t})$ we have,

$$F_s(t) = P(\tau_{1-n:F}(T) + \tau_{1-n:F}(U) \leq t)$$

$$=\int_0^t \left(1-e^{-0.5n\propto(t-x)}\right) d\left(1-e^{-n\propto x}\right)$$

$$=\int_0^t \left(1-e^{-0.5n\propto(t-x)}\right).\left(n\propto e^{-n\propto x}\right)dx$$

$$=1+e^{-n\propto t} - 2e^{-0.5n\propto t}$$

and

$$f_s(t) = \frac{d}{dt}F_s(t)$$

$$=n\propto e^{-0.5n\propto t}\left(1-e^{-0.5n\propto t}\right)$$

The distribution function of T_i+ U_i is defined by:

$$P(T_i + U_i \leq t) = \int_0^t \left(1-e^{-0.5\propto(t-x)}\right) d\left(1-e^{-\propto x}\right)$$

$$=1+e^{-\propto t} - 2e^{-0.5\propto t}$$

Thus,

$$F_c(t) = P(\tau_{1-n:F}(T+U) \leq t)$$

$$=1-\left(1-P(T_i+U_i \leq t)\right)^n$$

$$=1-\left(2e^{-0.5\propto t} - e^{-\propto t}\right)^n$$

and

$$f_c(t) = \frac{d}{dt}F_c(t)$$

$$= n\propto e^{-0.5n\propto t}\left(2-e^{-0.5\propto t}\right)^{n-1}.\left(1-e^{-0.5\propto t}\right)$$

Let,

$$g(t) = \frac{f_c(t)}{f_s(t)}$$

$$= \frac{n\propto e^{-0.5n\propto t}\left(2-e^{-0.5\propto t}\right)^{n-1}.\left(1-e^{-0.5\propto t}\right)}{n\propto e^{-0.5n\propto t}\left(1-e^{-0.5n\propto t}\right)}$$

$$= (2-e^{-0.5\alpha t})^{n-1}.\frac{(1-e^{-0.5\alpha t})}{(1-e^{-0.5n\alpha t})}$$

To prove the required result, we need to show that g (t) is an increasing function of t.

Letting,

$$L_1(t) = \left(2-e^{-0.5\propto t}\right)^{n-1}$$

and

$$L_2(t) = \frac{\left(1-e^{-0.5\propto t}\right)}{\left(1-e^{-0.5n\propto t}\right)}$$

$$g(t) = L_1(t).L_2(t)$$

It is obvious that $L_1(t)$ is an increasing function of t.

Now, by proving that $L_2(t)$ is an increasing function of t we get the required result.

The numerator of $\frac{d}{dt}L_2(t)$ is $0.5\alpha e^{-0.5\alpha t}$ - $0.5\alpha e^{-0.5\alpha(n+1)t}$ - $0.5n\alpha e^{-0.5n\alpha t}$ + $0.5n\alpha e^{-0.5\alpha(n+1)t}$

We can notice that

$(0.5n\alpha e^{-0.5\alpha(n+1)t} - 0.5\alpha e^{-0.5\alpha(n+1)t})$

$= 0.5\alpha e^{-0.5\alpha(n+1)t}(n-1)>0$

and that

$(0.5\alpha e^{-0.5\alpha t} - 0.5n\alpha e^{-0.5n\alpha t}) > 0$ if

$(1- ne^{-0.5\,(n-1)\,\alpha\,t}) > 0$

i.e., if $t > \frac{2\ln(n)}{\alpha(n-1)}$

Thus, $\frac{d}{dt}L_2(t) > 0$ under the condition that $t > \frac{2\ln(n)}{\alpha(n-1)}$ and $L_1(t)$ is increasing for all t.

Hence, g(t) is an increasing function of t under the condition $t > \frac{2\ln(n)}{\alpha(n-1)}$ and this completes the proof.

For 1-out-of-n:F system, suppose that the lifetimes of all the units in the redundant group (the original units and the spare units) are iid random variables with common distribution function $F(t)=1-e^{-rt}$, then cold-standby redundancy on component level is better in the sense of likelihood ratio ordering.

