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## Expected and Variance for a Single Machine Issue with Linear Decreasing Processing Time Deterioration

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### Abstract

We consider linear decreasing deteriorating of processing times of jobs on single machine, i.e.,  $P_i = P_i - \beta_i t_i$ ,  $0 < \beta_i < 1$  and  $t_i \geq 0$ , where  $P_i$  is actual process,  $P_i$  is a basic process for job  $i$ ,  $\beta_i$  is decreasing rate,  $t_i$  is a starting time of job  $i$  with  $n$ - jobs, non- preemptive, no – idle on a single machine. We study some classical criteria as make span, total completion time, total lateness times, total waiting times, maximum lateness times and maximum tardiness times, important conclusion dependent on the decreasing rate of expected of basic process of jobs with respect to its decreasing rate when a basic process is random variable in general case.

**Keywords:** Single machine scheduling, Linear decreasing deterioration, Various performance measure, Expected and variance analysis.

### التوقع والتباين لمسألة الماكينة الواحدة مع تناقص خطي متدهور لوقت المعالجة

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### الخلاصة

سوف نفترض التدهور المتناقص الخطي لأوقات معالجة الوظائف على ماكينة واحدة، أي ان  $P_i = P_i - \beta_i t_i$  ،  $0 < \beta_i < 1$  ، و  $t_i \geq 0$  ، حيث  $P_i$  هو وقت المعالجة الفعلي ،  $P_i$  هو وقت المعالجة الأساسي للوظيفة  $i$  ،  $\beta_i$  هو معدل التناقص ،  $t_i$  هو وقت بدء الوظيفة  $i$  ، مع  $n$  من الوظائف والتي تكون غير استباقية، وغير قابلة للتعطيل على ماكينة واحدة . كما ندرس بعض المعايير ودوال الهدف الكلاسيكية مثل فترة التصنيع، وإجمالي وقت الإكمال، وإجمالي أوقات التأخير، وإجمالي أوقات الانتظار، وإقصى أوقات التأخير المحدد وإقصى أوقات التأخير المعتاد، واستنتاج مهم يعتمد على معدل التناقص المتوقع لوقت المعالجة الأساسية للوظائف فيما يتعلق بمعدل تناقصها عندما يكون وقت المعالجة الأساسي هو متغيرًا عشوائيًا مستقلًا في الحالة العامة.

## 1. Introduction

In first-rate theory of scheduling, the duration of jobs processing is assumed to be set and well-known, for instance in resource [1] and [2], nevertheless, there are numerous

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circumstances when a job that is processed more slowly than the identical job at the same time earlier, in this context, scheduling is referred to as deteriorating jobs. However, most studies assume the assumption that the job's starting time and its actual processing time are related linearly. This presumption may only reflect a few circumstances in numerous reasonable circumstances, the deterioration rate of the job could grow or shrink over time, for instance, as the surface cools, the ingot's temperature in the rolling machine may decrease more slowly. As a result, the ingot's temperature drop may have decreased as show in this project we present decreasing linear processing times as time process.

Scheduling analysis when a job's processing depending on when it started was presented by Browne and Yechiali [3], and Mosheiov [4] in which they considered the minimizing of flow time problems in the linear degrading model assuming constant fundamental processing times. Researchers constructed models where the actual processing time of a job is a function of its starting time, Kubiak and Van De Velde [5] introduction scheduling deteriorating jobs to minimize makespan, Alidaee and Womer [6] divided models of scheduling with deteriorating jobs into three categories, namely linear non-linear and piecewise linear functions.

Bachman and Janiak showed in [7] that, with the linear deterioration assumption, the maximum lateness minimization is NP-complete, and two heuristic methods were consequently displayed. Bachman et al.[8] looked at three scheduling issues involving workloads that were deteriorating in order to reduce the overall time spent on a single machine.

Afterwards, Cheng and Ding [9] investigated the challenge of scheduling a series of start-time-dependent jobs on just one machine with the same starting processing times, predefined deadlines, and processing rates. After that Chung et al. [10] under the basic linear deterioration assumption, the single machine problem was taken into consideration in order to minimize the sum of squares of work lateness, i.e.,  $P_i = \beta_i t_i$  . Chen [11] examine a single-processor scheduling structure in which a task's execution time decreases linearly with its start time, and showed an  $O(n)^2$  to reduce the amount of late work, use the time dynamic programming approach. Gawiejnowicz, Kurc and Stanislaw [12] they discussed examination of a scheduling issue that varies in time using degradation rate sequence identities.

There is a comprehensive and accurate review in the research of both Alidaee and Womer [6] and Cheng et al. [13]. After that, Zhao and Tang [14] considered the situations involving single machine scheduling, in which the job processing time decreases linearly with the job start time. Considering this presumption, they offered the best way to solve the issues in order to reduce the total amount of early penalties associated with the late jobs, to minimize the makespan with the limits on total resource consumption and correspondingly, to minimize the resource consumption with makespan constraints.

Wang and Xia [15], Wang [16] agreement with single – machine scheduling issues involving decreasing linear deteriorating i.e., jobs where the processing times decrease in relation to their initial times. Furthermore, a series of precedence restrictions on a parallel graph and parallel chains link the jobs, respectively. For more current issues with decreasing linear deteriorating jobs in different setting, we invite the attention of the reader to Cai and Zhu [17], Zou- Zhang [18], Mosheiov [19], [20] and etc.

On the other hand, some researchers touched on the concept of learning, for instance, businesses are shifting to shorter production runs and more frequent product modifications as

the industrial environment grows more competitive and customers demand a wider range of products. Because employees often switch between activities and responsibilities before achieving mastery, the learning and forgetting that occurs in this setting has thus taken on more significance Yang and Kuo [21]. Biskup [22], in his review, he first go over the reasons and circumstances under which learning effects in scheduling contexts might arise and how planning can be approached.

D. Xu and Y. Yin [23] presented a comprehensive scheduling model that significantly generalizes several of the models that are already available in the literature by incorporating the impacts of learning and degradation at the same time. Job processing times are determined by functions of their start times and sequence positions, i.e., by the effects of learning and degradation.

W. lee and P. lai [24] offered a broad scheduling approach where the function's form is left open-ended. The actual job processing time, according to the suggested model, is a general function of the jobs' planned position and their previous processing times.

