MODULES HAVING (WEAK-S*) PROPERTY

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Abstract

Let R be a non zero ring with identity and let M be a non zero module over R. An R-module M is called cosingular if $Z^*(M)=M$ where $Z^*(M)=\{m \in M, mR<<E(M)\}$, in this paper we introduce the concept that an R-module M is weak-cosingular if $Z^*(M)\leq_e M$. and we call that an R-module has (weak-S*)property if every submodule N of M contains a direct summand K of M such that K≤N and N/K is weak-cosingular. And we study the properties of this kind of modules, and the relation between this modules and other kind of modules.

المقاســــــات ذات الخاصيــــــــة S^* الضعيفـــــــة

ســــاهرة محمــــــــود ياسيـــــن
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الخلاصة

لتكن R حلقة غير صفرية ذات عنصر محايد وليكن M مقاسا احاديا غير صفري ايمن معرف على R . وليكن $Z^*(M)=\{m \in M ,mR<<E(M)\}$. يقال للموديول M بأنه منفرد مضاد اذا كان $Z^*(M)=M$. في هذا البحث سنقول ان المقاس M بانه منفرد مضاد ضعيف اذا كان $Z^*(M)\leq_e M$. وان المقاس M له الخاصية (S^*) الضعيفة اذا كان لكل مقاس جزئي N من M توجد مركبة مجموع مباشر K من M حيث ان K مقاس جزئي من N وان N/K موديول منفرد مضاد ضعيف.درسنا الخواص الاساسية لهذا النوع من المقاسات وعلاقته مع بعض المقاسات الاخرى.

Introduction

Let R be a ring with identity and M be unital right R-module We write E(M), Rad(M) for injective envelope and radical submodule of M , respectively. We use N≤M to signify that N is submodule of M. N is essential in M, we write $N\leq_e M$, if N ∩ K ≠ 0 ∀ K non zero submodule of M.

A submodule N of M is called small submodule wherever N+L=M for some submodule L of M, we have M=L and in this case we write N<<M. In [1] Leonard defines a module M to be small if it is a small submodule of some R-module and he shows that M is small if and only if M is small in its injective hull. In [2] observed that $Z^*(M)$ = {m∈M, Rm is small module} is submodule of R-module M. This type of submodules was studied by Ozcan .In [3] it is shown that $Z^*(M)$ =M∩ Rad E (M), where Rad E(M) is the Jacobson radical of injective hull of M.

A module M is called lifting module if for every submodule N of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq N$ and $N \cap M_2 << M$ [4]. Let A and L be submodules of a module M, L is called a supplement of A in M if it is minimal with the property A + L = M and a submodule K is called a supplement in M if K is a supplement of some submodule of M. It is easy to check that L is supplement of A in M if and only if M = A + L and A ∩ L << L.

Let M be an R-module. The submodule. The following lemmas are proved in [3].

Lemma(1.1)

Let R be a ring and φ: M→M′ be homomorphism of R-modules M, M′, then $\varphi(Z^*(M)) \leq Z^*(M')$.

Lemma(1.2)

Let N be a submodule of R-module M, then $Z^*(N) = N\cap Z^*(M)$.

Lemma(1.3)

Let M_i ($i\in I$) be collection of R-modules and let $M=\bigoplus_{i\in I} M_i$, then $Z^*(M)=\bigoplus_{i\in I} Z^*(M_i)$.

Let R be a ring and M be an R-module M is called cosingular if $Z^*(M)=M$ .And R is called right cosingular if the (right) R-module R is cosingular. Small modules are cosingular and the converse is true if R is prefect ring [2]. Thus every Z-module is cosingular.

Weak-Cosingular Modules

In this section we introduce the concept of weak-cosingular module.

Definition(2.1)

Let R be a ring and M an R-module, M is called weak-cosingular if $Z^*(M)\leq_e M$, and R is called right weak-cosingular if the (right) R-module R is weak cosingular.

Remark(2.2)

Cosingular modules are weak-cosingulr, but the converse is not true as in the following example.

Let $R=\begin{bmatrix} F & 0 \\ F & F \end{bmatrix}$ be a lower tringular matrices over a field F, $J(R)=\begin{bmatrix} 0 & 0 \\ F & 0 \end{bmatrix}$. $Soc(R_R)=\begin{bmatrix} F & 0 \\ F & 0 \end{bmatrix}$, $Z^*(R_R)=Soc(R_R)$ by (8,ex 11). Thus $Z^*(R_R)\leq_e M$, then M is weak-cosingular but not cosingular.

