ISSN: 0067-2904

On e*_Singular Supplement Submodules

Ali A. Kabban*, **Wasan K. Hasan**
Department of Mathematics, University of Baghdad, College of Science, Baghdad, Iraq

Received: 21/9/2024 Accepted: 8/12/2024 Published: 30/12/2025

Abstract.
This paper aims present the main concepts of e*S–supplement submodules, weak e*S–supplement submodules, e*S–supplemented modules, weakly e*S–supplemented modules, cofinitely e*S–supplemented modules, and ⊕e*S–supplemented modules, as popularization of the concepts of supplement submodules, weakly supplement submodules, supplemented modules, weakly supplemented modules, cofinitely supplemented modules, and ⊕-supplemented modules respectively. We will prove some characteristics of these concepts.

Key words: e*S–supplement submodule, weak e*S–supplement submodule, e*S–supplemented module, weakly e*S–supplemented module, ⊕e*S–supplemented module.

حول المقاسات الجزئية المكملة المفردة من النمط *e

علي عبد عطية كبان، وسن خالد حسن
قسم الرياضيات، كلية العلوم، جامعة بغداد، بغداد، العراق

الخلاصة
تهدف هذه الورقة إلى تقديم المفاهيم الرئيسية للمقاسات الجزئية التكميلية من النمط e*S، والمقاسات الجزئية التكميلية الضعيفة من النمط e*S ، والمقاسات التكميلية من النمط e*S ، والمقاسات التكميلية الضعيفة من النمط e*S ، والمقاسات التكميلية المحددة من النمط e*S ، و المقاسات التكميلية التجميعية من النمط e*S، كتعميم على مفاهيم المقاسات الجزئية التكميلية، والمقاسات الجزئية التكميلية الضعيفة، والمقاسات التكميلية، والمقاسات التكميلية الضعيفة، والمقاسات التكميلية المحددة، والمقاسات التكميلية التجميعية، على التوالي. وسوف نثبت بعض خصائص هذه المفاهيم.

1. Introduction

In this paper W will be a unitary left R-module, and R is any ring with identity. Notationally, a submodule T of an R-module W is considered small, which is well known. Note that, $T \ll W$, if for each submodule of W, T + L = W, then L = W, [1] and [2]. A nonzero submodule T of W is considered essential if and only if, for every submodule L of W, L = {0} whenever $T \cap L = \{0\}$. Here, we denote $T \leq_e W$, where W is known as the essential extension of T [2] and [3].

* Email: alikuban5@gmail.com

A new submodule type was created by Baanoon in [4] and it is a generalization of an essential submodule called e∗-essential as follows. For any non-zero cosingular submodule B of W, if A ∩ B ≠ 0, we say that A is an e*-essential submodule in W. Denoted by $A \leq_{e*}$ W.
This is the definition of the singular submodule: Z(W) = {m ∈ W: ann(m) $\leq_e$R}. If Z(W) = W, then W is called a singular module and if Z(W) = 0, W is called non-singular by [5]. We generalized Z(W) to Z_{e*}(W), by applying an e*_essential submodule. Now, let W be a module define Z_{e*}(W) = {n ∈ W: ann(n) $\leq_{e*}$ R}, W is called an e*_singular module if Z_{e*}(W) = W, and W is called e*_non-singular module if Z_{e*}(W) = 0, [6].

In [6], the generalization of small submodule known as e*S–small submodule is introduced by A. Kabban and W. Khalid. A submodule T of W is called an e*S–small submodule of W (signified by T $\ll_{e*S}$ W) if whenever W = T + H, with $Z_{e*}(\frac{W}{H}) = \frac{W}{H}$ implies that W = H. A nonzero module W is called an e*S–hollow if every proper submodule of W is e*S–small, [6].
Let H ⊆ D ⊆ W, if $\frac{D}{H} \ll \frac{W}{H}$, then H is called a coessential submodule of D in W [7] [8]. For a generalization of the coessential submodule, we present the following as the e*S-coessential submodule in [6]. Let W and D be R–modules, and H ⊆ W, such that D ⊆ H ⊆ W, then D is called an e*S–coessential submodule of H in W (denoted by D $\subseteq_{e*S\_ce}$ H in W) if $\frac{H}{D} \ll_{e*S} \frac{W}{D}$. A submodule T of W is coclosed submodule of W (denoted by T $\subseteq_{cc}$ W) if whenever $\frac{T}{L} \ll \frac{W}{L}$ implies that T = L, see [9] [10] [11]. Based on this idea, we may provide the following idea. Let W be an R–module and H be submodule of W. We say that H is an e*S–coclosed submodule of W (denoted by H $\subseteq_{e*S_cc}$W) if whenever T $\subseteq_{e*S\_ce}$ H, (i.e., $\frac{H}{T} \ll_{e*S} \frac{W}{T}$) implies that T = H, [6].

As in [12] [13] [14] [15] [16]we will use e*S–small submodules to present a new generalization of supplement submodules, weak supplement submodules, supplemented modules, weakly supplemented modules, cofinitely supplemented modules, and ⊕-supplemented modules. Namely of e*S–supplement submodules, weak e*S–supplement submodules, e*S–supplemented modules, weakly e*S–supplemented modules, cofinitely e*S–supplemented modules, and ⊕e*S–supplemented respectively. We prove the main characteristics of these concepts.

2. e*-Singular Supplement Submodules.

A generalization of supplement submodules with certain characteristics is shown in this section. Remember that a sub-module T of a module W is called a supplement of a sub-module B in W, if W = T + B and T ∩ B is small in T [17] [8]. And in this section, we introduce a generalization of supplemented modules. We also show some properties of these generalized submodules. Recall that W is called a supplemented module if each sub-module of W has a supplement in W [17] and [8].

Firstly, we need to list basic properties of the concept of an e*S–small [6].

Lemma 2.1: [6] Let W be any R–module, so.
1) If D ⊆ C ⊆ W. Then C $\ll_{e*S}$ W if and only if D $\ll_{e*S}$ W and $\frac{C}{D} \ll_{e*S} \frac{W}{D}$.
2) Let D and C be submodules of W. Then D + C $\ll_{e*S}$ W if and only if D $\ll_{e*S}$ W and C $\ll_{e*S}$ W.

3) Let $N_1, N_2, \ldots, N_n \subseteq W$. Then $\sum_{j=1}^{n} N_j \ll_{e*S} W$ if and only if $N_j \ll_{e*S} W$, $\forall$ j = 1, 2, …, n.
4) Let $T \subseteq X$ be submodules of W. If $T \ll_{e*S} X$, then $T \ll_{e*S} W$.
5) Let $f: W \longrightarrow D$ be a homomorphisim. If $T \ll_{e*S} W$, then $f(T) \ll_{e*S} D$.
6) Let $W = T_1 \oplus T_2$ be an R–module and $H_1 \subseteq T_1$, $H_2 \subseteq T_2$. Then $H_1 \oplus H_2 \ll_{e*S} T_1 \oplus T_2$ if and only if $H_1 \ll_{e*S} T_1$ and $H_2 \ll_{e*S} T_2$.

Lemma 2.2: [6] Let W be any R-module, and let two submodules H and L of W. If $Z_{e*}(\frac{W}{H}) = \frac{W}{H}$ then $Z_{e*}(\frac{W}{L+H}) = \frac{W}{L+H}$.

The notion of an e*S–small submodule leads to the following:

Definition 2.3: Let T and H be submodules of an R-module W. If W = T + H and $T \cap H \ll_{e*S} T$, then T is called **e*-singular supplement** of H in W (in brief e*S–supplement). If each asubmodule of W has e*S–supplement, then W is called **e*S–supplemented module.**
It easy to show the following Lemma.