Result -9- (Use assumptions 1-3)
Let $T_1, T_2, \ldots, T_n$ be the lifetimes of the original units and $U_1, U_2, \ldots, U_n$ be the lifetimes of the spare units. And suppose that all the lifetimes are iid exponentially distributed with parameter r. Then

$$\tau_{1-n:F}(T) + \tau_{1-n:F}(U) \le_{lr} \tau_{1-n:F}(T+U)$$

Proof:
Define the distribution function of $\tau_{1-n:F}(T)$ by $(1-e^{-nrt})$. The distribution function of $\tau_{1-n:F}(U)$ is defined similarly.
We have,

$$F_s(t) = P(\tau_{1-n:F}(T) + \tau_{1-n:F}(U) \le t)$$

$$= \int_0^t \left(1 - e^{-nr(t-x)}\right) d(1 - e^{-nrx})$$

$$= \int_0^t \left(1 - e^{-nr(t-x)}\right).(nre^{-nrx})dx$$

$$= 1 - e^{-nrt}(1 + nrt)$$

and,

$$f_s(t) = \frac{d}{dt}F_s(t)$$

$$= n^2r^2te^{-nrt}$$

The distribution function of $T_i + U_i$ is defined to be

$$P(T_i + U_i \le t) = \int_0^t \left(1 - e^{-r(t-x)}\right) d(1 - e^{-rx})$$

$$= 1 - e^{-rt}(1 + rt)$$

Thus,

$$F_c(t) = P(\tau_{1-n:F}(T+U) \le t)$$

$$= 1 - \left(1 - P(T_i + U_i \le t)\right)^n$$

$$= 1 - e^{-nrt}(1 + rt)^n$$

and,

$$f_c(t) = nr^2te^{-nrt}(1 + rt)^{n-1}$$

Let,

$$g(t) = \frac{f_c(t)}{f_s(t)} = \frac{1}{n}(1 + rt)^{n-1}$$

Which is an increasing function of t and hence the result.
In the second step, we consider the case in which the reliability of the switching device is less than one. In this case the switching is described as imperfect.

To calculate the reliability of the resultant system, consider the case of two units in standby system the unit lifetimes are iid exponentially distributed random variables with parameter r, let T be the lifetime of the resultant system. Let A_1 denotes the event that the switching device fails and A_2 denotes the event that the switching device operates successfully and let P_s be the probability of successful operation of the switching device. Denote by Q_1 the unreliability of the original unit with successfully operating switching and by Q_2 the unreliability of the two units standby system with successfully operating switching. Then from the formula for total probabilities, and since Q_2 is a statement about the convolution of the distributions of the two units we have,

$$P(T \le t) = P(A_1).Q_1(t) + P(A_2).Q_2(t)$$

$$= (1 - P_s).(1 - e^{-rt}) + P_s.(1 - e^{-rt} - rte^{-rt})$$

$$= 1 - (e^{-rt} + P_srte^{-rt})$$

Thus,

$$P(T > t) = e^{-rt}.(1 + P_srt)$$

Result -10- (Use assumptions 1-3)
Let the spare and the original unit lifetimes be iid exponentially distributed random variables with parameter r. Then

$$\tau_{1-n:F}(T) + \tau_{1-n:F}(U) \le_{fr} \tau_{1-n:F}(T+U)$$

Where $T=(T_1, T_2, \ldots, T_n)$ be the vector of the original unit lifetimes and $U=(U_1, U_2, \ldots, U_n)$ be the vector of the spare unit lifetimes.

Proof:
Clearly, the distribution function of $\tau_{1-n:F}(T)$ and $\tau_{1-n:F}(U)$ is $(1-e^{-nrt})$, then the reliability function of $\tau_{1-n:F}(T) + \tau_{1-n:F}(U)$ is defined as:

$$R_s(t) = 1 - P(\tau_{1-n:F}(T) + \tau_{1-n:F}(U) \le t)$$

$$= 1 - ((1 - P_s).(1 - e^{-nrt}) + P_s.(1 - e^{-nrt} - nrte^{-nrt}))$$

$$= e^{-nrt}.(1 + P_snrt)$$

The reliability function of $T_i + U_i$ is defined as:

$$P(T_i + U_i > t) = e^{-rt}.(1 + P_srt)$$

Thus,

$$R_c(t) = P(\tau_{1-n:F}(T+U) > t)$$

$$= e^{-nrt}.(1 + P_srt)^n$$

Let,

$$g(t) - \frac{R_c(t)}{R_s(t)} - \frac{(1 + P_srt)^n}{(1 + P_snrt)}$$

The numerator of $\frac{d}{dt(g(t))}$ is given by:

$nP_s r(1+P_s rt)^{n-1}.(P_s rt-P_s rt)$, this function is positive.

