# THE INVERSE OF OPERATOR MATRIX A WHERE A≥I

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**Abstract**

Let H and K be Hilbert spaces and let H$\oplus$K be the cartesian product of them.Let B(H),B(K),B(H$\oplus$K),B(K,H),B(H,K) be the Banach spaces of bounded(continuous) operators on H,K,H$\oplus$K,and from K into H and from H into K respectively. In this paper we find the inverse of operator matrix $A=\begin{bmatrix} B & C \\ D & E \end{bmatrix}$ $\in$ B(H$\oplus$K) where B$\in$B(H) ,C$\in$B(K,H), D$\in$B(H,K), E$\in$B(K) and A≥ $I_{H\oplus K}$ where $I_{H\oplus K}$ is the identity operator on H$\oplus$K.

## معكوس مصفوفة المؤثر A حيث A≥I

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**الخلاصة**

ليكن كل من H,K فضاء هلبرت وليكن H$\oplus$Kهو الضرب الديكارتي لهما وليكن
B(H),B(K),B(H$\oplus$K),B(K,H),B(H,K)
فضاءات باناخ لكل المؤثرات المقيده(المستمره)على H,K,H$\oplus$K ، ومن K الى H ومن H الى K على الترتيب.في هذا البحث سنجد معكوس مصفوفة المؤثر

$A=\begin{bmatrix} B & C \\ D & E \end{bmatrix}$ $\in$ B(H$\oplus$K)

حيث أن B$\in$B(H) ,C$\in$B(K,H), D$\in$B(H,K), E$\in$B(K) وأن $I_{H\oplus K}$ ≥A حيث $I_{H\oplus K}$ هو المؤثر المحايد على H$\oplus$K.

## Introduction

Let $<,>$ denotes an inner product on a Hilbert space, and we will denote Hilbert spaces by H, K, $H_i$, $K_i$ and H ⊕ K denotes the Cartesian product of the Hilbert spaces H, K ,and B(H) ,B(H ⊕ K),B(K,H),be the Banach spaces of bounded(continuous) operators on H, H ⊕ K and from K into H respectively[see2]. The inner product on H⊕K is define by: $\langle (x,y),(w,z)\rangle \geq \langle x,w\rangle + \langle y,,z\rangle$ x,w ∈ H, y,z ∈ K.

we say that A is positive operator on H and denote that by $A \geq 0$ if $\langle Ax,x\rangle \geq 0$ for all x in H,and in this case it has a unique positive square root ,we denote this square root by $\sqrt{A}$ [see2],it is easy to check that $A$ is invertible if and only if $\sqrt{A}$ is invertible.

$A^*$ denotes the adjoint of $A$ and $I_H$ denotes the identity operator on the Hilbert space H.We define the operator matrix $A=\begin{bmatrix} B & C \\ E & D\end{bmatrix} \in$ B(H ⊕ K,L ⊕ M) where $B \in B(H,L), C \in B(K,L)$, $E \in B(H,M), D \in B(K,M)$ as following $A\begin{pmatrix} x \\ y\end{pmatrix} = \begin{bmatrix} B & C \\ E & D\end{bmatrix}\begin{pmatrix} x \\ y\end{pmatrix} = \begin{bmatrix} Bx+Cy \\ Ex+Dy\end{bmatrix}$,where $\begin{pmatrix} x \\ y\end{pmatrix} \in H \oplus K$ ,and similar for the case $m \times n$ operator matrix [see 1,3,6].

If $A=\begin{bmatrix} B & C \\ E & D\end{bmatrix}$ then $A^* = \begin{bmatrix} B^* & E^* \\ C^* & D^*\end{bmatrix}$.

If $A=\begin{bmatrix} B & C \\ E & D\end{bmatrix} \geq 0$ then A is a self- adjoint and so has the form $A=\begin{bmatrix} B & C \\ C^* & D\end{bmatrix}$ and similar for the case $n \times n$ operator matrix [see 1,3]. For elementary facts about matrices [see5 ,8] and for elementary facts about Hilbert spaces and operator theory [see 2,6].

**Remark:** we will sometimes denote $I_{H\oplus K}$ (the identity operator on H ⊕ K) or $I_H$ (the identity on H ) or $I_K$ (the identity on K ) or any identity operator by ,and also we will sometimes denote any zero operator by 0

## Preliminaries

**Proposition1.1**.: Let T ∈ B(H,K) then

1)if $T^*T \geq I$ and $TT^* \geq I$ then T is invertible,

2)if T is self-adjoint, $T^2 \geq I$ then T is invertible,

3)if $T \geq 0$ then T is invertible if and only if $\sqrt{T}$ is invertible, and in this case we have $((\sqrt{T}))^2)^{-1} = ((\sqrt{T}))^{-1})^2$ .