Lemma 2.4: For any R–module W, let T and D be asubmodules of W. Then T is an e*S–supplement of D in W if and only if for each $S \subseteq T$ with $Z_{e*}(\frac{T}{S}) = \frac{T}{S}$, and W = S + D implies T = S.
Proof: ⇒) Since W = T + D and $T \cap D \ll_{e*S} T$, for $S \subseteq T$ with W = S + D, we have $T = W \cap T = (S + D) \cap T = S + (T \cap D)$, since $T \cap D \ll_{e*S} T$ with $Z_{e*}(\frac{T}{S}) = \frac{T}{S}$, thus T = S.
⇐) Clare.

Examples and Remarks 2.5:
1) Every supplemented module is an e*S–supplemented. Conversely need not be accurate since an e*S–small need not be a small submodule, [6].
2) In the Z–module Z_{12}, the sub-module $\langle\bar{4}\rangle$ is not an e*S–supplement of $\langle\bar{2}\rangle$, since $\langle\bar{4}\rangle + \langle\bar{2}\rangle = \langle\bar{2}\rangle \neq Z_{12}$.
3) If W is a uniform cosingular R–module, then supplemented and e*S–supplemented modules coincide. In particular, the Q as Z–module is a uniform cosingular, not supplemented module ([8] P.238). So, Q as Z–module is no e*S–supplemented.
4) For any R–module W, the submodule $\{0\}$ is the e*S–supplement of W and W is the e*S–supplement of $\{0\}$ in W.
5) Every semi-simple module is e*S–supplemented. In particular, the Z–module Z_6 is e*S–supplemented.
6) The e*S–supplement submodule need not be existing. For example, the Z–module Z, a submodule 2Z has no an e*S–supplement submodule, since $\{0\}$ the only e*S–small of Z.
7) The e*S–supplement is not commute. For example, in Z_{24} as Z–module, the submodule $\langle\bar{2}\rangle$ has an e*S–supplement $\langle\bar{3}\rangle$. But $\langle\bar{2}\rangle$ is not an e*S–supplement of $\langle\bar{3}\rangle$, since $\langle\bar{2}\rangle \cap \langle\bar{3}\rangle = \langle\bar{6}\rangle$ and $\langle\bar{6}\rangle$ is not an e*S–small in $\langle\bar{2}\rangle$. Because $\langle\bar{6}\rangle + \langle\bar{4}\rangle = \langle\bar{2}\rangle$ and $Z_{e*}(\frac{\langle\bar{2}\rangle}{\langle\bar{4}\rangle}) \cong \frac{\langle\bar{2}\rangle}{\langle\bar{4}\rangle}$, but $\langle\bar{2}\rangle \neq \langle\bar{4}\rangle$.
8) If $M = A \oplus B$, then A is e*S–supplement of B and B is an e*S–supplement of A. For example, Z_6 as Z–module $\langle\bar{3}\rangle$ is e*S–supplement of $\langle\bar{2}\rangle$ and $\langle\bar{2}\rangle$ is e*S–supplement of $\langle\bar{3}\rangle$.
9) The Z as Z–module isn't an e*S–supplemented, since the submodule 2Z has no e*S–supplement submodule. See (6).
10) Every e*S–hollow module is e*S–supplemented.

To see that let W be e*S–hollow, and T be a submodule of W. If W = T, so T has e*S–supplement $\{0\}$. If T is a proper submodule of W. Hence, T is e*S–Small submodule of W. Since T + W = W and $T \cap W = T \ll_{e*S} W$, so T has e*S–supplement. Therefore, W is e*S–supplemented.
11) The convers of (10) isn't accurate in general, for example Z_6 as Z–module.

Proposition 2.6: Let H and Y be submodules of a module W such that $Y \subseteq H \subseteq W$. If Y is an e*S–supplement in W, then Y is an e*S–supplement in H.
Proof: Since Y is an e*S–supplement in W, there exists K a submodule of W, where $Y + K = W$ and $Y \cap K \ll_{e*S} Y$. The submodule $H = H \cap W = H \cap (Y + K)$ and by the Modular law, $H = Y + (H \cap K)$. Hence, Y is an e*S–supplement of $H \cap K$ in H, since $Y \cap (H \cap K) = Y \cap K \ll_{e*S} Y$. Therefore, Y is an e*S–supplement in H.

Proposition 2.7: Let D and Y be sub-modules of a module W such that $D \subseteq Y \subseteq W$. If Y is an e*S–supplement in W, then $\frac{Y}{D}$ is an e*S–supplement in $\frac{W}{D}$.
Proof: Since Y is an e*S–supplement in W, there exists $K \subseteq W$, such that $Y + K = W$ and $Y \cap K \ll_{e*S} Y$. Now, $\frac{W}{D} = \frac{Y+K}{D} = \frac{Y}{D} + \frac{K+D}{D}$ and $\frac{Y}{D} \cap \frac{K+D}{D} = \frac{Y \cap (K+D)}{D} = \frac{D+(Y \cap K)}{D}$, by Modular law since $Y \cap K \ll_{e*S} Y$, by Lemma 2.1 (1), we have that $\frac{D+(Y \cap K)}{D} \ll_{e*S} \frac{Y}{D}$. Therefore, $\frac{Y}{D}$ is an e*S–supplement of $\frac{K+D}{D}$ in $\frac{W}{D}$.

Proposition 2.8: For any R–module W, let Y be an e*S–hollow submodule of W. Then Y is an e*S–supplement of each proper sub-module H of W such that $W = Y + H$.
Proof: Let H be a proper submodule of W such that $W = Y + H$. So, $Y \cap H$ is a proper submodule of Y if $Y \cap H = Y$. Hence, $Y \subseteq H$ and W = H, which contradicts. Now, since Y is an e*S–hollow, thus $H \cap Y$ is an e*S–small in Y. Therefore, Y is an e*S–supplement of H in W.

Proposition 2.9: For any R–module W, let T, H be sub-modules of W such that H is an e*S–supplement of T in W. If W = Y + H, for some submodule Y of T, then H is an e*S–supplement of Y in W.
Proof: Assume that W = Y + H, for some submodule Y of T and H is an e*S–supplement of T in W. So, we have W = T + H, and $T \cap H \ll_{e*S} H$. Since $Y \subseteq T$, so $Y \cap H \subseteq T \cap H \ll_{e*S} H$, by Lemma 2.1, $Y \cap H \ll_{e*S} H$, and W = Y + H. Therefore, H is an e*S–supplement of Y in W.

Proposition 2.10: For any R–module W, let H, T be sub-modules of W, and T be an e*S–supplement of H in W if $C \ll_{e*S} W$, then T is an e*S–supplement of H + C.
Proof: Let T + (H + C) = W, to show $T \cap (H + C) \ll_{e*S} T$, let $T \cap (H + C) + X = T$, with $Z_{e*}(\frac{T}{X}) = \frac{T}{X}$, $W = T + (H + C) = T \cap (H + C) + X + (H + C) = X + (H + C) = (H + X) + C$, to show $Z_{e*}(\frac{W}{H+X}) = \frac{W}{H+X}$, since $\frac{W}{H+X} = \frac{T+(H+C)+X}{H+X} = \frac{T+(H+X)}{(H+X)} \cong \frac{T}{T \cap (H+X)} = \frac{T}{X+(H \cap T)}$, by Second Isomorphism and Modular law. Since $Z_{e*}(\frac{T}{X}) = \frac{T}{X}$, then we get $Z_{e*}(\frac{T}{X+(H \cap T)}) = \frac{T}{X+(H \cap T)}$, hence $Z_{e*}(\frac{W}{H+X}) = \frac{W}{H+X}$, since $C \ll_{e*S} W$, then W = H + X, but W = H + T, and $X \subseteq T$ and $Z_{e*}(\frac{T}{X}) = \frac{T}{X}$, then T = X, by Lemma 2.4.