Obviously, g(t) is an increasing function of t and this completes the proof.

In the next result we prove that, if $t > \frac{1}{rP_s}$ then the previous result is still true when the comparison is carried out according to the likelihood ratio ordering.

Result -11- (Use assumptions 1-3)

Let the spare and the original unit lifetimes be iid exponentially distributed random variables with parameter r. Then

$$\tau_{1-n:F}(T) + \tau_{1-n:F}(U) \leq_{lr} \tau_{1-n:F}(T+U)$$

Proof:

We have,

$$f_c(t) = -\frac{d}{dt} R_c(t)$$

$$= -\frac{d}{dt}[(1+P_s rt)^n . e^{-nrt}]$$

and,

$$f_s(t) = -\frac{d}{dt} R_s(t)$$

$$= -\frac{d}{dt}[(1+P_s nrt). e^{-nrt}]$$

To prove that, $\tau_{1-n:F}(T) + \tau_{1-n:F}(U) \leq_{lr} \tau_{1-n:F}(T+U)$ it is enough to show that

$$g(t) = \frac{f_c(t)}{f_s(t)} = \frac{-\frac{d}{dt}\left[(1+P_s rt)^n . e^{-nrt}\right]}{-\frac{d}{dt}\left[(1+P_s rt). e^{-nrt}\right]}$$ is increasing in t.

Simplifying g(t) we get,

$$g(t) = \frac{(1+P_s rt)^{n-1}.(1-P_s+P_s rt)}{(1-P_s+P_s nrt)}$$

After simplification, the numerator of $\frac{d}{dt(g(t))}$ will be

$$P_s r(n-1)(1-P_s+P_s nrt)(1-P_s+P_s rt)(1+P_s rt)^{n-2} + P_s r(1+P_s rt)^{n-1}(1-P_s+P_s nrt) - nrP_s(1-P_s+P_s rt)(1+P_s rt)^{n-1}$$

Therefore, g(t) will be increasing if the numerator is positive which is reduced to

$$-nP_s(1-P_s) + n^2 rtP_s(1-P_s) + P_s(1-P_s) - nrtP_s(1-P_s) + nr^2t^2P_s^2(n-1) \geq 0$$

Now, assuming that $t > \frac{1}{rP_s}$ we have,

$$-nP_s(1-P_s) + P_s(1-P_s) + nr^2t^2P_s^2(n-1) > -nP_s(1-P_s) + P_s(1-P_s) + n(n-1) = n(n-1)(n - P_s(1-P_s))$$

Which is positive always.

Partially Energized Standby redundancy

In the case of partially energized standby redundancy, the spare units are in a partially energized state up to the instant they replace the primary units and then begin to operate in normal operating conditions. During period they are in standby state, they can fail but their probability of failure is less than that while in an operating state.

Here, we discuss partially energized standby redundancy on component and system levels using an automatic switching device, which instantaneously inserts the next unit when the original unit has failed.

In the first step, we consider the case of two exponentially distributed units in standby system for which r_1 is the failure rate of the normal operating unit, r_2 is the failure rate of the standby unit when operating and r_3 is its failure rate when in a standby situation. Also, we consider that the switching device is absolutely reliable (has reliability one) this switching is described as perfect switching.

The standby system operates successfully during (0, t] if the first unit does not fail during the interval (0, t) or the first unit fails by time t_1, t_1<t and the standby unit does not fail during

(0, t_1] and does not fail while operating during (t_1, t].

Thus, the reliability function of this system is

$$P(T>t) = e^{-r_1 t} + \int_{t_1=0}^{t} r_1 e^{-r_1 t_1}. e^{-r_3 t_1}. e^{-r_2(t-t_1)}\, dt_1$$

$$= e^{-r_1 t} + r_1 \int_0^t e^{-((r_3+r_1)t_1 + r_2(t-t_1))}\, dt_1$$

$$= e^{-r_1 t} + r_1 \int_0^t e^{-(r_1-r_2+r_3)t_1}. e^{-r_2 t}\quad dt_1$$

$$= e^{-r_1 t} + r_1. e^{-r_2 t}. \left(\frac{1-e^{-(r_1-r_2+r_3)t}}{r_1-r_2+r_3}\right)$$

$$= e^{-r_1 t} + \frac{r_1 e^{-r_2 t}}{r_1-r_2+r_3} - \frac{r_1 e^{-(r_1+r_3)t}}{r_1-r_2+r_3}$$

Result -12- (Use assumptions 1-3)

Consider 1-out-of-2:F system. Let T_1 ,T_2 be the lifetimes of the original units and U_1 ,U_2 be the lifetimes of the spare units. If $T_i \sim NE(r)$, $U_i \sim NE(r)$, $r = \alpha\ r_3$, $\alpha > 1$. Then partially energized standby redundancy on component level is better than that on system level under the usual stochastic ordering.