W. Yang and S. Chand [25] , suggested researching how learning and forgetting affect the issue of scheduling several jobs on a single machine in order to shorten the time it takes for each job to be completed. When a device switches from one family to another for task processing, setup time is involved. Examination of how learning and forgetfulness impact the group scheduling issue. Wang et al. [26] investigated resource allocation and single-machine scheduling issues with a reduced learning effect. Li et al.[27] thought about scheduling a flow shop with broad shortened learning effects. They suggested a branch-and-bound approach along with a few heuristics for makespan minimization. However, Ren et al.[28] took into account both the delivery time and the shortened learning impact. Z. Liu and J. Wang [29] studied the single machine scheduling problem with a shortened learning effect establish the ideal work plan to reduce the overall weighted completion time and increase tardiness. Additionally, we provide the reader with a selection of research sources that focus on scheduling in a typical situation and their allocation in the degradation sector, see [30] , [31] and [32].

The use of the linear type of decreasing rate in processing time in many classical objective functions is emphasized in this study, as is the significance of the relative amount between the expected main processing time (in the case of a random variable) and the decreasing rate. This amount will have a major influence on many theories and the determination of the best scheduling of target functions within a single machine, however, this study stands out for expanding the theoretical understanding of a significant set of theories and conclusions. In order to facilitate the reader's access to the evidence, we also planned to provide a thorough explanation.

The arrangement of this paper is as follows: Section 2 describe the issues that are subject to the condition of linear decreasing deterioration, a set of theories that are subject to expectation and variance were derived. Whereas the conclusions can be found in Section 3.

## 2. Problem description

In this section we describe and formulate our considered problem. There are  $n$  jobs must be processing by single machine, each job equipped at the time zero to be processed and a basic processing time  $P_i$  of job  $i$  ( $i = 1, 2, \dots, n$ ) without disruption or precedence. The machine is always accessible and it can handle no more the one job on a time , nonetheless, the activity real processing times decreasing (deteriorating) with its starting time  $t_i$  , each job have different decreasing rate  $\beta_i$  , i.e.,  $P_i = P_i - \beta_i t_i$  ,  $0 < \beta_i < 1$  and  $t_i \geq 0$  is

when the job  $i$  officially begins,  $w_i, d_i, r_i$  ( weighted, due date and release time, respectively ) are independent in any sequence and the basic processing time is independent random variable.

In below some notations and definition which we need in the remainder of this paper:

- $P_i$  Basic Processing time for job  $i$ , shows how long job  $j$  took to process on machine  $i$ .
- $\beta_i$  Decreasing rate for job  $i$ .
- $d_i$  Due date of job  $i$ , the date which job  $i$  should be complete.
- $w_i$  Weight of job  $i$ , as a measure of job  $j$ 's importance in relation to other jobs in the system.
- $E_i$  Earliness of job  $i, E_i = \max\{d_i - C_i, 0\}$ .
- $C_i$  Completion time for job  $i$ .
- $L_i$  Lateness of job  $i, L_i = C_i - d_i$ .
- $T_i$  Tardiness of job  $i, T_i = \max \{L_i, 0\}$ .
- $C_{max}$  Makespan, the maximum time a work may be completed in the schedule.
- $L_{max}$  Maximum lateness, represent the greatest lateness, or  $L_{max}$ . The greatest date-related infraction is measured.
- $T_{max}$  Maximum tardiness.
- $\sum_{i=1}^n C_i$  The total amount of time that each job must wait until the system processes it completely is indicated by this objective function.
- $\sum_{i=1}^n w_i C_i$  Total weighted completion time represent the overall holding or inventory costs incurred by the schedule may be determined by adding up the weighted completion times of the  $n$  jobs.

$$\begin{aligned}
 P_1 &= P_1 \text{ (the actual processing of job 1 when the starting time } t_1 = 0\text{).} \\
 P_2 &= P_2 - \beta_2 P_1 \text{ (the starting time for job 2 is actually when the first job is done).} \\
 P_3 &= P_3 - \beta_3(P_1 + P_2) \\
 &= P_3 - \beta_3(P_1 + P_2 - \beta_2 P_1) \\
 &= P_3 - \beta_3 P_1 - \beta_3 P_2 + \beta_2 \beta_3 P_1. \\
 P_4 &= P_4 - \beta_4(P_1 + P_2 + P_3) \\
 &= P_4 - \beta_4(P_1 + P_2 - \beta_2 P_1 + P_3 - \beta_3 P_1 - \beta_3 P_2 + \beta_2 \beta_3 P_1) \\
 &= P_4 - \beta_4 P_1 - \beta_4 P_2 + \beta_2 \beta_4 P_1 - \beta_4 P_3 + \beta_3 \beta_4 P_1 + \beta_3 \beta_4 P_2 - \beta_2 \beta_3 \beta_4 P_1. \\
 &\dots \\
 P_i &= P_i - \beta_i P_1 - \beta_i P_2 \dots - \beta_i P_{i-1} + \beta_i \beta_{i-1} P_1 + \beta_i \beta_{i-1} P_2 + \dots + \beta_i \beta_{i-1} P_{i-2} \\
 &\quad - \beta_i \beta_{i-1} \beta_{i-2} P_1 - \beta_i \beta_{i-1} \beta_{i-2} P_2 - \dots - \beta_i \beta_{i-1} \beta_{i-2} P_{i-3} + \dots + \beta_i \beta_{i-1} \beta_{i-2} \dots \beta_1 P_1.
 \end{aligned}$$

To determine the time required for completion of job  $i$  :

$$\begin{aligned}
 C_i &= \sum_{l=1}^i P_l \\
 &= P_1 + P_2 + P_3 + \dots + P_i. \\
 &= P_1 + P_2 - \beta_2 P_1 + P_3 - \beta_3 P_1 - \beta_3 P_2 + \beta_2 \beta_3 P_1 + P_4 - \beta_4 P_1 - \beta_4 P_2 + \beta_2 \beta_4 P_1 - \\
 &\quad \beta_4 P_3 + \beta_3 \beta_4 P_1 + \beta_3 \beta_4 P_2 - \beta_2 \beta_3 \beta_4 P_1 + \dots + P_i - \beta_i P_1 - \beta_i P_2 \dots - \beta_i P_{i-1} + \\
 &\quad \beta_i \beta_{i-1} P_1 + \beta_i \beta_{i-1} P_2 + \dots + \beta_i \beta_{i-1} P_{i-2} - \beta_i \beta_{i-1} \beta_{i-2} P_1 - \beta_i \beta_{i-1} \beta_{i-2} P_2 - \dots - \\
 &\quad \beta_i \beta_{i-1} \beta_{i-2} P_{i-3} + \dots + \beta_i \beta_{i-1} \beta_{i-2} \dots \beta_2 P_1. \\
 &= P_1(1 - \beta_2 - \beta_3 + \beta_2 \beta_3 - \beta_4 + \beta_2 \beta_4 + \beta_3 \beta_4 - \beta_2 \beta_3 \beta_4 + \dots + \beta_i \beta_{i-1} \beta_{i-2} \dots \beta_2) + \\
 &\quad P_2(1 - \beta_3 - \beta_4 + \beta_3 \beta_4 - \dots + \beta_i \beta_{i-1} \beta_{i-2} \dots \beta_3) + \dots + P_{i-1}(1 - \beta_i) + P_i.
 \end{aligned}$$

Represents the completion time of the job  $j$  by

$$C_j = \sum_{l=1}^j P_l \prod_{r=l+1}^j (1 - \beta_r).$$

$$C_{j+1} = (1 - \beta_{j+1})C_j + P_{j+1}.$$

$$C_{j+2} = (1 - \beta_{j+2})(1 - \beta_{j+1})C_j + (1 - \beta_{j+2})P_{j+1} + P_{j+2}.$$

We initially declare that will be employed in the sequel's proof of the qualities before providing the primary findings.