Proposition 2.11: For any R–module W, let X, T be sub-modules of W. If T is an e*S–supplement of X in W, then $\frac{T+L}{L}$ is an e*S–supplement of $\frac{X}{L}$ in $\frac{W}{L}$, for L ⊆ X.
Proof: Since T is an e*S–supplement of X in W. Then W = X + T and $X \cap T \ll_{e*S} T$, for L ⊆ X, we have X ∩ (T + L) = (X ∩ T) + L, by Modular Law, and $\frac{X}{L} \cap (\frac{T+L}{L}) = \frac{(X \cap T)+L}{L}$, since $X \cap T \ll_{e*S} T$, it follows that $\frac{(X \cap T)+L}{L} \ll_{e*S} \frac{T+L}{L}$. Now, $\frac{W}{L} = \frac{X+T}{L} = \frac{X}{L} + \frac{T+L}{L}$. Therefore, $\frac{T+L}{L}$ is e*S–supplement of $\frac{X}{L}$ in $\frac{W}{L}$.

Proposition 2.12: For any R–module W, let T be an e*S–supplement of C in W, K ⊆ T, then $K \ll_{e*S} W$ if and only if $K \ll_{e*S} T$.
Proof: ⇒) Let K + Y = T with $Z_{e*}(\frac{T}{Y}) = \frac{T}{Y}$, but T + C = W and $T \cap C \ll_{e*S} T$, then W = (K + Y) + C, hence W = K + (Y + C) to show $Z_{e*}(\frac{W}{Y+C}) = \frac{W}{Y+C}$, since $\frac{W}{Y+C} = \frac{T+(Y+C)}{(Y+C)} \cong \frac{T}{T \cap (Y+C)} = \frac{T}{Y+(T \cap C)}$, by Modular law and Second Isomorphism Theorem. Since $Z_{e*}(\frac{T}{Y}) = \frac{T}{Y}$, then we get $Z_{e*}(\frac{T}{Y+(T \cap C)}) = \frac{T}{Y+(T \cap C)}$, hence $Z_{e*}(\frac{W}{Y+C}) = \frac{W}{Y+C}$, but $K \ll_{e*S} W$, then W = Y + C, since W = T + C, Y ⊆ T and $Z_{e*}(\frac{T}{Y}) = \frac{T}{Y}$, then by Lemma 2.4, T = Y.
⇐ Clearly by Lemma 2.1.

Proposition 2.13: For any R–module W, let V be an e*S–supplement of U in W, and H, T be sub-modules of V. Then T is e*S–supplement of H in V if and only if T is e*S–supplement of H + U in W.
Proof: ⇒) Let T be an e*S–supplement of H in V, then V = T + H and $T \cap H \ll_{e*S} T$. Let (H + U) + L = W for L ⊆ T with $Z_{e*}(\frac{T}{L}) = \frac{T}{L}$. Now, H + L ⊆ V. Since $\frac{V}{H+L} = \frac{T+(H+L)}{H+L} \cong \frac{T}{T \cap (H+L)} = \frac{T}{L+(H \cap T)}$, by Modular law and Second Isomorphism Theorem, and $Z_{e*}(\frac{T}{L}) = \frac{T}{L}$, then we get $Z_{e*}(\frac{T}{L+(H \cap T)}) = \frac{T}{L+(H \cap T)}$, hence $Z_{e*}(\frac{V}{H+L}) = \frac{V}{H+L}$, and because V is e*S–supplement of U in W, then W = V + U and by Lemma 2.4, H + L = V. Since L ⊆ T and T is an e*S–supplement of H in V, then T = L.
⇐) Let T be an e*S–supplement of H + U in W. Then T + (U + H) = W and $T \cap (U + H) \ll_{e*S} T$. Let T + H = V, to prove $T \cap H \ll_{e*S} T$, since $T \cap H \subseteq T \cap (U + H) \ll_{e*S} T$, then $T \cap H \ll_{e*S} T$, hence T is an e*S–supplement of H in V.

For any R–module W, let V and T be sub-modules of W. We said T and V are **mutual e*S–supplements**, if T is an e*S–supplement of V in W and V is e*S–supplement of T in W.

Corollary 2.14: For any R–module W, let V, B be mutual e*S–supplements in W. L be e*S–supplement of U in V, and H be an e*S–supplement of T in B, then L + H is an e*S–supplement of T + U in W.
Proof: Since V = U + L and B is e*S–supplement of V in W, then by proposition 2.13, H is e*S–supplement of U + L + T in W and then $(U + L + T) \cap H \ll_{e*S} H$, since B = T + H and V is e*S–supplement of B in W, then by proposition 2.13, L is e*S–supplement of U + T + H in W and then $(U + T + H) \cap L \ll_{e*S} L$, because V = U + L , B = T + H , and W = V + B, then we have W = U + L + T + H = U + T + L + H, then $(U + T) \cap (L + H) \subseteq L \cap (U + T + H) + H \cap (U + T + L) \ll_{e*S} L + H$, hence L + H is e*S–supplement of T + U in W.

Proposition 2.15: For any R–module W, let T, V be submodules of W, then the following statements are equivalent.

1) V is an e*S–supplement of T in W;
2) W = T + V and for every proper sub-module X of V with $Z_{e*}(\frac{V}{X}) = \frac{V}{X}$, then W ≠ T + X.

Proof: (1)⇒(2) Assume that V is an e*S–supplement of T in W and suppose that W = T + X, where X is a proper sub-module of V such that $Z_{e*}(\frac{V}{X}) = \frac{V}{X}$. Then by Modular law, V = V ∩ W = V ∩ (T + X) = X + (T ∩ V). Since V is an e*S–supplement of T in W and $Z_{e*}(\frac{V}{X}) = \frac{V}{X}$, then V = X, which is a contradiction. Thus W ≠ T + X.

(2)⇒(1) Suppose that W = V + T. To show that V is an e*S–supplement of T in W, it is sufficient to show that $V \cap T \ll_{e*S} V$, let C be a submodule of V such that V = (T ∩ V) + C, with $Z_{e*}(\frac{V}{C}) = \frac{V}{C}$. If C is a proper sub-module of V, then by our assumption W ≠ T + C. But W = T + V = T + (T ∩ V) + C = T + C, which is a contradiction. Thus, V is an e*S–supplement of T in W.

Proposition 2.16: For any R–module W, let T, V and C be submodules of W. If T is an e*S–supplement of V in W, and V is an e*S–supplement of C in W, then V is an e*S–supplement of T in W.

Proof: Let W = T + V = V + C, $T \cap V \ll_{e*S} T$ and $V \cap C \ll_{e*S} V$. To prove that $T \cap V \ll_{e*S} V$. Let D be a sub-module of V such that V = (T ∩ V) + D, with $Z_{e*}(\frac{V}{D}) = \frac{V}{D}$. Since W = V + C = (T ∩ V) + D + C, and $T \cap V \ll_{e*S} T$, then $T \cap V \ll_{e*S} W$. Note that, $\frac{W}{D+C} = \frac{V+(D+C)}{D+C} \cong \frac{V}{V\cap(D+C)} = \frac{V}{D+(V\cap C)}$, by Second Isomorphism and Modular law. Since $Z_{e*}(\frac{V}{D}) = \frac{V}{D}$, then we get $Z_{e*}(\frac{V}{D+(V\cap C)}) = \frac{V}{D+(V\cap C)}$, hence $Z_{e*}(\frac{W}{D+C}) = \frac{W}{D+C}$, and $T \cap V \ll_{e*S} W$, then W = D + C. Now, V = V ∩ W = V ∩ (D + C) = D + (V ∩ C), by Modular law. But $V \cap C \ll_{e*S} V$, and $Z_{e*}(\frac{V}{D}) = \frac{V}{D}$, therefore V = D. Thus, V is an e*S–supplement of T in W.