Lemma(2.3)

For any ring R let M be weak-cosingular R-module, then every submodule of M is weak-cosingular.

Proof

Let N be submodule of M, thus $Z^*(N)=N\cap Z^*(M)$ lemma (1.2) to show that $Z^*(N)\leq_e N$, let $K\neq\{0\}$ be submodule of N thus $K\cap Z^*(N)=K\cap N\cap Z^*(M)=K\cap Z^*(M)\neq\{0\}$, since $Z^*(M)\leq_e M$.then $Z^*(N)\leq_e N$.

Lemma(2.4)

Let M_i ($i\in I$) be collection of weak-cosingular modules, and let $M=\bigoplus_{i\in I} M_i$ then M is weak-cosingular.

Proof

M_i is weak-cosingular, then $Z^*(M_i)\leq_e M_i$ $\forall i\in I$, $Z^*(M)=\bigoplus Z^*(M_i)$ by lemma (1.3). Then $Z^*(M)\leq_e M$ [6].

Modules with (weak- S^*) property

In [3] an R-module M has (S^*) property if for every submodule N of M, there exists a direct summand K of M such that $K\leq N$ and N/K is cosingular. In this section we introduce (weak- S^*) property.

Definition(3.1)

Let M be an R-module, M is said to satisfy (weak- S*) property, if for every submodule N of M , there exists a direct summand K of M such that $K\leq N$ and N/K is weak-cosingular. A ring R satisfies (weak- S^*) property if the right R –module R satisfies weak- S^* property.

Remark(3.2)

Every modules satisfies S^* property satisfies (weak- S^*) property.

The following Lemma follows immediately from the definition.

Lemma(3.3)

Let M be an R-module satisfies (weak- S^*) property, then every submodule of M has (weak- S^*) property.

Proof

Let N be a submodule of M. Forany $K\leq N$, $\{0\}\leq K$ and $\{0\}\oplus N=N$ and $Z^*(K/\{0\}) = Z^*(K) = K\cap Z^*(M)$, $Z^*(K)\leq_e K$. Then N is (weak- S^*).

Remark (3.4)

1. Every weak cosingular module satisfies (weak- S^*) property.
   Proof: Clear.
2. Every lifting module satisfies (weak-S^*) property.
   Proof: Since every lifting module has S^* property [7].

Lemma(3.5)

Let M be a module satisfies (weak- S^*) property. Such that $Z^*(M)$ is small in M, then M is lifting-module.

Proof

Let N be submodule of M, then there exists a direct summand K of M such that $K\leq N$ and N/K is weak cosingular i.e. $Z^*(N/K)\leq_e N/K$. Let L be a submodule of M such that $M=K\in L$, then $N=K\oplus(N\cap L)$, i.e. $N/K = N\cap L$. Thus $Z^*(N\cap L)\leq_e N\cap L$. Thus

$N\cap L$ is weak-cosingular, $N\cap L << Z^*(M)$, i.e. $N\cap L<<M$. Hence M is lifting module.

Lemma(3.6)

Let M be a weak-cosingular such that M_1, M_2 direct summands of M with $M_1\leq M_2$, then $Z^*(M_1) = Z^*(M_2)$ if and only if $M_1= M_2$.

Proof

see [7.lemma 16].

Lemma(3.7)

Let M be an R-module, then the following statements are uquivalent

1. M satisfies (weak- S^*) property.
2. For every submodule N of M, N has decomposition $N=A\oplus B$ such that $A\leq N$ and $N\cap B$ is weak-cosingular.
3. For every submodule N of M, N has a decomposition $N=A\oplus B$ such that A is direct summand of M and B is weak-cosingular.

Proof

(1⇒2) Let N be submodule of M, then by (1), there exists $A\leq\oplus$ M, $M= A\oplus B$, $A\leq N$ and $Z^*(N/A)\leq_e N/A$. $N=A\oplus(N\cap B)$, $N\cap B \cong N/A$. Thus $Z^*(N\cap B) \leq {}_e N\cap B$. Then $N\cap B$ is cosingular**.**

(2⇒1) is clear. Since $N\cap B \cong N/A$. i.e. N/A is weak-cosingular.

(1⇒3) Let N be submodule of M. Since M satisfies (weak- S^*) property and $N= A\oplus B$, by hypothesis, then there exists $A \leq\oplus M$, $A\leq N$; N/A is (weak-cosingular).
Hence there exist H submodule of M such that $M= A\oplus H$, then $N=A \oplus (N\cap H)$, then $N\cap H$ is weak-cosingular. But $N\cap H\cong B$, thus B is weak-cosingular.