4)if T is self-adjoint then T is invertible from right if and only if it is invertible from left,

5)if $T \geq I$ then T is invertible,

6) if $T \geq 0$ and it is invertible then $T^{-1} \geq 0$ ,and in this case we have $\sqrt{T^{-1}} = (\sqrt{T})^{-1}$

7) $T \geq I$ if and only if $0 \leq T^{-1} \leq I$ .

**Proof**:1)see[2]p.156

2) From 1.

3) if T is invertible then there exists an operator S such that ST =TS=I ,so (S $\sqrt{T}$ ) $\sqrt{T}$ = $\sqrt{T}$ ( $\sqrt{T}$ S)=I i.e. $\sqrt{T}$ is invertible . Conversely if $\sqrt{T}$ is invertible then there exists an operator R such that R $\sqrt{T}$ = $\sqrt{T}$ R= $\sqrt{T}$ ,so I=I.I= ( $\sqrt{T}$ R)( $\sqrt{T}$ R) = $\sqrt{T}$ (R $\sqrt{T}$ )R = $\sqrt{T}$ ( $\sqrt{T}$ R)R =TR$^2$= R$^2$T ,hence T is invertible, and in this case we have $((\sqrt{T}))^2)^{-1}$ =T$^{-1}$=R$^2$=$((\sqrt{T}))^{-1})^2$ .□

4) if T is self-adjoint then T=T$^*$ ,but T is invertible from right if and only if T$^*$ is invertible from left.□

5) if $T \geq I$ then $T \geq 0$ ,so $\sqrt{T}$ exists and it is self-adjoint and $(\sqrt{T})^2 \geq I$ ,so $\sqrt{T}$ is invertible and hence T is invertible.□

6)if $T \geq 0$ and it is invertible then $\langle Tx,x\rangle \geq 0$ ,so $\langle TT^{-1}x, T^{-1}x\rangle \geq 0$ . i.e. $\langle x, T^{-1}x\rangle \geq 0$ , $\forall x$ .Hence . $T^{-1} \geq 0$ Now $\sqrt{I}$ =$I$ because $\sqrt{I}$ . $\sqrt{I}$ =$I$, and $I.I$=$I$,but the positive square root is unique(see[2]p.149) so $\sqrt{I}$ =I.and since $T \geq 0, T^{-1} \geq 0, T^{-1}T = I \geq 0$ ,we have $\sqrt{T^{-1}}$ $\sqrt{T}$ = $\sqrt{T^{-1}T}$ (see[2]p.149),so $\sqrt{T^{-1}}$ $\sqrt{T}$ = $\sqrt{I}$ =I,hence $\sqrt{T^{-1}} = (\sqrt{T})^{-1}$ .□

7) If $T \geq I$ then $T \geq 0$ and it is invertible .so [from 6)] we have $T^{-1} \geq 0$ .Now $T^{-1} \geq 0$ & $T-I \geq 0$ &T$^{-1}$(T-I) = (T-I) T$^{-1}$ [because

$T^{-1}$ (T-I) $=T^{-1}T- T^{-1}=I-T^{-1}$ and (T-I) $T^{-1}$ = $TT^{-1}-T^{-1}$ =I-$T^{-1}$ So, $T^{-1}(T-I)\geq 0$ (see [2]p.149) ,hence $T^{-1}\leq I$ .□ Conversely if $0\leq T^{-1}\leq I$ then [from 6)]we have $T\geq 0$ but $I-T^{-1}\geq 0$ and T(I-$T^{-1}$) =(I-$T^{-1}$)T so $T(I-T^{-1})\geq 0$ ,hence $T\geq I$ .

**Proposition1.2**:1) if $A=\begin{bmatrix} B & C \\ D & E \end{bmatrix}\geq 0$ then C=$D^*$ and B≥0 & E≥0

2) If $A=\begin{bmatrix} B & C \\ D & E \end{bmatrix}\geq I$ then C= $D^*$ and B≥I & E≥I

3) if $A=\begin{bmatrix} B & C \\ D & E \end{bmatrix}\leq I$ then C=$D^*$ and $B\leq I$ & $E\leq I$

**Proof**:1)see[1]p.18

2) if $T\geq I$ then $T-I\geq 0$ but $I=\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$ ,so $\begin{bmatrix} B & C \\ D & E \end{bmatrix}-\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}=\begin{bmatrix} B-I & C \\ D & E-I \end{bmatrix}\geq 0$ .Then from 1) we have that C=$D^*$, B-I≥0 & E-I≥0 i.e. B≥I & E≥I.□
3)Similar to 2)

**Proposition1.3.:** if $A=\begin{bmatrix} B & C \\ C^* & E \end{bmatrix}$ is invertible, A≥I then B,E are invertible
**Proof**: from Proposition1.2. 2) we have B≥I & E≥I,so B,E are invertible.□
To show that the converse is not true we need the following theorem from[1]p.19:-
**Theorem1.4**.:Let B∊B(H),E∊B(K),C∊B(K,H) such that B≥0 & E≥0 then:
$\begin{bmatrix} B & C \\ C^* & E \end{bmatrix}\geq 0$ if and only if there exists a contraction X∊B(K,H) such that C=$\sqrt{B}X\sqrt{E}$
Now the following example show that the converse of proposition1.3. is not true

**Example1.5**: Let $A=\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$,so B=2≥1,E=2≥1 and they are invertible but A is not invertible[since detA=0].Note that A≥0 [since C=2= $\sqrt{2}\sqrt{2}=\sqrt{B}X\sqrt{E}$ where X=1,hence $|X|\leq I$ ],but $A\ngeq I$[since $A-I=\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$,and if $\exists X$ such that 2=$\sqrt{1}X\sqrt{1}$,so X=2,hence $|X|\neq 1$ i.e. $A-I\ngeq 0$,hence $A\ngeq I$].