Now, we will present a few properties of e*S–supplemented modules.

Proposition 2.17: Let A and B be submodules of W such that A is an e*S–supplemented module. If A + B has an e*S–supplement in W then B does.

Proof: Let D be an e*S–supplement submodule of A + B in W. Then (A + B) + D = W and $D \cap (A + B) \ll_{e*S} D$. Since A is an e*S–supplemented module, (D + B) ∩ A is a submodule of A. Hence, there exists Y ⊆ A such that (D + B) ∩ A + Y = A and $(D + B) \cap A \cap Y = (D + B) \cap Y \ll_{e*S} Y$. Thus, we have D + B + Y = W, and $(D + B) \cap Y \ll_{e*S} Y$, that is Y is an e*S–supplement of D + B in W. Next, we will show that D + Y is an e*S–supplement of B in W, itis clear that (D + Y) + B = W, so it suffices to show that $(D + Y) \cap B \ll_{e*S} D + Y$, since Y + B ⊆ A + B, by Lemma 2.1, $D \cap (Y + B) \subseteq D \cap (A + B) \ll_{e*S} D$. Thus, $(D + Y) \cap B \subseteq D \cap (Y + B) + Y \cap (D + B) \ll_{e*S} D + Y$. Hence, $(D + Y) \cap B \ll_{e*S} D + Y$. Therefore, B has an e*S–supplement in W.

Remember that a **fully invariant** submodule D of W is defined as follows: g (D) ⊆ D, for each g ∈ End (W) and W is called **duo module** if each submodule of W is a fully invariant. W is called **distributive** module if for every D, V and U are submodule of W, then D ∩ (V + U) = (D ∩ V) + (D ∩ U) [8].

Proposition 2.18: Let W be an e*S–supplemented module and let T is a fully invariant of W, then $\frac{W}{T}$ is an e*S–supplemented.

Proof: Let $\frac{K}{T} \subseteq \frac{W}{T}$, to prove $\frac{K}{T}$ has e*S–supplement in $\frac{W}{T}$, K ⊆ W, since W is e*S–supplemented, then there exists Y ⊆ W such that W = K + Y, and $K \cap Y \ll_{e*S} Y$. Now, $\frac{W}{T} =$

$\frac{K+Y}{T} = \frac{K}{T} + \frac{Y+T}{T}$, to prove $\frac{K}{T} \cap \frac{Y+T}{T} \ll_{e*S} \frac{Y+T}{T}$, let $(\frac{K}{T} \cap \frac{Y+T}{T}) + \frac{V}{T} = \frac{Y+T}{T}$, with $Z_{e*}(\frac{Y+T}{V}) = \frac{Y+T}{V}$, to prove $\frac{V}{T} = \frac{Y+T}{T}$, so $\frac{K \cap (Y+T)}{T} = \frac{T+(K \cap Y)}{T}$, then $\frac{T+(K \cap Y)}{T} + \frac{V}{T} = \frac{Y+T}{T}$, and T + (K ∩ Y) + V = Y + T, since T ⊆ V, then (K ∩ Y) + V = Y + T, but $Z_{e*}(\frac{Y+T}{V}) = \frac{Y+T}{V}$, and K ∩ Y $\ll_{e*S}$ Y ⊆ Y + T, then K ∩ Y $\ll_{e*S}$ Y + T, therefore V = Y + T and $\frac{V}{T} = \frac{Y+T}{T}$.

Corollary 2.19: The homomorphic image of an e*S–supplemented module is an e*S-supplemented.
Proof: Since every homomorphic image is isomorphic a quotient module.
Remark 2.20: The convers of proposition 2.18, need not be accurate in general. For example, $\frac{Z}{6Z} \cong Z_6$ as a Z–module is an e*S–suplemented module. But the Z–module Z isn't e*S–supplemented module. See, Examples and remarks 2.5.

Proposition 2.21: Let W = $W_1 \oplus W_2$ be aduo module, then W_1 and W_2 are e*S–supplemented modules if and only if W is an e*S–supplemented.
Proof: ⇒) Let H ⊆ W, since W = $W_1 + W_2$ + H, trivially has an e*S–supplement in W. By Proposition 2.17, then W_2 + H has an e*S–supplement in W, by Proposition 2.17, again, H has an e*S–supplement in W. So, W is an e*S–supplemented module.
⇐) $W_2 \cong \frac{W}{W_1}$, since W is an e*S–supplemented module, by proposition 2.18, $\frac{W}{W_1}$ is an e*S–supplemented module. Thus, by corollary 2.19, W_2 is an e*S–supplemented module. Similarity W_1 is an e*S–supplemented module.

Corollary 2.22: Let $W = \oplus_{i=1}^{n} W_i$. Wis an e*S–supplemented module if and only if $W_1, W_2, \dots, W_n$ are e*S–supplemented modules.
Corollary 2.23: Let $W_1 \oplus W_2$ = W be aduo module, H and V are sub-modules of W_1, if H is an e*S–supplement of V in W_1, then H ⊕ W_2 is an e*S–supplement of V in W.
Proof: Let H be an e*S–supplement submodule of V in W_1, then W_1 = H + V and H ∩ V $\ll_{e*S}$ H, since W = $W_1 \oplus W_2$, then W = (H + V) ⊕ W_2, hence W = V + (H ⊕ W_2) but (H ⊕ W_2) ∩ V = (H ⊕ W_2) ∩ W_1 ∩ V = H ∩ V $\ll_{e*S}$ H. And by Lemma 2.1, then H ∩ V $\ll_{e*S}$ H ⊕ W_2, hence H ⊕ W_2 is an e*S–supplement of V in W.
The following explain the relation between e*S–supplemented modules and e*S–coessential submodules.

Proposition 2.24: Let T, V and X be sub-modules of a distributive R–module W. If T is an e*S–supplement of V in W and V is an e*S–supplement of X in W with T ⊆ X, then T $\subseteq_{e*S\_ce}$ X in W.
Proof: Assume that T is an e*S–supplement of V in W, and V is an e*S–supplement of X in W with T ⊆ X. To show that T $\subseteq_{e*S\_ce}$ X in W, let $\frac{W}{T} = \frac{X}{T} + \frac{Y}{T}$, with $Z_{e*}(\frac{W}{Y}) = \frac{W}{Y}$, then W = X + Y. So, by Modular law, Y = Y ∩ W = Y ∩ (T + V) = T + (Y ∩ V). Hence, W = X + Y = X + T + (Y ∩ V) = X + (Y ∩ V), and V = V ∩ W = V ∩ (X + Y). Hence, V = (V ∩ X) + (V ∩Y). To show $Z_{e*}(\frac{V}{V \cap Y}) = \frac{V}{V \cap Y}$, by Second Isomorphism Theorem, $\frac{V}{V \cap Y} \cong \frac{V+Y}{Y} = \frac{W}{Y}$, where T ⊆ Y. But $Z_{e*}(\frac{W}{Y}) = \frac{W}{Y}$, hence $Z_{e*}(\frac{V}{V \cap Y}) = \frac{V}{V \cap Y}$, and (V ∩ X) $\ll_{e*S}$ V, then V = Y ∩ V, so V ⊆ Y, since Y = T + (Y ∩ V), then Y = T + V = W. Thus T $\subseteq_{e*S\_ce}$ X in W.

3. Weak e*-singular Supplement Submodules.

Now, we present one generalization of weak supplement submodules, and we show some of its properties. Recall that a submodule A of an R-module W is called a weak supplement of a submodule B in W, if W = A + B and A ∩ B is small in W, [8]. As well as we introduce the generalization of weakly supplemented modules with some properties. Recall that when every sub-module of an R-module W has a weak supplement, then W called weakly supplemented, [8].