The next lemma (will prove it in our way) by Rau [33] who stated it with summation reversed.

**Lemma 2.1:** The summation  $\sum_{i=1}^n \mu_{\pi(i)} \prod_{r=i+1}^n \gamma_{\pi(r)}$  is minimizing when computed throughout the permutation ordered by increasing values of  $\frac{\mu_i}{\gamma_{i-1}}$  (i.e.  $\frac{\mu_1}{\gamma_1-1} < \frac{\mu_2}{\gamma_2-1} < \dots < \frac{\mu_n}{\gamma_n-1}$ ), where  $\mu_i$  and  $\gamma_i$  positive function .

**Proof:**

Let  $\pi = (1,2,3, \dots, k, k + 1, \dots, n)$  a permutation which is not order by increasing value of  $\frac{\mu_i}{\gamma_{i-1}}$ , that means, there is adjacent pair (two numbers that are next to each other in a sequence or list) of index  $k, k + 1$  such that  $\frac{\mu_{\pi(k)}}{\gamma_{\pi(k)-1}} > \frac{\mu_{\pi(k+1)}}{\gamma_{\pi(k+1)-1}}$ .

Let  $\pi^* = (1,2,3, \dots, k + 1, k, \dots, n)$  is other a permutation has same order of  $\pi$  with interchange adjacent pair of index  $k + 1, k$ , now

$$\sum_{i=1}^n \mu_{\pi(i)} \prod_{r=i+1}^n \gamma_{\pi(r)}.$$

We will divide this summation into four parts according to the index, from 1 to  $k - 1, k, k + 1$  and  $k + 2$  to  $n$ .

$$= \sum_{i=1}^{k-1} \mu_{\pi(i)} \prod_{r=i+1}^n \gamma_{\pi(r)} + \mu_{\pi(k)} \prod_{r=k+1}^n \gamma_{\pi(r)} + \mu_{\pi(k+1)} \prod_{r=k+2}^n \gamma_{\pi(r)} + \sum_{i=k+2}^n \mu_{\pi(i)} \prod_{r=i+1}^n \gamma_{\pi(r)}$$

and

$$\sum_{i=1}^n \mu_{\pi^*(i)} \prod_{r=i+1}^n \gamma_{\pi^*(r)}$$

$$= \sum_{i=1}^{k-1} \mu_{\pi^*(i)} \prod_{r=i+1}^n \gamma_{\pi^*(r)} + \mu_{\pi^*(k+1)} \prod_{r=k+2}^n \gamma_{\pi^*(r)} + \mu_{\pi^*(k)} \prod_{r=k+1}^n \gamma_{\pi^*(r)}$$

$$+ \sum_{i=k+2}^n \mu_{\pi^*(i)} \prod_{r=i+1}^n \gamma_{\pi^*(r)}.$$

All the terms of this index of  $\pi, \pi^*$  in this summation are same except  $k, k + 1$ , then

$$\sum_{i=1}^n \mu_{\pi(i)} \prod_{r=i+1}^n \gamma_{\pi(r)} - \sum_{i=1}^n \mu_{\pi^*(i)} \prod_{r=i+1}^n \gamma_{\pi^*(r)}.$$

The only result of this statement are the different terms,

$$= \mu_{\pi(k)} \prod_{r=k+1}^n \gamma_{\pi(r)} + \mu_{\pi(k+1)} \prod_{r=k+2}^n \gamma_{\pi(r)} - \mu_{\pi^*(k+1)} \prod_{r=k+2}^n \gamma_{\pi^*(r)} - \mu_{\pi^*(k)} \prod_{r=k+1}^n \gamma_{\pi^*(r)}$$

with  $\mu_{\pi(k)} = \mu_{\pi^*(k+1)}$  and for any  $k = 1,2, \dots, n$  (cause  $\pi^*$  have same order of  $\pi$  with interchange adjacent pair of index  $k + 1$  and  $k$ ), then

$$\mu_{\pi(k)} \left[ \prod_{r=k+1}^n \gamma_{\pi(r)} - \prod_{r=k+2}^n \gamma_{\pi^*(r)} \right] - \mu_{\pi(k+1)} \left[ \prod_{r=k+1}^n \gamma_{\pi^*(r)} - \prod_{r=k+2}^n \gamma_{\pi(r)} \right]$$

$$= \prod_{r=k+2}^n \gamma_{\pi(r)} \mu_{\pi(k)} [\gamma_{\pi(k+1)} - 1] - \prod_{r=k+2}^n \gamma_{\pi(r)} \mu_{\pi(k+1)} [\gamma_{\pi(k+1)} - 1].$$

And by  $\frac{\mu_k}{\gamma_{k-1}} > \frac{\mu_{k+1}}{\gamma_{k+1-1}}$ , it implies  $\mu_k(\gamma_{k+1} - 1) > \mu_{k+1}(\gamma_k - 1)$  then

$$\sum_{i=1}^n \mu_{\pi(i)} \prod_{r=i+1}^n \gamma_{\pi(r)} - \sum_{i=1}^n \mu_{\pi^*(i)} \prod_{r=i+1}^n \gamma_{\pi^*(r)}$$

will be must great than zero, that means the summation  $\sum_{i=1}^n \mu_{\pi(i)} \prod_{r=i+1}^n \gamma_{\pi(i)}$  is not minimizing without ordered by increasing values of  $\frac{\mu_i}{\gamma_{i-1}}$ .

**Theorem 2.2: Makespan**

The Longest Deteriorating Basic Processing Time (LDBPT) yields on optimal schedule for problem  $1/P_i = P_i - \beta_i t_i / C_{max}$ , where the make span is

$$C_{max} = \sum_{i=1}^n P_i \prod_{r=i+1}^n (1 - \beta_r).$$

**Proof:**

The proof is direct from Lemma 2.1 (note that the condition of positive  $(1 - \beta_i)$  is satisfy with consideration  $0 < \beta_i < 1$ ),

if  $\frac{P_1}{(1-\beta_1)-1} < \frac{P_2}{(1-\beta_2)-1} < \dots < \frac{P_n}{(1-\beta_n)-1} = \frac{P_1}{-\beta_1} < \frac{P_2}{-\beta_2} < \dots < \frac{P_n}{-\beta_n}$  and that implies to  $\frac{P_1}{\beta_1} > \frac{P_2}{\beta_2} > \dots > \frac{P_n}{\beta_n}$ , then the problem is minimizing.

