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CENTRALIZING MAPPINGS OF PRIME AND SEMIPRIME *-RINGS

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Abstract

In this paper we prove the following result. Let R be a non-commutative prime*-ring of characteristic different from 2, then R is normal *-ring if and only if there exists a nonzero Jordan*-derivation d: R→R be which satisfies [d(x), x] ∈ Z(R) for all x ∈ R, and [d(h),s] ∈ Z(R) or [d(s), h] ∈ Z(R) for all h ∈ H(R), s ∈ S(R).

الــدوال في الحلقــات-* الاوليــة وشبــه الاوليــة

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الخلاصة

في هذا البحث سنبرهن التالي: لتكن R حلقة-* اولية (غير ابدالية) طليقة الالتواء من الـنمط ٢، فـان R تكون حلقة-* سوية اذا وفقط اذا وجدd: R→R دالة مشتقة-* جوردان غير الصفرية وتحقـق [d(x), x] في Z(R) لكل x في Z(R)، وان [d(h), s] في Z(R) او [d(s), h] في Z(R) لكل h في H(R)، s في S(R).

1. Introduction

This note is motivated by the work of M. Breŝar and J. Vukman [1]. Throughout, R will represent an associative ring with center $Z(R)$. A ring R is n-torsion free, if $nx = 0$, $x \in R$ implies $x = 0$, where n is a positive integer. Recall that R is prime if $aRb = (0)$ implies $a = 0$ or $b = 0$, and semiprime if $aRa = (0)$ implies $a = 0$. An additive mapping $x \to x^*$ on a ring R is called an involution if $(xy)^* = y^* x^*$ and $(x)^{**} = x$ for all $x, y \in R$. A ring equipped with an involution is called *-ring. An element x in a *-ring R is said to be hermitian if $x^* = x$ and skew-hermitian if $x^* = -x$. The sets of all hermitian and skew-hermitian elements of R will be denoted by $H(R)$ and $S(R)$, respectively. If R is 2-torsion free then every $x \in R$ can be uniquely represented in the form $2x = h + k$ where $h \in H(R)$ and $k \in S(R)$. An element $x \in$ R is called normal element if $xx^* = x^*x$, and if all the elements of R are normal then R is called a normal ring. As usual the commutator xy-yx will be denoted by $[x, y]$. We shall use basic commutator identities $[xy, z] = [x, z]\, y + x[y, z]$ and $[x, yz] = [x, y]z + y[x, z]$ for all $x,y,z \in R$. An additive mapping d: $R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all pairs $x,y \in R$, and is called a Jordan derivation in case $d(x^2) = d(x)x + xd(x)$ is fulfilled for all $x \in R$. An additive mapping d: $R \to R$ is called a *-derivation if $d(xy) = d(x)y^* + xd(y)$ holds for all pairs x , $y \in R$, and is called a Jordan *-derivation in case $d(x^2) = d(x)x^* + xd(x)$ is fulfilled for all $x \in R$, the concepts of *-derivation and Jordan*-derivation were first mentioned in [1]. It is clear that Every *-derivation is a Jordan *-derivation but the converse in general not true, for example let R be a 2-torsion free semiprime *-ring and let $a \in$ R such that $[a,x] \neq 0$, for some $x \in R$, define a map d: $R\to R$ as follows, d(x)=ax^*-xa for all $x\in$ R, then d is a Jordan *-derivation but not a *-derivation. Let S be a nonempty subset of R, a function f: $R\to R$ is said to be a centralizing function on S (resp. commuting on S) if $[f(x), x] \in Z(R)$, for all $x \in S$ (resp. $[f(x), x]=0$, for all $x \in S$). The fundamental result on commuting and related mappings is due to E. Posner [2]. He proved that, if a derivation D of a prime ring satisfies $[D(x), x]\in Z(R)$ for all $x \in$ R, then R is commutative. Recently, many authors studied Posner's theorem in more generalized versions. J. Mayne [3] obtained the analogous result for automorphisms. J. Vukman [4] proved if R be a 2-torsion free semiprime ring and d: $R\to R$ be a derivation. Suppose that $[[d(x),x],x]=0$ holds for all $x\in$ R. In this case $[d(x),x]=0$ holds for all $x\in$ R. M.Brešar[5] show that R is commutative if there exist derivation d and g, not both zero, such that $(xd(x)-g(x)x)\in Z(R)$ for all $x \in$ R. The purpose of this paper is to prove a result concerning a Jordan *-derivations. More precisely, we study a centralizing of this map on non-commutative prime ring.