Definition 3.1: Let B and T be submodules of the R-module W. If W = T + B and $T \cap B \ll_{e*S}$ W, then T is called a **weak e*S–supplement** of B in W. A module W is called **weakly e*S–supplemented** if each asubmodule of W has a weak e*S–supplement in W.

Examples and Remarks 3.2:

1) If W is a uniform cosingular R–module, then weakly supplemented and weakly e*S–supplemented modules coincide.

2) Every e*S–supplemented is a weakly e*S–supplemented module. The converse need not be accurate in general. For example, the Q as Z–module is a uniform cosingular which is weakly supplemented ([8] P.238). From (1), the Z–module Q is weakly e*S–supplemented. But not e*S–supplemented.

3) In Z_{12} as Z–module, the submodule $\langle\bar{4}\rangle$ is not a weakly e*S–supplement of $\langle\bar{2}\rangle$ in Z_{12} since $\langle\bar{4}\rangle + \langle\bar{2}\rangle = \langle\bar{2}\rangle \neq Z_{12}$.

4) Z as Z–module isn't a weakly e*S–supplemented, since 2Z has no an e*S–supplement (weak e*S–supplement) submodule. See Examples and remarks 2.5 (9).

5) Every e*S–supplement submodule is a weak e*S–supplement. The converse need not be accurate in general. For example, in Z_{12} as Z–module, the submodule $\langle\bar{2}\rangle$ is a weak e*S–supplement of $\langle\bar{3}\rangle$ while $\langle\bar{2}\rangle$ is not e*S–supplement of $\langle\bar{3}\rangle$ in Z_{12}. Since $\langle\bar{2}\rangle \cap \langle\bar{3}\rangle = \langle\bar{6}\rangle$ and $\langle\bar{6}\rangle + \langle\bar{4}\rangle = \langle\bar{2}\rangle$, and $Z_{e*}(\frac{\langle\bar{2}\rangle}{\langle\bar{4}\rangle}) = \frac{\langle\bar{2}\rangle}{\langle\bar{4}\rangle}$, but $\langle\bar{4}\rangle \neq \langle\bar{2}\rangle$. Thus, $\langle\bar{6}\rangle$ is not e*S–small in $\langle\bar{2}\rangle$.

Proposition 3.3: Let T, B be two submodules of W, and let T be a weakly e*S–supplemented module. If T + B has a weak e*S–supplement in W then B does.

Proof: By assumption there exists N ⊆ W, such that N + (B + T) = W, and $N \cap (B + T) \ll_{e*S}$ W, since T is weakly e*S–supplemented module there exists D ⊆ T, such that (N + B) ∩ T + D = T and $(N + B) \cap D \ll_{e*S}$ T, thus B + N + D = W, and $(N + B) \cap D \ll_{e*S}$ T, and by Lemma 2.1, $(N + B) \cap D \ll_{e*S}$ W, that is D is a weak e*S–supplement of N + B in W, we will show that N + D is a weak e*S–supplement of B in W, it is clear that (N + D) + B = W, so it enough to show that $(N + D) \cap B \ll_{e*S}$ W. Since $(N + D) \cap B \subseteq N \cap (T + B) + (N + B) \cap D \ll_{e*S}$ W, then $(N + D) \cap B \ll_{e*S}$ W. Therefore, N + D is a weak e*S–supplement of B in W.

Corollary 3.4: Let $W = T_1 + T_2$, if T_1 and T_2 are a weakly e*S–supplemented modules then W is a weakly e*S–supplemented.

Proof: Let D be asubmodule of W. Since $T_1 + T_2 + D = W$, trivially has weak e*S–supplement in W. By Proposition 3.3, $T_2 + D$ has a weak e*S–supplement in W. And again, by proposition 3.3, D has aweak e*S–supplement in W. So, W is aweakly e*S–supplemented.

Proposition 3.5: Let W be a weakly e*S–supplemented module and Y ⊆ D ⊆ W, if $Y \ll_{e*S}$ W implies that $Y \ll_{e*S}$ D, then D is an e*S–supplement submodule of W.

Proof: Assume that W is a weakly e*S–supplemented. So, W = D + L, L ⊆ W and $D \cap L \ll_{e*S} W$. By our assumption we get $D \cap L \ll_{e*S} D$. Hence, D is an e*S–supplement of L in W.

Proposition 3.6: Let W be a weakly e*S–supplemented module then for every T, B ⊆ W, with T + B =W, there exists a weak e*S–supplement K of T in W with K ⊆ B.
Proof: Suppose T, B ⊆ W, with W = T + B. Since W is weakly e*S–supplemented, T ∩ B has a weak e*S–supplement D in W. In this case W = (T ∩ B) + D and $(T \cap B) \cap D \ll_{e*S} W$. Since W = T + B, and B = (T ∩ B) + (B ∩ D), then W = T + (T ∩ B) + (B ∩ D) = T + (B ∩ D). Let K = B ∩ D. Then W = T + K and $T \cap K = T \cap B \cap D \ll_{e*S} W$. Hence, K is a weak e*S–supplement of T in W with K ⊆ B.

Proposition 3.7: For any R–module W, let T is a weak e*S–supplement of V in W. Then for L ⊆ V, $\frac{T+L}{L}$ is a weak e*S–supplement of $\frac{V}{L}$ in $\frac{W}{L}$.
Proof: Since T is aweak e*S–supplement of V in W. Then W = V + T and $V \cap T \ll_{e*S} W$, for L ⊆ V. Now, $\frac{W}{L} = \frac{V+T}{L} = \frac{V}{L} + \frac{T+L}{L}$ and $\frac{V}{L} \cap (\frac{T+L}{L}) = \frac{(V \cap T)+L}{L}$, by Modular law, and since $V \cap T \ll_{e*S} W$, then $\frac{(V \cap T)+L}{L} \ll_{e*S} \frac{W}{L}$. Therefore, $\frac{T+L}{L}$ is a weak e*S–supplement of $\frac{V}{L}$ in $\frac{W}{L}$.

Corollary 3.8: Let A and B be submodules of R–module W, with A ⊆ B. If A and B have the same weak e*S–supplement submodule in W, then A is an e*S–coessential submodule of B.
Proof: Clearly by Proposition 2.24, and Examples and remarks 3.2 (2).
Corollary 3.9: Epimorphic image of a weakly e*S–supplemented module is a weakly e*S–supplemented module.
Proof: It follows from Corollary 2.19, and Examples and remarks 3.2 (2).

Corollary 3.10: Let W be aweakly e*S–supplemented module and a submodule D of W, then $\frac{W}{D}$ is a weakly e*S–supplemented module.
Proof: Clearly by Proposition 2.18, and Examples and remarks 3.2 (2).