The question of uncertainty is crucial to production scheduling, just like it is in every other area of the manufacturing industry. The scheduling system's randomness may be caused by, among other things, inconsistent processing times, equipment failures, and imprecise client due date information. Some negative consequences of inadequately addressing or accounting for the variability in the scheduling parameters would be excess inventory, unstable systems, and dissatisfied customers owing to missed deadlines.

Since the 1960s, researchers have examined the issue of project scheduling in specific or ambiguous situations. Initially, Probability theory was brought to the project scheduling problem by Freeman [34]. Through chance-constrained programming, Charnes and Cooper [35] investigated the stochastic project scheduling problem. McKay, Safayeni and Buzacott [36] argued that the main reason why scheduling theory is not very applicable in real-world situations is that it cannot adequately account for large fluctuations in processing times.

B. Dodin [37] argued that the objectives of stochastic analysis of a schedule are not completely reflected in the pseudo-deterministic sequence that is produced by sequencing the tasks when all activities are considered to take their predicted timeframes deterministically. Additionally, he recommends using a different sequence that was selected using an Optimality Indices (OI) rating method. OI is the probability that each sequence has of being the best. Mittenthal and Raghavachari [38] posed the challenge of scheduling in order to minimize an expected sum of no regular penalty functions on a single machine that is prone to unpredictable failures. When the punishment function is the squared divergence of work completion times from a common due date, a deterministic equivalent objective function is established and a basic recourse model is examined.

The issue of scheduling tasks on a single machine with a multi-objective function with triangular fuzzy due date was introduced by Cheachan and Kadhim [39], as well as the branch and bound approach was utilized to solve it. M. Geurtsen et al.[40] offered a thorough analysis of the three scheduling issues taken together, as well as the combine of resources,

maintenance and production. D. Briskorn et al.[41] examined a maintenance scheduling issue involving job-dependent, stochastic deterioration on a single machine with a fixed job sequence.

Now, we can calculate the expected and the variance of make span when the basic processes are random, independent (the processing time of any job that is being measured is one that is independent of the other processing times) and with general distribution as,

$$E(C_{max}) = \sum_{i=1}^n E(P_i) \prod_{r=i+1}^n (1 - \beta_r).$$

$$Var(C_{max}) = \sum_{i=1}^n Var(P_i) \prod_{r=i+1}^n (1 - \beta_r)^2.$$

**Corollary 2.3:**

The Longest Deteriorating Expected Basic Processing Time (LDEBPT) yields on optimal schedule for  $1/P_i = P_i - \beta_i t_i / E(C_{max})$  problem (i.e.,  $\frac{E(P_1)}{\beta_1} > \frac{E(P_2)}{\beta_2} > \dots > \frac{E(P_n)}{\beta_n}$ ).

**Proof:**

The proof is identical to Theorem 2.2.

**Corollary 2.4:**

The Shortest Deteriorating Variance Basic Processing Time (SDVBPT) yields on optimal schedule for  $1/P_i = P_i - \beta_i t_i / Var(C_{max})$  problem (i.e.,  $\frac{Var(P_1)}{(1-\beta_1)^2-1} < \frac{Var(P_2)}{(1-\beta_2)^2-1} < \dots < \frac{Var(P_n)}{(1-\beta_n)^2-1}$ ).

**Proof:**

The proof is directly from to Lemma 2.1.

For special case when  $\beta_r = \beta$ ,  $r = 1, 2, \dots, n$ , then

$$E(C_{max}) = \sum_{i=1}^n E(P_i) (1 - \beta)^{n-i},$$

and

$$Var(C_{max}) = \sum_{i=1}^n Var(P_i) (1 - \beta)^{2(n-i)},$$

then the expected of make span and its variance will be minimize when the jobs are schedule by decreasing value of  $E(P_i)$  and increasing value of  $Var(P_i)$ , respectively (seem as classical case of scheduling).

Before the next theorem, we will state here inequality (it has not been confirmed whether it was previously published or not) which is have useful in this paper.

**Proposition 2.5:**

Let  $x, y \in (0,1)$ , then  $\frac{x}{1-x} < \frac{y}{1-y} \Leftrightarrow x < y$ .

**Proof:**

Let  $x, y \in (0,1)$  so that  $1 - x$  and  $1 - y$  are positive, therefore, it will not affect the inequality when multiply both sides by theses quantities,

now if  $\frac{x}{1-x} < \frac{y}{1-y}$ , then

$$x(1 - y) < y(1 - x)$$

implies to  $x - xy < y - xy \Rightarrow x < y$ .

Now, if  $x < y$  then  $-x > -y \rightarrow 1 - x > 1 - y$

Then

$$\frac{1}{1-x} < \frac{1}{1-y}$$

The reason is if  $a < b$  then  $\frac{1}{a} > \frac{1}{b}$ .

Since  $x, y, 1-x$  and  $1-y$  are all positive and  $x < y$  (given) and  $\frac{1}{1-x} < \frac{1}{1-y}$  then,

$$\frac{x}{1-x} < \frac{y}{1-y}$$

The reason is (if  $a < b$  and  $c < d$  and  $a, b, c$  and  $d$  are all positive then  $ac < bd$ ).

**Theorem 2.6: Expected total weighted completion times  $E(\sum_{i=1}^n w_i C_i)$ :**

If all jobs satisfy LDEBPT, then the problem  $1/P_i = P_i - \beta_i t_i / E(\sum_{i=1}^n w_i C_i)$  is minimize when the jobs order by  $\frac{\beta_1}{w_1} < \dots < \frac{\beta_n}{w_n}$ .