2. Main Result

In the following theorem a centralizing Jordan *-derivation d on 2-torsion free semiprime *-ring, such that $[d(h),s] \in Z(R)$ or $[d(s), h] \in Z(R)$ for all $h \in H(R)$, $s \in S(R)$, force d is commuting.

Theorem 2.1.

Let R be a 2-torsion free semiprime *-ring, and d: R→R be a Jordan *-derivation which satisfies $[d(x), x] \in Z(R)$ for all $x \in$ R, and $[d(h),s] \in Z(R)$ or $[d(s), h] \in Z(R)$ for all $h \in H(R)$, $s \in S(R)$, then $[d(x), x]=0$ for all $x \in$ R.

To prove the above theorem we need following lemmas.

Lemma 2.2.

Let R be a 2-torsion free *-semiprime ring, and d: R→R be a Jordan *-derivation witch satisfies $[d(x), x] \in Z(R)$ for all $x \in$ R, then $[d(h), h]=0$ for all $h \in H(R)$.

Proof

We have

$$[d(x), x] \in Z(R) \text{ for all } x \in R. \quad (1)$$

Putting x^2 for x in (1) we get

$$[d(x^2), x^2] \in Z(R) \quad \text{for all } x \in R. \quad (2)$$

Therefore,

$$[d(x)x^*+xd(x),x^2] \in Z(R) \quad \text{for all } x \in R.$$

Setting $x= h \in H(R)$ in the above relation, we get

$$[d(h)\, h + h\, d(h),\, h^2] \in Z(R) \quad \text{for all } h \in H(R). \quad (3)$$

Because of,

$$d(h)\, h+ h\, d(h)=2\, h\, d(h)-[\, h,\, d(h)] \quad \text{for all } h \in H(R), \quad (4)$$

According to (3) and (4) we get

$$[2h\, d(h)-[\, h,\, d(h)],\, h^2] \in Z(R) \quad \text{for all } h \in H(R). \quad (5)$$

From relation (5) we obtain

$$4\, h^2\, [h,\, d(h)] \in Z(R) \quad \text{for all } h \in H(R). \quad (6)$$

Therefore,

$$h^2\, [\, h,\, d(h)],\, d(h)]=0 \quad \text{for all } h \in H(R). \quad (7)$$

Then from (7) one obtain

$$8\, h\, [h,d(h)]^2 =0 \quad \text{for all } h \in H(R). \quad (8)$$

Therefore,

$$8[h\, [h,d(h)]^2,\, d(h)]=0 \quad \text{for all } h \in H(R). \quad (9)$$

Since $[x,d(x)]^2 \in Z(R)$, then we get

$$8[h,d(h)]^2\, [\, h,d(h)]=0 \quad \text{for all } h \in H(R). \quad (10)$$

R is a 2-torsion free we get

$$[h,d(h)]^2\, [\, h,d(h)]=0 \quad \text{for all } h \in H(R). \quad (11)$$

Right multiplication by $z[h,d(h)]$, we get

$$[h,d(h)]^2\, z\, [h,d(h)]^2=0 \quad \text{for all } z \in R, \text{ and for all } h \in H(R). \quad (12)$$

By the semiprimness of R, we have

$$[h,d(h)]^2=0 \quad \text{for all } h \in H(R). \quad (13)$$

Left multiplication by z, we get

$$[h,d(h)]\, z\, [h,d(h)] =0 \quad \text{for all } h \in H(R). \quad (14)$$

Since R is a semiprime *-ring we get

$[d(h), h] =0$ for all $h \in H(R)$.