(3⇒1) is clear, since $N/A\cong B$.

Lemma(3.7)

Let M be an R-module that satisfies (weak-S^*). Suppose that there exists a supplement of $Z^*(M)$ in M, then there is decomposition $M= A\oplus B$ such that A is lifting module and B is weak- cosingular.

Proof

By hypothesis, there existsa submodule A of M such that $M=A+Z^*(M)$, $A\cap Z^*(M)<<A$. Then $Z^*(A) = Rad(A)<<A$. Since M satisfies weak- S^*, there exists a direct summand K of M such that $K\leq A$, $Z^*(A/K) \leq_e K$. Let B be submodule of M such that $M=K\oplus B$. $A=K\oplus (A\cap B)$. Since M is satisfies (weak-S^*) by (lemma 3.6).Then $A\cap B\leq_e Z^*(A\cap B) \leq Z^*(A) \leq A$ but $A\cap B$ is direct summand of A, then $A\cap B=0$, hence $M=A\oplus B$, by lemma(3.3) and lemma(3.5), A is lifting module, we have $M=A+ Z^*(M)= A+ Z^*(A)+ Z^*(B)= A+Z^*(B)$, hence $Z^*(B) \leq_e B$.

Corollary(3.8)

Let M be an R-module satisfies weak-S^*, then there is a decomposition $M=A\oplus B$ such that A is semisimple with $Z^*(A)=0$ and B is weak-cosingular. (see 3)

Proposition (3.9)

Let R be a ring. An injective R-module M satisfies (weak-S*) property. If and only if every submodule of M is a direct sum of an injective module and a weak-cosingular module.

Proof

Suppose that M satisfies (weak-S*) property. Let N be a submodule of M. There exist submodules K, K′ of M such that $M=K\oplus K'$, $K\leq N$ and N/K is weak-cosingular. Then $N = K\oplus (N\cap K')$ where K is injective and $N\cap K'$ is weak-cosingular since $N\cap K'\cong N/K$. Conversely, suppose that every submodule of M is a direct sum of an injective module and a weak-cosingular module. Let L be any submodule of M. Then $L = L_1 \oplus L_2$ for some injective module L_1 and weak- cosingular module L_2. Clearly L_1 is a direct summand of M and $Z^*(L/L_1) \leq_e L/L_1$ because $L/L_1 \cong L_2$.

Theorem (3.10)

The following statements are equivalent for a ring R.

i) Every right R-module satisfies (weak-S*) property.,
ii) Every injective right R-module satisfies (weak-S*) property.
iii) Every right R-module is a direct sum of an injective module and a weak-cosingular module.

Proof

(i) ⇔ (ii) It is clear because every sub-module of a module with (weak-S*) also has (weak-S*).

(ii) ⇔ (iii) by Proposition (3.9)

Lemma (2.3.14)

Let P_i $(1\leq i \leq n)$ be a finite collection of projective injective R- modules satisfying (weak-S*) and let P= $P_1 \oplus \ldots\ldots \oplus P_n$. Then P satisfies (weak-S*) property.

Proof

By induction on n it is sufficient to prove the result when n = 2. Let P = $P_1 \oplus P_2$ and let f_i: $P \rightarrow P_i$ (i = 1; 2) denote the canonical projections. Let N be a submodule of P. By hypothesis, the submodule $f_1(N)=Q_1 \oplus L_1$ for some direct summand Q_1 of P_1 and weak-cosingular submodule L_1 of P_1. Let Ø: $f_1(N) \rightarrow Q_1$ denote the canonical projection. Then $Øf_1$: N $\rightarrow$ Q_1 is an epimorphism with kernel H= $\{m \in N: f_1(m) \in L_1\}$. Note that Q_1 is a projective module and hence $N=N_1 \oplus H$ for some submodule $N_1 \cong Q_1$. by the same argument for $f_2(H)$ we see that $H=N_2 \oplus N'$ for some sub-module N_2 isomorphic to a direct summand of P_2 and submodule N' where $N' = \{m \in N: f_1(m) \in L_1; f_2(m) \in L_2\}$ for some weak-cosingular submodule L_2 of P_2. $N = N_1 \oplus N_2 \oplus N'$ where $N_1 \oplus N_2$ is injective and hence a direct summand of P. and $N' \leq L_1 \oplus L_2$ so that N' is weak- cosingular by [lemma 2.4] then P satisfies (weak-S*) property.

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