**Proof:**

Let  $\pi = (1, 2, \dots, j, j+1, \dots, n)$  be the optimal sequence which is not order by  $\frac{\beta_1}{w_1} < \dots < \frac{\beta_n}{w_n}$ , that means there are two jobs that are next to each other  $j$  and  $j+1$  such that  $\frac{\beta_j}{w_j} > \frac{\beta_{j+1}}{w_{j+1}}$ , now let  $\pi^*$  have same order jobs as  $\pi$  with interchange  $j, j+1$  jobs, then

$$E\left(\sum_{i=1}^n w_{\pi(i)} C_{\pi(i)}\right) - E\left(\sum_{i=1}^n w_{\pi^*(i)} C_{\pi^*(i)}\right).$$

It is clear all terms its equal in  $\pi$  and  $\pi^*$  before  $j$ , then we have,

$$\begin{aligned} & E\left(\sum_{i=1}^n w_{\pi(i)} C_{\pi(i)}\right) - E\left(\sum_{i=1}^n w_{\pi^*(i)} C_{\pi^*(i)}\right) = w_{\pi(j)} E(C_{\pi(j)}) + w_{\pi(j+1)} E(C_{\pi(j+1)}) \\ & + \sum_{i=j+2}^n w_{\pi(i)} E(C_{\pi(i)}) - w_{\pi^*(j+1)} E(C_{\pi^*(j+1)}) - w_{\pi^*(j)} E(C_{\pi^*(j)}) - \sum_{i=j+2}^n w_{\pi^*(i)} E(C_{\pi^*(i)}). \end{aligned}$$

To ensure that simplify procedures, we will take each expression separately,

$$\begin{aligned} w_{\pi(j)} E(C_{\pi(j)}) &= w_j E(P_1(1-\beta_2)(1-\beta_3) \dots (1-\beta_{j-1})(1-\beta_j) \\ &+ P_2(1-\beta_3)(1-\beta_4) \dots (1-\beta_{j-1})(1-\beta_j) + \dots + P_{j-1}(1-\beta_j) + P_j) \\ &= w_j(1-\beta_j) E(P_1(1-\beta_2)(1-\beta_3) \dots (1-\beta_{j-1}) \\ &+ P_2(1-\beta_3)(1-\beta_4) \dots (1-\beta_{j-1}) + \dots + P_{j-1}) + w_j E(P_j) \\ &= w_j(1-\beta_j) E(C_{\pi(j-1)}) + w_j E(P_j). \end{aligned}$$

$$\begin{aligned} w_{\pi(j+1)} E(C_{\pi(j+1)}) &= w_{j+1} E(P_1(1-\beta_2)(1-\beta_3) \dots (1-\beta_{j-1})(1-\beta_j)(1-\beta_{j+1}) \\ &+ P_2(1-\beta_3)(1-\beta_4) \dots (1-\beta_{j-1})(1-\beta_j)(1-\beta_{j+1}) + \dots \\ &+ P_{j-1}(1-\beta_j)(1-\beta_{j+1}) + P_j(1-\beta_{j+1}) + P_{j+1}) \\ &= w_{j+1}(1-\beta_j)(1-\beta_{j+1}) E(P_1(1-\beta_2)(1-\beta_3) \dots (1-\beta_{j-1}) \\ &+ P_2(1-\beta_3)(1-\beta_4) \dots (1-\beta_{j-1}) + \dots + P_{j-1}) + w_{j+1}(1-\beta_{j+1}) E(P_j) \\ &+ w_{j+1} E(P_{j+1}) \\ &= w_{j+1}(1-\beta_j)(1-\beta_{j+1}) E(C_{\pi(j-1)}) + w_{j+1}(1-\beta_{j+1}) E(P_j) \\ &+ w_{j+1} E(P_{j+1}). \end{aligned}$$

$$\begin{aligned}
 &w_{\pi^*(j+1)}E(C_{\pi^*(j+1)}) \\
 &= w_{j+1}E(\mathcal{P}_1(1 - \beta_2)(1 - \beta_3) \dots (1 - \beta_{j-1})(1 - \beta_{j+1}) \\
 &+ \mathcal{P}_2(1 - \beta_3)(1 - \beta_4) \dots (1 - \beta_{j-1})(1 - \beta_{j+1}) + \dots + \mathcal{P}_{j-1}(1 - \beta_{j+1}) \\
 &+ \mathcal{P}_{j+1}) \\
 &= w_{j+1}(1 - \beta_{j+1})E(\mathcal{P}_1(1 - \beta_2)(1 - \beta_3) \dots (1 - \beta_{j-1}) \\
 &+ \mathcal{P}_2(1 - \beta_3)(1 - \beta_4) \dots (1 - \beta_{j-1}) + \dots + \mathcal{P}_{j-1}) + w_{j+1}E(\mathcal{P}_{j+1}) \\
 &= w_{j+1}(1 - \beta_{j+1})E(C_{\pi(j-1)}) + w_{j+1}E(\mathcal{P}_{j+1}).
 \end{aligned}$$

$$\begin{aligned}
 &w_{\pi^*(j)}E(C_{\pi^*(j)}) \\
 &= w_jE(\mathcal{P}_1(1 - \beta_2)(1 - \beta_3) \dots (1 - \beta_{j-1})(1 - \beta_{j+1})(1 - \beta_j) \\
 &+ \mathcal{P}_2(1 - \beta_3)(1 - \beta_4) \dots (1 - \beta_{j-1})(1 - \beta_{j+1})(1 - \beta_j) + \dots \\
 &+ \mathcal{P}_{j-1}(1 - \beta_{j+1})(1 - \beta_j) + \mathcal{P}_{j+1}(1 - \beta_j) + \mathcal{P}_j) \\
 &= w_j(1 - \beta_{j+1})(1 - \beta_j)E(\mathcal{P}_1(1 - \beta_2)(1 - \beta_3) \dots (1 - \beta_{j-1}) \\
 &+ \mathcal{P}_2(1 - \beta_3)(1 - \beta_4) \dots (1 - \beta_{j-1}) + \dots + \mathcal{P}_{j-1}) + w_jE(\mathcal{P}_{j+1})(1 - \beta_j) \\
 &+ w_jE(\mathcal{P}_j) \\
 &= w_j(1 - \beta_{j+1})(1 - \beta_j)E(C_{\pi(j-1)}) + w_jE(\mathcal{P}_{j+1})(1 - \beta_j) + w_jE(\mathcal{P}_j).
 \end{aligned}$$