Lemma 2.3.

Let R be a 2-torsion free semiprime *-ring, and let d: R→R be a Jordan *-derivation which satisfies $[d(x), x] \in Z(R)$ for all $x \in R$, then $[d(s), s] = 0$ for all $s \in S(R)$.

Proof

Putting $x+y$ in (1) we get

$$([d(x), y] + [d(y), x]) \in Z(R) \quad \text{for all } x, y \in R. \quad (15)$$

Replace x by x^2 and y by x^* we obtain

$$([d(x^2), x^*] + [d(x^*), x^2]) \in Z(R) \quad \text{for all } x \in R. \quad (16)$$

Setting $x=s \in S(R)$, we get

$$([d(s^2), s^*] + [d(s^*), s^2]) \in Z(R) \quad \text{for all } s \in S(R). \quad (17)$$

But,

$$d(s^2)=sd(s)-d(s)s=[s,d(s)] \in Z(R) \quad \text{for all } s \in S(R). \quad (18)$$

Then from (17), (18) we get

$$[s^2,d(s)] \in Z(R) \quad \text{for all } s \in S(R). \quad (19)$$

Therefore,

$$2s[s,d(s)] \in Z(R) \quad \text{for all } s \in S(R). \quad (20)$$

Since $[s,d(s)] \in Z(R)$, we obtain

$$0=2[s[s,d(s)],d(s)]=2[s,d(s)]^2 \quad \text{for all } s \in S(R). \quad (21)$$

R is a 2-torsion free we get

$$[d(s), s]^2=0 \quad \text{for all } s \in S(R). \quad (22)$$

Right multiplication by z, we get

$$[s,d(s)]\, z\, [s,d(s)] = 0 \quad \text{for all } s \in S(R). \quad (23)$$

By the semiprimness of R, $[d(s),s]=0$ for all $s \in S(R)$.

Proof of Theorem 2.1

Assume that $[d(h),s] \in Z(R)$ for all $h \in H(R), s \in S(R)$, By using Lemma2.2, we have

$$[d(h),h]=0 \quad \text{for all } h \in H(R). \quad (24)$$

For $h_1, h_2 \in H(R)$, putting h_1+h_2 for h, we get

$$[d(h_1), h_2]+[d(h_2),h_1]=0 \quad \text{for all } h_1, h_2 \in H(R). \quad (25)$$

Since $s^2 \in H(R)$ for all $s \in S(R)$, then replace h_2 by s^2 in (25) we get

$$[d(h_1), s^2]+[d(s^2),h_1]=0 \quad \text{for all } s \in S(R), \text{ and } h_1 \in H(R). \quad (26)$$

By using Lemma 2.3, we have

$$d(s^2)=sd(s)-d(s)s=[s,d(s)] = 0 \quad \text{for all } s \in S(R). \quad (27)$$

According to the relation (26), (27) we get

$$[d(h_1),s^2]=0 \quad \text{for all } s \in S(R), \text{ and } h_1 \in H(R). \quad (28)$$

Therefore since $[d(h_1),s] \in Z(R)$, we obtain

$$2s[d(h_1),s]=0 \quad \text{for all } s \in S(R), \text{ and } h_1 \in H(R). \quad (29)$$

Hence,

$$2[d(h_1), s[d(h_1),s]]=0 \quad \text{for all } s \in S(R), \text{ and } h_1 \in H(R). \quad (30)$$

Therefore,

$$2[d(h_1),s]^2=0 \quad \text{for all } s \in S(R), \text{ and } h_1 \in H(R). \quad (31)$$