Proof:

$$R_c(t) = e^{-2rt}.\left(1+\frac{r}{r_3}-\frac{r}{r_3}e^{-r_3 t}\right)^2$$

$$R_s(t) = e^{-2rt}.\left(1+\frac{r}{r_3}-\frac{r}{r_3}e^{-2r_3 t}\right)$$

$R_c(t) \geq R_s(t)\ iff\ (1+\propto -\propto e^{-r_3 t})^2 \geq 1+\propto -\propto e^{-2r_3 t}$

After simplification we conclude that $R_c(t) \geq R_s(t)$ iff $\alpha^2\left(1-e^{-r_3 t}\right)^2+\alpha\left(1-e^{-r_3 t}\right)^2 \geq 0$ and the result.

In the second step, we consider the case in which the reliability of the switching device is less than one. In this case the switching is described as imperfect.

To calculate the reliability of the resultant system, consider the case of two exponentially distributed units in standby system for which r_1 is the failure rate of the normal operating unit, r_2 is the failure rate of the standby unit when operating and r_3 is its failure rate when in a standby situation. We also consider that the switching device is imperfect, has reliability less than one, and the probability of successful operation of the switching device is denoted by P_s.

The standby system may survive (0, t] if the first unit does not fail in (0, t] or the first unit fails by time t_1, $t_1<t$, the switching device is able to activate the standby unit and the standby unit does not fail during $(0, t_1]$, is activated at time t_1, and does not fail while operating during $(t_1, t]$. Let T be the lifetime of the resultant system.

Thus, the reliability function of this system is $P(T>t)$.

$$= e^{-r_1 t}+\int_{t_1=0}^{t} P_s \cdot r_1 e^{-r_1 t_1}.e^{-r_3 t_1}.e^{-r_2(t-t_1)}\,dt_1$$

$$= e^{-r_1 t}+\frac{P_s \cdot r_1}{r_1-r_2+r_3}\cdot\left(e^{-r_2 t}-e^{-(r_1+r_3)t}\right)$$

Result -13- (Use assumptions 1-3)

Consider 1-out-of-2: F system. Let T_1, T_2 be the lifetimes of the original units and U_1, U_2 be the lifetimes of the spare units. If $T_i \sim NE(r)$, $U_i \sim NE(r)$, $r = \alpha\ r_3$, $\alpha > 1$. Then partially energized standby redundancy on component level is better than that on system level under the usual stochastic ordering.

Proof:

$$R_s(t) = e^{-2rt}.\left(1+\frac{r}{r_3}P_s-\frac{r}{r_3}P_s\, e^{-2r_3 t}\right)$$

$$R_c(t) = e^{-2rt}.\left(1+\frac{r}{r_3}P_s-\frac{r}{r_3}P_s\, e^{-r_3 t}\right)^2$$

$R_c(t) \geq R_s(t)\ iff\ (1+\propto P_s-\propto P_s\, e^{-r_3 t})^2 \geq 1+\propto P_s-\propto P_s\, e^{-2r_3 t}$

After simplification we conclude that $R_c(t) \geq R_s(t)$ iff $\alpha^2 P_s^2\left(1-e^{-r_3 t}\right)^2+\alpha P_s\left(1-e^{-r_3 t}\right)^2 \geq 0$ iff and the result.

Conclusions

In this research, we put some conditions under which we conclude that providing cold-standby redundancy on component level may be more effective, according to some types of stochastic orderings, than that on system level. The reverse may be true under some other suggested conditions. If we consider the case of active redundancy, we conclude that, under some conditions, active redundancy on component level is the best according to some types of stochastic orderings. With respect to partially energized standby redundancy, we have the result that, under some conditions, partially energized standby redundancy on component level is the best according to the usual stochastic ordering.

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