Then we have

$$\begin{aligned}
 &= w_j(1 - \beta_j)E(C_{\pi(j-1)}) + w_jE(\mathcal{P}_j) + w_{j+1}(1 - \beta_j)(1 - \beta_{j+1})E(C_{\pi(j-1)}) \\
 &+ w_{j+1}(1 - \beta_{j+1})E(\mathcal{P}_j) + w_{j+1}E(\mathcal{P}_{j+1}) - w_{j+1}(1 - \beta_{j+1})E(C_{\pi(j-1)}) \\
 &- w_{j+1}E(\mathcal{P}_{j+1}) - w_j(1 - \beta_{j+1})(1 - \beta_j)E(C_{\pi(j-1)}) - w_jE(\mathcal{P}_{j+1})(1 - \beta_j) - w_jE(\mathcal{P}_j) \\
 &= E(C_{\pi(j-1)})[w_j(1 - \beta_j) + w_{j+1}(1 - \beta_j)(1 - \beta_{j+1}) - w_{j+1}(1 - \beta_{j+1}) \\
 &- w_j(1 - \beta_{j+1})(1 - \beta_j)] + w_{j+1}(1 - \beta_{j+1})E(\mathcal{P}_j) - w_jE(\mathcal{P}_{j+1})(1 - \beta_j) \\
 &= E(C_{\pi(j-1)})[w_j(1 - \beta_j)[1 - (1 - \beta_{j+1})] - w_{j+1}(1 - \beta_{j+1})[1 - (1 - \beta_j)]] \\
 &+ w_{j+1}(1 - \beta_{j+1})E(\mathcal{P}_j) - w_jE(\mathcal{P}_{j+1})(1 - \beta_j) \\
 &= E(C_{\pi(j-1)})[w_{j+1}(1 - \beta_{j+1})\beta_j - w_j(1 - \beta_j)\beta_{j+1}] \\
 &+ [E(\mathcal{P}_j)w_{j+1}(1 - \beta_{j+1}) - E(\mathcal{P}_{j+1})w_j(1 - \beta_j)].
 \end{aligned}$$

Now, for last terms

$$\sum_{i=j+2}^n w_{\pi(i)}E(C_{\pi(i)}) - \sum_{i=j+2}^n w_{\pi^*(i)}E(C_{\pi^*(i)})$$

$$\begin{aligned}
 \sum_{i=j+2}^n w_{\pi(i)} E(C_{\pi(i)}) &= w_{j+2} E(C_{\pi(j+2)}) + w_{j+3} E(C_{\pi(j+3)}) + \dots + w_n E(C_{\pi(n)}) \\
 &= w_{j+2} E(\mathcal{P}_1)(1 - \beta_2)(1 - \beta_3) \dots (1 - \beta_j)(1 - \beta_{j+1})(1 - \beta_{j+2}) \\
 &\quad + w_{j+2} E(\mathcal{P}_2)(1 - \beta_3)(1 - \beta_4) \dots (1 - \beta_j)(1 - \beta_{j+1})(1 - \beta_{j+2}) + \dots \\
 &\quad + w_{j+2} E(\mathcal{P}_j)(1 - \beta_{j+1})(1 - \beta_{j+2}) + w_{j+2} E(\mathcal{P}_{j+1})(1 - \beta_{j+2}) \\
 &\quad + w_{j+2} E(\mathcal{P}_{j+2}) \\
 &\quad + w_{j+3} E(\mathcal{P}_1)(1 - \beta_2)(1 - \beta_3) \dots (1 - \beta_j)(1 - \beta_{j+1})(1 - \beta_{j+2})(1 - \beta_{j+3}) \\
 &\quad + \dots \\
 &\quad + w_{j+3} E(\mathcal{P}_2)(1 - \beta_3)(1 - \beta_4) \dots (1 - \beta_j)(1 - \beta_{j+1})(1 - \beta_{j+2})(1 - \beta_{j+3}) \\
 &\quad + \dots + w_{j+3} E(\mathcal{P}_j)(1 - \beta_{j+1})(1 - \beta_{j+2})(1 - \beta_{j+3}) \\
 &\quad + w_{j+3} E(\mathcal{P}_{j+1})(1 - \beta_{j+2})(1 - \beta_{j+3}) + w_{j+3} E(\mathcal{P}_{j+2})(1 - \beta_{j+3}) \\
 &\quad + w_{j+3} E(\mathcal{P}_{j+3}) + \dots \\
 &\quad + w_n E(\mathcal{P}_1)(1 - \beta_2)(1 - \beta_3) \dots (1 - \beta_j)(1 - \beta_{j+1})(1 - \beta_{j+2}) \dots (1 - \beta_n) \\
 &\quad + w_n E(\mathcal{P}_2)(1 - \beta_3)(1 - \beta_4) \dots (1 - \beta_j)(1 - \beta_{j+1})(1 - \beta_{j+2}) \dots (1 - \beta_n) \\
 &\quad + \dots + w_n E(\mathcal{P}_j)(1 - \beta_{j+1})(1 - \beta_{j+2}) \dots (1 - \beta_n) \\
 &\quad + w_n \mathcal{P}_{j+1}(1 - \beta_{j+2}) \dots (1 - \beta_n) + w_n E(\mathcal{P}_{j+2})(1 - \beta_{j+3}) \dots (1 - \beta_n) + \dots \\
 &\quad + w_n E(\mathcal{P}_{n-1})(1 - \beta_n) + w_n E(\mathcal{P}_n).
 \end{aligned}$$

$$\begin{aligned}
 \sum_{i=j+2}^n w_{\pi^*(i)} E(C_{\pi^*(i)}) &= w_{j+2} E(C_{\pi^*(j+2)}) + w_{j+3} E(C_{\pi^*(j+3)}) + \dots + w_n E(C_{\pi^*(n)}) \\
 &= w_{j+2} E(\mathcal{P}_1)(1 - \beta_2)(1 - \beta_3) \dots (1 - \beta_{j+1})(1 - \beta_j)(1 - \beta_{j+2}) \\
 &\quad + w_{j+2} E(\mathcal{P}_2)(1 - \beta_3)(1 - \beta_4) \dots (1 - \beta_{j+1})(1 - \beta_j)(1 - \beta_{j+2}) + \dots \\
 &\quad + w_{j+2} E(\mathcal{P}_{j+1})(1 - \beta_j)(1 - \beta_{j+2}) + w_{j+2} E(\mathcal{P}_j)(1 - \beta_{j+2}) + w_{j+2} E(\mathcal{P}_{j+2}) \\
 &\quad + w_{j+3} E(\mathcal{P}_1)(1 - \beta_2)(1 - \beta_3) \dots (1 - \beta_{j+1})(1 - \beta_j)(1 - \beta_{j+2})(1 - \beta_{j+3}) \\
 &\quad + \dots \\
 &\quad + w_{j+3} E(\mathcal{P}_2)(1 - \beta_3)(1 - \beta_4) \dots (1 - \beta_{j+1})(1 - \beta_j)(1 - \beta_{j+2})(1 - \beta_{j+3}) \\
 &\quad + \dots + w_{j+3} E(\mathcal{P}_{j+1})(1 - \beta_j)(1 - \beta_{j+2})(1 - \beta_{j+3}) \\
 &\quad + w_{j+3} E(\mathcal{P}_j)(1 - \beta_{j+2})(1 - \beta_{j+3}) + w_{j+3} E(\mathcal{P}_{j+2})(1 - \beta_{j+3}) \\
 &\quad + w_{j+3} E(\mathcal{P}_{j+3}) + \dots \\
 &\quad + w_n E(\mathcal{P}_1)(1 - \beta_2)(1 - \beta_3) \dots (1 - \beta_{j+1})(1 - \beta_j)(1 - \beta_{j+2}) \dots (1 - \beta_n) \\
 &\quad + w_n E(\mathcal{P}_2)(1 - \beta_3)(1 - \beta_4) \dots (1 - \beta_{j+1})(1 - \beta_j)(1 - \beta_{j+2}) \dots (1 - \beta_n) \\
 &\quad + \dots + w_n E(\mathcal{P}_{j+1})(1 - \beta_j)(1 - \beta_{j+2}) \dots (1 - \beta_n) \\
 &\quad + w_n E(\mathcal{P}_j)(1 - \beta_{j+2}) \dots (1 - \beta_n) + w_n E(\mathcal{P}_{j+2})(1 - \beta_{j+3}) \dots (1 - \beta_n) + \dots \\
 &\quad + w_n E(\mathcal{P}_{n-1})(1 - \beta_n) + w_n E(\mathcal{P}_n).
 \end{aligned}$$