Since R 2-torsion free we get

$$[d(h_1),s]^2=0 \quad \text{for all } s \in S(R), \text{ and } h_1 \in H(R). \quad (32)$$

Right multiplication by z, we get

$$[d(h_1),s]\, z\, [d(h_1),s] = 0 \quad \text{for all } s \in S(R), \text{ and } h_1 \in H(R). \quad (33)$$

By the semiprimness of R, we have

$$[d(h_1),s] = 0 \quad \text{for all } s \in S(R), \text{ and } h_1 \in H(R). \quad (34)$$

Putting s for x, and h for y in the relation (15) we get

$$[d(s), h] + [d(h), s] \in Z(R), \text{ for all } s \in S(R), \text{ and } h \in H(R). \quad (35)$$

Comparing the relation (34) and (35) we get

$$[d(s),h] \in Z(R), \text{ for all } s \in S(R), \text{ and } h \in H(R). \quad (36)$$

Since $h^2 \in H(R)$, for all $h \in H(R)$, then from (36) we obtain

$$[d(s), h^2] \in Z(R), \text{ for all } s \in S(R), \text{ and } h \in H(R). \quad (37)$$

By assumption $[d(h),s] \in Z(R)$ for all $h \in H(R)$, $s \in S(R)$, Then from relation (37) one obtains ((see how (34) was obtained from (28))

$$[d(s),h] = 0 \text{ for all } s \in S(R), \text{ and } h \in H(R). \quad (38)$$

To prove $[d(x),x]=0$, Since R be a 2-torsion free we only show

$$4[d(x),x]=0 \text{ for all } x \in R. \quad (39)$$

We have for all $x \in R$ then ($2x=s+h$ for $s \in S(R)$, and $h \in H(R)$). Therefore,

$$4[d(x),x]=[d(2x),2x]=[d(s+h), s+h] \text{ for } s \in S(R), \text{ and } h \in H(R).$$

Hence,

$$4[d(x),x]=[d(s),s]+[d(s),h]+[d(h), h]+[d(h), s] \text{ for } s \in S(R), \text{ and } h \in H(R).$$

By using Lemma2.2, and Lemma2.3, and relation (34), (38) we get

$$[d(x), x]=0 \text{ for all } x \in R.$$

Now assume

$$[d(s),h] \in Z(R) \text{ for all } h \in H(R), s \in S(R).$$

Then from relation (36) we get

$$[d(s),h] = 0 \text{ for all } s \in S(R), \text{ and } h \in H(R).$$

Then from (35) we get

$$[d(h),s] \in Z(R) \text{ for all } h \in H(R), s \in S(R),$$

Then we get, similar as a first assumption

$$[d(x),x]=0 \text{ for all } x \in R,$$

Then the proof of Theorem2.1 is complete. Now, we'll mention the third result in [1].

Theorem 2.4. [1].
Let R be a non-commutative prime *-ring of characteristic different from 2, then R is normal ring if and only if there exists a nonzero commuting Jordan *-derivation.

The main goal of this paper is to prove the following corollary. This corollary says that the existence of a non-zero centralizing Jordan *-derivation d on non-commutative prime *-ring R, such that $[d(h),s] \in Z(R)$ or $[d(s), h] \in Z(R)$ for all $h \in H(R)$, $s \in S(R)$, implies that R is a normal *-ring.

Corollary 2.5.
Let R be a non-commutative prime *-ring of characteristic different from 2, then R is normal *-ring if and only if there exists a nonzero Jordan *-derivation $d: R \to R$ be which satisfies $[d(x), x] \in Z(R)$ for all $x \in R$, and $[d(h),s] \in Z(R)$ or $[d(s), h] \in Z(R)$ for all $h \in H(R)$, $s \in S(R)$.

Proof:
If R is a normal *-ring then by using Theorem2.4, then prove is a clear, to prove the converse, we have by using Theorem2.1, that d is a nonzero commuting Jordan *-derivation, hence by Theorem2.5, we get R is a normal *-ring.

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