Proposition 3.11: Let T be a submodule of an R–module W. Consider the following statement
1) T is e*S–supplement submodule of W;
2) T is e*S–coclosed in W;
3) For every submodule Y of T, if $Y \ll_{e*S} W$, then $Y \ll_{e*S} T$.
Then (1) ⇒ (2) ⇒ (3). If W is weakly e*S–supplemented, then (3) ⇒ (1).
Proof: (1) ⇒ (2) Let T, be an e*S–supplement of B in W, then W = T + B and $T \cap B \ll_{e*S} T$. To prove that T is an e*S–coclosed, assume that $Z_{e*}(\frac{T}{Y}) = \frac{T}{Y}$ and $\frac{T}{Y} \ll_{e*S} \frac{W}{Y}$ for some submodule Y of T. Since W = T + B, $\frac{W}{Y} = \frac{T}{Y} + \frac{B+Y}{Y}$. We have $\frac{W}{B+Y} = \frac{T+(B+Y)}{B+Y} \cong \frac{T}{T \cap (B+Y)} = \frac{T}{Y+(T \cap B)}$, since $Z_{e*}(\frac{T}{Y}) = \frac{T}{Y}$, then $Z_{e*}(\frac{W}{B+Y}) = \frac{W}{B+Y}$. But $\frac{T}{Y} \ll_{e*S} \frac{W}{Y}$ so $\frac{W}{Y} = \frac{B+Y}{Y}$ which implies that W = B + Y. Not that, T = T ∩ W = T ∩ (B + Y) = Y + (T ∩ B). But $T \cap B \ll_{e*S} T$ and $Z_{e*}(\frac{T}{Y}) = \frac{T}{Y}$, therefore T = Y. Thus, T is an e*S–coclosed.
(2) ⇒ (3) Suppose that T is an e*S–coclosed in W and $Y \ll_{e*S} W$, let T = Y + D, with $Z_{e*}(\frac{T}{D}) = \frac{T}{D}$. Since T is an e*S–coclosed in W, it is sufficient to show that $\frac{T}{D} \ll_{e*S} \frac{W}{D}$, let $\frac{W}{D} = \frac{T}{D}$

$+ \frac{B}{D}$ with $Z_{e*}(\frac{W}{B}) = \frac{W}{B}$, then W = T + B = Y + D + B = Y + B. But $Y \ll_{e*S} W$ and $Z_{e*}(\frac{W}{B}) = \frac{W}{B}$, therefore W = B.
(3) ⇒ (1) Since W is weakly e*S–supplemented, there exists a submodule D of W such that W = T + D and $T \cap D \ll_{e*S} W$. By (3) $T \cap D \ll_{e*S} T$. Thus, T is an e*S–supplement submodule of W.

4. Cofinitely e*-singular Supplemented and ⊕e*-singular Supplemented Modules.

It is known that a sub-module B of W is called cofinite if $\frac{W}{B}$ is finitely generated. We present one generalization of cofinitly supplement modules, as well as showing some of its properties. Recall that for any R–module W. If each cofinite sub-module of W owns a supplement in W, it is called a cofinitly supplemented [8]. Recall that if each sub-module of an R-module W owns a supplement, which is a direct summand of W, it is called ⊕-supplemented module. [18]. We introduce the generalization of ⊕-supplemented module with some properties.

Definition 4.1: An R–module W is called cofinitely e*S–supplemented (briefly cof e*S–supplemented) if each cofinite submodule of W has an e*S–supplement in W.
Examples and Remarks 4.2:
1) Z_6 as Z–module is cof e*S–supplemented module.
2) Z as Z–module isn't cof e*S–supplemented. Because 2Z is a cofinite submodule of Z which has no e*S–supplement. By Examples and remarks 2.5 (9).
3) Clearly that every e*S–supplemented module is cof e*S–supplemented. The converse isn't accurate in general. For example, the Z–module Q is cof e*S–supplemented, since the only cofinite submodule of Q, is Q which has an e*S–supplement, but Q we know that isn't e*S–supplemented.
The following proposition gives a condition under which the e*S–supplemented module and cof e*S–supplemented are equivalent.

Proposition 4.3: Let W be a finitely generated R–module. Then W is an e*S–supplemented if and only is W is cof e*S–supplemented.
Proof: Assum that W is cof e*S–supplemented to show that W is an e*S–supplemented. Let T be asubmodule of W, since W is finitely generated then $\frac{W}{T}$ is finitely generated, hence T is cofinite sub-module of W. But W is cof e*S–supplemented, hence T has e*S–supplement in W. Thus, W is e*S–supplemented. The converse is clear by Example and remarks 4.2 (3).
Next, we present certain cof e*S–supplemented module properties.

Proposition 4.4: Let W be a cof e*S–supplemented, and let T be a submodule of W, then $\frac{W}{T}$ is cof e*S–supplemented. for each fully invariant submodule T of W.
Proof: By the same arguments of Proposition 2.18.
The converse of Proposition 4.4, is not accurate in general, for example, the Z as Z-module. $\frac{Z}{6Z} \cong Z_6$ is cof e*S–supplemented but Z isn't cof e*S–supplemented.

Corollary 4.5: Let W be a cof e*S–supplemented, then any direct summand of W is cof e*S–supplemented.
We need the following standard lemma. To show that arbitrary sum of cof e*S–supplemented is cof e*S–supplemented.

Lemma 4.6: Let K, B be sub-modules of a module W such that K is cof e*S–supplemented, B is cofinite in W and K + B has an e*S–supplement T in W. Then K ∩ (B + T) has an e*S–supplement V in K. Moreover, V + T is an e*S–supplement of B in W.
Proof: Let T be an e*S–supplement of K + B in W. Thus W = T + (K + B) and T ∩ (K + B) $\ll_{e*S}$ T. Now, $\frac{K}{K \cap (B+T)} \cong \frac{K+(B+T)}{B+T} = \frac{W}{B+T} \cong \frac{\frac{W}{B}}{\frac{B+T}{B}}$, which is fintely generated, hence K ∩ (B + T) is cofinite in K. But K is cof e*S–supplemented. A submoduleV of K exists. Such that V is an e*S–supplement of K∩ (B + T) in K. Thus K = V + [K ∩ (B + T)] and V ∩ [K ∩ (B + T)] = V ∩ (B + T) $\ll_{e*S}$ V. Now, to show that T + V is an e*S–supplement of B in W, we have W = T + K + B = T + V + [K ∩ (B + T)] + B = T + V + B, one can easily show that B ∩ (T + V) ⊆ [T ∩ (B + V)] + [V ∩ (B + T)] $\ll_{e*S}$V + T. Therefore, V + T is an e*S–supplement of B in W.

Proposition 4.7: Arbitrary sum of cof e*S–supplemented modules is cof e*S–supplemented.
Proof: Assume that $\{W_i\}_{i\in I}$ isa family of cof e*S–supplemented modules, and let W = $\sum_{i\in I} Wi$. Let T, be a cofinite submodule of W, so W = T + W_{i1} + … + W_{in} for some n ∈ N, i_k ∈ I. Since T is cofinite in W and W has a zero e*S–supplement. Appling Lemma 4.6, we see by induction that T has an e*S–supplement in W. Thus, W is cof e*S–supplemented module.

Definition 4.8: An R-module W is called **⊕e*S–supplemented module** if each submodule of W has an e*S–supplement which is a direct summand of W.
Examples and Remarks 4.9:
1) Every semisimple is ⊕e*S–supplemented. For example, Z_6 as Z–module.
2) Z as Z–module isn't ⊕e*S–supplemented.
3) Obviously, that every ⊕e*S–supplemented is e*S–supplemented.
4) Every ⊕–supplemented is ⊕e*S–supplemented.

An R–module W is said to have **property (D3)**, if there are direct summands W_1 and W_2 ofW with W_1 + W_2 = W, implies W_1 ∩ W_2 is also adirect summand of W [19] [17].

Proposition 4.10: Let W be a ⊕e*S–supplemented module with D3 property. Then every direct summand of W is a ⊕e*S–supplemented module.
Proof: Let W be a ⊕e*S–supplemented with D3 property and let V be a direct summand of W. To show that V is a ⊕e*S–supplemented, let T, be a submodule of V. Then there exists a direct summand N of W such that N is an e*S–supplement of T in W, then W = T + N and T ∩ N $\ll_{e*S}$ N. But T ⊆ V, therefore W = V + N. Since V and N are direct summand of W and W = V + N, then V ∩ N is a direct summand of W and hence it is a direct summand of V. By modularity, we have V ∩ W = V ∩ (T + N) = T + (V ∩ N). Not that T ∩ (V ∩ N) = T ∩ N $\ll_{e*S}$ N. But, V ∩ N is a direct summand of W, therefore by Proposition13,[6]. T ∩ N $\ll_{e*S}$ V ∩ N, thus V is ⊕e*S–supplemented module.