Then  $\sum_{i=j+2}^n w_{\pi(i)} E(C_{\pi(i)}) - \sum_{i=j+2}^n w_{\pi^*(i)} E(C_{\pi^*(i)})$

$$\begin{aligned}
 &= w_{j+2}E(\mathcal{P}_j)(1 - \beta_{j+1})(1 - \beta_{j+2}) + w_{j+2}E(\mathcal{P}_{j+1})(1 - \beta_{j+2}) \\
 &+ w_{j+3}E(\mathcal{P}_j)(1 - \beta_{j+1})(1 - \beta_{j+2})(1 - \beta_{j+3}) \\
 &+ w_{j+3}E(\mathcal{P}_{j+1})(1 - \beta_{j+2})(1 - \beta_{j+3}) + \dots \\
 &+ w_nE(\mathcal{P}_j)(1 - \beta_{j+1})(1 - \beta_{j+2}) \dots (1 - \beta_n) \\
 &+ w_nE(\mathcal{P}_{j+1})(1 - \beta_{j+2}) \dots (1 - \beta_n) - w_{j+2}E(\mathcal{P}_{j+1})(1 - \beta_j)(1 - \beta_{j+2}) \\
 &- w_{j+2}E(\mathcal{P}_j)(1 - \beta_{j+2}) - w_{j+3}E(\mathcal{P}_{j+1})(1 - \beta_j)(1 - \beta_{j+2})(1 - \beta_{j+3}) \\
 &- w_{j+3}E(\mathcal{P}_j)(1 - \beta_{j+2})(1 - \beta_{j+3}) - \dots \\
 &- w_nE(\mathcal{P}_{j+1})(1 - \beta_j)(1 - \beta_{j+2}) \dots (1 - \beta_n) \\
 &- w_nE(\mathcal{P}_j)(1 - \beta_{j+2}) \dots (1 - \beta_n) \\
 &= \sum_{i=j+2}^n w_i \prod_{r=j+2}^i (1 - \beta_r) (E(\mathcal{P}_j)\beta_{j+1} - E(\mathcal{P}_{j+1})\beta_j).
 \end{aligned}$$

The last result is

$$\begin{aligned}
 &E\left(\sum_{i=1}^n w_{\pi(i)}C_{\pi(i)}\right) - E\left(\sum_{i=1}^n w_{\pi^*(i)}C_{\pi^*(i)}\right) \\
 &= E(C_{\pi(j-1)})[w_{j+1}(1 - \beta_{j+1})\beta_j - w_j(1 - \beta_j)\beta_{j+1}] \\
 &+ [E(\mathcal{P}_j)w_{j+1}(1 - \beta_{j+1}) - E(\mathcal{P}_{j+1})w_j(1 - \beta_j)] + \sum_{i=j+2}^n w_i \prod_{r=j+2}^i (1 - \beta_r) (E(\mathcal{P}_j)\beta_{j+1} \\
 &- E(\mathcal{P}_{j+1})\beta_j) > 0.
 \end{aligned}$$

By Proposition 2.5 (where  $0 < \beta_i < 1$  satisfying the necessary condition), that is a contradiction with optimality of  $\pi$ .

In special case when the basic processing time for all jobs are identical then the problem

$$1/P_i = P - \beta_i t_i / E(\sum_{i=1}^n w_i C_i) \text{ minimize with decreasing order by } \frac{\beta_i}{w_i}.$$

**Corollary 2.7: Expected of total completion times  $E(\sum_{i=1}^n C_i)$**

If all jobs satisfy LREBPT, then the problem  $1/P_i = P_i - \beta_i t_i / E(\sum_{i=1}^n C_i)$  is minimize when the jobs order by  $\beta_1 < \beta_2 < \dots < \beta_n$ .

**Proof:**

By Theorem 2.6 (the problem  $1/P_i = P_i - \beta_i t_i / E(\sum_{i=1}^n w_i C_i)$  is minimize when the jobs order by  $\frac{\beta_1}{w_1} < \dots < \frac{\beta_n}{w_n}$ ) with  $w_i = 1$  or  $w_i = w$  (for all  $i$ ), we get the problem  $1/P_i = P_i - \beta_i t_i / E(\sum_{i=1}^n w C_i)$  is minimize when the jobs order by  $\frac{\beta_1}{w} < \dots < \frac{\beta_n}{w}$ , that implies the jobs order by  $\beta_1 < \beta_2 < \dots < \beta_n$  and same result when  $w = 1$ .

The similar arguments leads to the problem of expected total flow times  $E(\sum_{i=1}^n F_i) = E(\sum_{i=1}^n (C_i - r_i))$  and expected total lateness times  $E(\sum_{i=1}^n L_i) = E(\sum_{i=1}^n (C_i - d_i))$  are minimize by  $\beta_1 < \beta_2 < \dots < \beta_n$ .

Another criterion the waiting time (WT) define as the time between the actual processes or it is the time spend in ready queue.

**Corollary 2.8: The expected of total waiting times (WT),  $E(\sum_{i=1}^n (C_i - P_i))$**

If all jobs satisfy LREBPT, then the problem  $1/P_i = P_i - \beta_i t_i / E(\sum_{i=1}^n (C_i - P_i))$  is minimize when the jobs order by  $\beta_1 < \beta_2 < \dots < \beta_n$  satisfy.

**Proof:**

$$E\left(\sum_{i=1}^n (C_i - P_i)\right) = \sum_{i=1}^n E(C_i) - \sum_{i=1}^n E(P_i).$$

The first term is minimizing by Corollary 2.7, the second term is minimizing by Theorem 2.2 with, the arguments will be led to minimize problem.

**Theorem 2.9: The expected of maximum lateness  $E(L_{max})$**

If all jobs satisfy LDEBPT, then the problem  $1/P_i = P_i - \beta_i t_i / E(L_{max})$  is minimize when the jobs order by EDD (Earliest Due Date: The job with the earliest due date is selected first  $d_1 < d_2 < \dots < d_n$ ).

**Proof:**

Let  $\pi = (1, 2, \dots, j, j + 1, \dots, n)$  optimal sequence that is not EDD sequence, that means there are two adjacent jobs that are next to each other  $j$  and  $j + 1$  such that  $d_j > d_{j+1}$ , now let  $\pi^*$  have same order jobs as  $\pi$  with interchange the jobs  $j, j + 1$ , then

$$L_{\pi(j)} = C_{\pi(j)} - d_j$$

$$\begin{aligned} L_{\pi(j+1)} &= C_{\pi(j+1)} - d_{j+1} = (1 - \beta_{j+1})C_{\pi(j)} + P_{j+1} - d_{j+1} \\ &= (1 - \beta_{j+1}) \left( (1 - \beta_j)C_{\pi(j-1)} + P_j \right) + P_{j+1} - d_{j+1} \\ &= (1 - \beta_{j+1})(1 - \beta_j)C_{\pi(j-1)} + (1 - \beta_{j+1})P_j + P_{j+1} - d_{j+1} \\ &= (1 - \beta_{j+1})(1 - \beta_j)C_{\pi(j-1)} + P_j - \beta_{j+1}P_j + P_{j+1} - d_{j+1} \\ &= (1 - \beta_{j+1})(1 - \beta_j)C_{\pi(j-1)} + P_j + P_{j+1} - (\beta_{j+1}P_j + d_{j+1}). \end{aligned}$$

$$\begin{aligned} L_{\pi^*(j)} &= C_{\pi^*(j)} - d_j = (1 - \beta_j)C_{\pi^*(j+1)} + P_j - d_j \\ &= (1 - \beta_j) \left( (1 - \beta_{j+1})C_{\pi(j-1)} + P_{j+1} \right) + P_j - d_j \\ &= (1 - \beta_j)(1 - \beta_{j+1})C_{\pi(j-1)} + (1 - \beta_j)P_{j+1} + P_j - d_j \\ &= (1 - \beta_j)(1 - \beta_{j+1})C_{\pi(j-1)} + P_{j+1} - \beta_j P_{j+1} + P_j - d_j \\ &= (1 - \beta_{j+1})(1 - \beta_j)C_{\pi(j-1)} + P_j + P_{j+1} - (\beta_j P_{j+1} + d_j). \end{aligned}$$

$$L_{\pi^*(j+1)} = C_{\pi^*(j+1)} - d_{j+1} = (1 - \beta_{j+1})C_{\pi(j-1)} + P_{j+1} - d_{j+1}.$$

Since,  $d_j > d_{j+1}$  and  $\frac{P_j}{\beta_j} > \frac{P_{j+1}}{\beta_{j+1}}$  then

$$L_{\pi(j+1)} > L_{\pi^*(j)} \text{ and } L_{\pi(j+1)} > L_{\pi^*(j+1)}$$

That implies

$$L_{\pi(j+1)} > \max\{L_{\pi^*(j)}, L_{\pi^*(j+1)}\}.$$

As the maximum lateness for all jobs before  $j$  in the both sequences its same, let  $L_{\pi(\beta)} = \max\{L_{\pi(i)}, i = j + 2, \dots, n\}$  and  $L_{\pi^*(\beta)} = \max\{L_{\pi^*(i)}, i = j + 2, \dots, n\}$ ,

$$\begin{aligned} L_{\pi(j+2)} &= C_{\pi(j+2)} - d_{j+2} \\ &= (1 - \beta_{j+2})(1 - \beta_{j+1})(1 - \beta_j)C_{\pi(j-1)} + (1 - \beta_{j+2})(1 - \beta_{j+1})P_j + (1 - \beta_{j+2})P_{j+1} \\ &\quad + P_{j+2} - d_{j+2} \end{aligned}$$

$$\begin{aligned} L_{\pi^*(j+2)} &= C_{\pi^*(j+2)} - d_{j+2} \\ &= (1 - \beta_{j+2})(1 - \beta_j)(1 - \beta_{j+1})C_{\pi(j-1)} + (1 - \beta_{j+2})(1 - \beta_j)P_{j+1} + (1 - \beta_{j+2})P_j + P_{j+2} \\ &\quad - d_{j+2} \end{aligned}$$

that leads  $L_{\pi(j+2)} \geq L_{\pi^*(j+2)}$ , by the same way it easy to show that  $L_{\pi(\beta)} > L_{\pi^*(\beta)}$ , then

$$L_{max}(\pi) = \max\{L_{\pi(j)}, L_{\pi(j+1)}, L_{\pi(\beta)}\} > \max\{L_{\pi^*(j)}, L_{\pi^*(j+1)}, L_{\pi^*(\beta)}\} = L_{max}(\pi^*).$$

That leads  $E(L_{max}(\pi)) > E(L_{max}(\pi^*))$  which is a contradiction with optimality of  $\pi$ , then the problem  $1/P_i = P_i - \beta_i t_i / E(L_{max})$  is minimize when the jobs order by EDD.

**Corollary 2.10:** At the same argument of Theorem 2.9, the expected of maximum tardiness of linear decreasing deteriorating  $1/P_i = P_i - \beta_i t_i / E(T_{max})$  is minimize when the jobs order by EDD.

## 1. Conclusions

In this research, we assumed that the processing times are not constant during the initial time and we considered a single machine scheduling problem. It was our assumption that , the job processing times are decreasing functions of their starting times , for the objective make span we find the exception and variance such that  $1/P_i = \mathcal{P}_i - \beta_i t_i / E(C_{max})$  and  $1/P_i = \mathcal{P}_i - \beta_i t_i / Var(C_{max})$ , as for the following ( total weighted completion times , total completion times , total flow times, total lateness times, total waiting times, maximum lateness and maximum tardiness). The expectation was taken for them as shown in the following formulation:

$$1/P_i = \mathcal{P}_i - \beta_i t_i / E(\sum_{i=1}^n w_i C_i), \quad 1/P_i = \mathcal{P}_i - \beta_i t_i / E(\sum_{i=1}^n C_i) \quad , \quad 1/P_i = \mathcal{P}_i - \beta_i t_i / E(\sum_{i=1}^n F_i), \quad 1/P_i = \mathcal{P}_i - \beta_i t_i / E(\sum_{i=1}^n L_i), \quad 1/P_i = \mathcal{P}_i - \beta_i t_i / E(\sum_{i=1}^n (C_i - P_i)), \\ 1/P_i = \mathcal{P}_i - \beta_i t_i / E(L_{max}), \quad 1/P_i = \mathcal{P}_i - \beta_i t_i / E(T_{max}).$$

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