Proposition 4.11: Let W be ⊕e*S–supplemented module and B be a fully invariant submodule of W. If B is a direct summand of W, then B is ⊕e*S–supplemented.

Proof: Let a direct summand T of W and Y be a submodule of T. Since W is ⊕e*S–supplemented, there exists a direct summand V of W, such that W = Y + V and Y ∩ V $\ll_{e*S}$ V and W = V ⊕ X, X ⊆ W. We have T = T ∩ W = T ∩ (V ⊕ X) = (T ∩ V) ⊕ (T ∩ X). If we show that T ∩ V is e*S–supplement of Y in T, then we complete the proof. By modularity, we

have T = T ∩ W = T ∩ (Y + V) = Y + (T ∩ V). Now, Y ∩ V $\ll_{e*S}$ V. Because T ∩ V is a direct summand of W, we obtain Y ∩ V $\ll_{e*S}$ T ∩ V. Hence, T ∩ V is an e*S–supplement of Y in B. So, it implies that T is ⨁e*S–supplemented.
The following theorem shows that the direct sum of ⨁e*S–supplemented modules is ⨁e*S–supplemented.

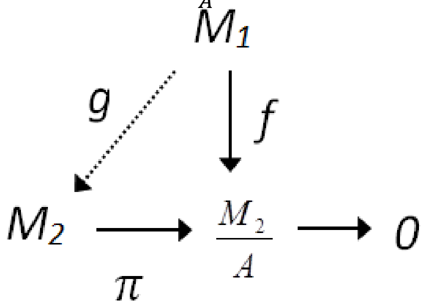
Theorem 4.12: Let W = $W_1 \oplus W_2$. If W_1 and W_2 be ⨁e*S–supplemented modules, then W is ⨁e*S–supplemented module.

Proof: Let B be any sub-module of W. Since W_1 is ⨁e*S–supplemented module, W_1 ∩ (W_2 + B) has an e*S–supplement Y in W_1 , then we have W_1 = [W_1 ∩ (W_2 + B)] + Y and W_1 ∩ (W_2 + B) ∩ Y = (W_2 + B) ∩ Y $\ll_{e*S}$ Y such that Y is adirect summand of W_1. Claim that Y is an e*S–supplement of W_2 + B in W. Since W_1 = [W_1 ∩ (W_2 + B)] + Y, then W = W_1 + W_2 = [W_1 ∩ (W_2 + B)] + Y + W_2 = Y + B + W_2 and (W_2 + B) ∩ Y $\ll_{e*S}$ Y, hence Y is an e*S–supplement of W_2 + B in W. Now, since W_2 ∩ (B + Y) ⊆ W_2 and W_2 is ⨁e*S–supplemented, then W_2 ∩ (B + Y) has an e*S–supplement X in W_2 and X is a direct summand of W_2 , then we have W_2 = X ⊕ K, K ⊆ W_2 , W_2 = [W_2 ∩ (B + Y)] + X and W_2 ∩ (B + Y) ∩ X = (B + Y) ∩ X $\ll_{e*S}$ X. Since W = W_2 + B + Y = [W_2 ∩ (B + Y) + X] + (B + Y) = X + B + Y and Y ∩ (X + B) ⊆ Y ∩ [X + [W_2 ∩ (B + Y)] + B] ⊆ Y ∩ (W_2 + B) $\ll_{e*S}$ Y and W_2 ∩ (B + Y) ∩ X = (B + Y) ∩ X $\ll_{e*S}$ X. One can easily show that B ∩ (Y + X) ⊆ [Y ∩ (X + B)] + [X ∩ (B + Y)] $\ll_{e*S}$ Y + X. So, Y + X is e*S–supplement of Bin W. Thus, W is ⨁e*S–supplemented.

Corollary 4.13: Any finite direct sum of ⨁e*S–supplemented modules are ⨁e*S–supplemented.

Proof: By induction.

Recall that if M_1 and M_2 be R–module. Then called M_1 is **M_2–projective** if for each submodule A of M_2 and any homomorphism f: $M_1 \longrightarrow \frac{M_2}{A}$, there is a homomorphism g : $M_1 \longrightarrow M_2$ such that $\pi \circ g = f$, where π : $M_2 \longrightarrow \frac{M_2}{A}$ is the natural epimorphosis, see [20].



$M_1$and $M_2$are say to be relatively projective if M_1 is M_2–projective and M_2 is M_1–projective, see [20].
Lemma 4.14: Let M = S ⊕ T = N + T where S is T–projective. Then M = K ⊕ T where K ⊆N.

Proof: See ([20] Lemma 4.47).

Theorem 4.15: Let $W_i (1 \le i \le n)$ be relatively projective, any finite collection modules. The module W = $W_1 \oplus W_2 \oplus \ldots \oplus W_n$ is a ⨁e*S–supplemented module if and only if W_i is ⨁e*S–supplemented module for all ($1 \le i \le n$).

Proof: The necessity part is proved in Theorem 4.12.
Conversely, it is enough to prove that W_1 is ⊕e*S–supplemented. Let S be any submodule of W_1. Then there exists a direct summand B of W such that W = S + B and S ∩ B $\ll_{e*S}$ B. Not that W = S + B = W_1 + B. By lemma 4.14, there exists a submodule K of B such that W = W_1 ⊕ K. Now, B = B ∩ W = B ∩ (W_1 ⊕ K) = K ⊕ (B ∩ W_1), then (B ∩ W_1) is a direct summand of W and hence it is a direct summand of W_1 . Now, we have W_1 = W_1 ∩ W = W_1 ∩ (S + B) = S + (B ∩ W_1), and S ∩ B ∩ W_1 = S ∩ B $\ll_{e*S}$ B, then S ∩ B ∩ W_1 $\ll_{e*S}$ B ∩ W_1. Therefore B ∩ W_1 is e*S–supplement of S in W_1 which is adirect summand. Thus W_1 is ⊕e*S–supplemented.

Proposition 4.16: Let W be ⊕e*S–supplemented nonzero module and let S be a fully invariant submodule of W. Then the factor module $\frac{W}{S}$ is a ⊕e*S–supplemented.
Proof: To show that $\frac{W}{S}$ is ⊕e*S–supplemented, let $\frac{B}{S}$ be any submodule of $\frac{W}{S}$. Since W is ⊕e*S–supplemented module, there exists a direct summand D of W such that W = B + D, B ∩ D $\ll_{e*S}$ D and W = D ⊕ A, A ⊆ W. By Proposition 2.11, $\frac{D+S}{S}$ is e*S–supplement of $\frac{B}{S}$ in $\frac{W}{S}$. Since S is a fully invariant submodule of W, then $\frac{D+S}{S}$ is adirect summand of $\frac{W}{S}$. Thus, $\frac{W}{S}$ is ⊕e*S–supplemented.

Corollary 4.17: Let W be a ⊕e*S–supplemented duo module. Then each factor module of W is a ⊕e*S–supplemented module.

Theorem 4.18: Let W be a module such that W = $W_1 \oplus W_2$ is a direct sum of sub-modules W_1 and W_2. Then W_2 is a ⊕e*S–supplemented module if and only if there exists a direct summand X of W such that X ⊆ W_2 , W = T + X and T ∩ X $\ll_{e*S}$ X, for every submodule $\frac{T}{W_1}$ of $\frac{W}{W_1}$.
Proof: ⇒) Let $\frac{T}{W_1}$ be any submodule of $\frac{W}{W_1}$. Since W_2 ∩ T ⊆ W_2 and W_2 is ⊕e*S–supplemented, then T ∩ W_2 has e*S–supplement say X in W_2, where X ⊕ K = W_2, W_2 = (T ∩ W_2) + X and T ∩ W_2 ∩ X = T ∩ X $\ll_{e*S}$ X. Clearly, X is a direct summand of W and W = W_1 + W_2 = W_1 + (T ∩ W_2) + X ⊆ W_1 + T + X. But W_1 ⊆ T, therefore W = T + X. So, we obtain a result.
⇐) Let T be a submodule of W_2. Consider the submodule $\frac{T \oplus W_1}{W_1}$ of $\frac{W}{W_1}$. By our hypothesis there exists a direct summand X of W such that X ⊆ W_2 , W = (T + W_1) + X and (T + W_1) ∩ X $\ll_{e*S}$ X. Since W_2 = W_2 ∩ W = W_2 ∩ [(T + W_1) + X] = X + [(T + W_1) ∩ W_2] = X + T + (W_1 ∩ W_2) = X + T, by Modular law, and since T ∩ X ⊆ (T + W_1) ∩ X $\ll_{e*S}$ X, then X is e*S–supplement of T in W_2 . Thus W_2 is ⊕e*S–supplemented.

Proposition 4.19: Let W be a ⊕e*S–supplemented module. Then W = $W_1 \oplus W_2$, where $Z_{e*}(W_1) \ll_{e*S} W_1$ and $Z_{e*}(W_2) = W_2$.
Proof: Since $Z_{e*}(W)$ ⊆ W, and W is ⊕e*S–supplemented module, then there exists W_1 such that W = $W_1 \oplus W_2$ for some submodule W_2 of W, W = $Z_{e*}(W) + W_1$ and $Z_{e*}(W) \cap W_1 \ll_{e*S} W_1$. But $Z_{e*}(W_1) = Z_{e*}(W) \cap W_1 \ll_{e*S} W_1$. Since $Z_{e*}(W) = Z_{e*}(W_1) \oplus Z_{e*}(W_2)$, then W = $Z_{e*}(W_1) \oplus Z_{e*}(W_2) + W_1$, with $Z_{e*}(\frac{W}{Z_{e*}(W_2) + W_1}) = \frac{W}{Z_{e*}(W_2) + W_1} = \frac{Z_{e*}(W_1)+(Z_{e*}(W_2) + W_1)}{(Z_{e*}(W_2) + W_1)} \cong \frac{Z_{e*}(W_1)}{Z_{e*}(W_1)\cap(Z_{e*}(W_2) + W_1)} = \frac{Z_{e*}(W_1)}{Z_{e*}(W_1)} = 0 = Z_{e*}(0)$ by Second Isomorphism Theorem. So, W =

$Z_{e*}(W_2) \oplus W_1$. But by Modular law, $W_2 = W_2 \cap W = W_2 \cap (Z_{e*}(W_2) \oplus W_1) = Z_{e*}(W_2)$. Thus, we get the result.

Theorem 4.20: For an R–module W with (D3) the following statements are equivalent.
1) Every direct summand of W is ⊕e*S–supplemented;
2) W is ⊕e*S–supplemented;
3) W = $W_1 \oplus W_2$, where W_1 is ⊕e*S–supplemented with $Z_{e*}(W_1) \ll_{\mathbf{e*S}} W_1$ and W_2 is ⊕e*S–supplemented with $Z_{e*}(W_2) = W_2$.
Proof: (1 ⇒ 2) Clear by the Definition.
(2 ⇒ 1) By Proposition 4.10.
(2 ⇒ 3) By Proposition 4.19.
(3 ⇒ 2) By Theorem 4.12.

5. Conclusions.

We confirm the following outcomes:
1) Every supplemented module is e*S–supplemented.
2) The image homomorphism of the e*S–supplemented module is e*S–supplemented.
3) If W is a uniform cosingular R-module, then weakly supplemented and weakly e*S–supplemented modules coincide.
4) Every e*S–supplemented module is cof e*S–supplemented.
5) A random sum of cof e*S–supplemented modules are cof e*S–supplemented.
6) Every e*S–hollow modules are ⊕e*S–supplemented.
7) Any finite direct sum of ⊕e*S–supplemented modules are ⊕e*S–supplemented.
8) Many properties have been presented of an e*S–supplemented, weak e*S–supplemented, cof e*S–supplemented and ⊕e*S–supplemented modules.

References:

[1] K. Ahmmed, N. S. Al. Mothafar "Pr-small R-submodules of modules and Pr-radicals," *Journal of Interdisciplinary Mathematics,* vol. 26, no. 7, pp. 1511-1516, 2023.

[2] F. Kasch, Modules and Rings, London: Acad. Press, 1982.

[3] D. X. Zhou, X. R. Zhang,"Small-Essential Submodule and Morita Duality," *Southeast Asian Bulletin of Mathematic ,* vol. 35, pp. 1051-1062, 2011.

[4] H. Baanoon, I_{F*}-Lifting Module With Respect to Image of Fully Invariant Submodules, University of Baghdad: Ph.D. Thesis, 2023.

[5] K. R. Goodearl, Ring theory: Nonsingular ring and modules, Newyork: Marcel Dekker, 1976.

[6] A. Kabban, W. Khalid, "e*Singular–Hollow Modules and e*Singular–Coclosed Submodules," *Iraqi Journal of Science.*, Vol. 66, No. 6, pp. 2477-2486, 2025.

[7] L. Ganesan, N. Vanaja. "Module for which every submodule has a unique coclosure," *Communications in Algebra,* vol. 30, no. 5, pp. 2355-2377, 2002.

[8] J. Clark, C. Lomp, N. Vanaja and R. Wisbauer. Lifting Modules, Supplements and Projectivity in Module Theory, Boston: Frontiers in Mathematics, 2006.

[9] Y. Talebi, B. Talaee. "On δ -Coclosed Submodules," *Far East Journal of Mathematical Sciences,* vol. 35, no. 1, pp. 19-31, 2009.

[10] S. Crivei, H. Inankil."Correspondences of Coclosed Submodules," *Communications in Algebra,* vol. 41, pp. 3635-3647, 2013.

[11] A. T. Hamad, A. A. Elewi. "On Z-coessential submodules and Z-coclosed submodules," *AIP Conference Proceedings.,* vol. 2414, pp. 251-260, 2023.

[12] M. A. Sagban, S. M. Yaseen. "Supplement-Hollow and Supplement-Lifting Modules," *Journal of Al-Qadisiyah for Computer Science and Mathematics,* vol. 16, no. 4, pp. 76-81, 2024.

[13] R. S. Wadbude, "Generalization of Pure-Supplemented Modules," *IJRIAS,* pp. 2454-6194, 2020.

[14] T. Amouzegar, "On *I*-⊕-supplemented Modules," *Revisat Colombiana de Matematicas,* vol. 54, no. 2, pp. 129-139, 2020.

[15] A. A. Alawi, T. A. Ibrahiem. "On δ^*-Supplemented Modules," *Baghdad Science Journal,* vol. 17, no. 1, pp. 136-140, 2020.

[16] S. M. Yaseen, M. Tawfiq. "Y-Supplement Extending Modules," *General Mathematics Notes,* vol. 29, no. 2, pp. 48-54, 2015.

[17] R. Wisbauer, Foundation of module and ring theory, Philadelphia: Gordon and Breach, 1991.

[18] A. Haramanci, D. Keskin, and P. F. Smith "On ⊕–Supplemented Modules," *Acta Mathematica Hungarica,* vol. 83, no. 1-2, pp. 161-169, 1999.

[19] D. Keskin, "Discrete and quasi-discrete modules," *Communications in Algebra,* vol. 30, no. 11, pp. 5273-5282, 2002.

[20] S. H. Mohamed, B. J. Muller. Continuous and discrete Modules, Soc. LNS 147 Cambridge University Press Cambridge: London Math., 1990.