**Remark1.6:** it is easy to check that:
1) If A is invertible $m\times n$ operator matrix (i.e. ∃ an $n\times m$ operator matrix B s.t. AB=$I_m$&BA=$I_n$
Where $I_m$ & $I_n$ are the $m\times m$ and the $n\times n$ identity operator matrices respectively) and if matrix C results from A by interchanging two rows (columns) of A then C is also invertible.
2) If two rows (columns) of an $m\times n$ operator matrix A are equal then A is not invertible.
3) If a row (column) of an $m\times n$ operator matrix A consists entirely of zero operators then A is not invertible.
4) $A=\begin{bmatrix} B & 0 \\ 0 & E \end{bmatrix}$ is invertible if and only if B,E are invertible, and in this case $A^{-1}=\begin{bmatrix} B^{-1} & 0 \\ 0 & E^{-1} \end{bmatrix}$.

**Remark1.7**.: from remark1.6. 1) We can conclude: if $A=\begin{bmatrix} B & C \\ D & E \end{bmatrix}$ ∊ B (H⊕K,L⊕M)then $A=\begin{bmatrix} B & C \\ D & E \end{bmatrix}$ is invertible if and only if $\begin{bmatrix} C & B \\ E & D \end{bmatrix}$ is invertible if and only if $\begin{bmatrix} D & E \\ B & C \end{bmatrix}$ is invertible if and only if $\begin{bmatrix} E & D \\ C & B \end{bmatrix}$ is invertible.

**2) The inverse of a 2x2 operator matrix A where A≥I**

**Theorem2.1.:**1)if $A=\begin{bmatrix} B & C \\ C^* & E \end{bmatrix}\geq I$ then B,E, B-$CE^{-1}C^*$, E-$C^*B^{-1}C$ are invertible and

$$A^{-1}=\begin{bmatrix}(B-CE^{-1}C^*)^{-1} & -(B-CE^{-1}C^*)^{-1}CE^{-1}\\ E^{-1}C^*(B-CE^{-1}C^*)^{-1} & (E-C^*B^{-1}C)^{-1}\end{bmatrix}$$

In fact :2)if $A=\begin{bmatrix}B & C\\ C^* & E\end{bmatrix}\geq I$ then B≥I,E≥I, $B-CE^{-1}C^*\geq I$, $E-C^*B^{-1}C\geq I$ .

Proof 1) if $A=\begin{bmatrix}B & C\\ C^* & E\end{bmatrix}\geq I$ then $A$ is invertible[proposition1.1.5)] and B≥I,E≥I [proposition1.2.2)] ,so B,E are invertible [proposition1.1.5)]

Now, let $A^{-1}=\begin{bmatrix}J & G\\ G^* & F\end{bmatrix}$ i.e. $AA^{-1}$ $=I=\begin{bmatrix}I_H & 0\\ 0 & I_K\end{bmatrix}$,then J≥0 & F≥0 since $A^{-1}\geq 0$

And i)$BJ+CG^*=I_H$ ii) BG+CF=0 iii)C*J+EG*=0 iv)$C^*G+EF=I_K$

So from iii) we haveJC+GE=0.So,$G=-JCE^{-1}=-B^{-1}CF$ Then we have from iv) that $(E-C^*B^{-1}C)F=I_K$ i.e. $(E-C^*B^{-1}C)$ is invertible , $F=(E-C^*B^{-1}C)^{-1}$

and from i) we have $J(B-CE^{-1}C^*)=I_H$,so $B-CE^{-1}C^*$ is invertible, and $J=(B-CE^{-1}C^*)^{-1}$,

$G=-(B-CE^{-1}C^*)^{-1}CE^{-1}=-B^{-1}C(E-C^*B^{-1}C)^{-1}$

Then it is clear that,

$$A^{-1}=\begin{bmatrix}(B-CE^{-1}C^*)^{-1} & -(B-CE^{-1}C^*)^{-1}CE^{-1}\\ -E^{-1}C^*(B-CE^{-1}C^*)^{-1} & (E-C^*B^{-1}C)^{-1}\end{bmatrix}$$

2) if $A=\begin{bmatrix}B & C\\ C^* & E\end{bmatrix}\geq I$ then

$$0\leq A^{-1}=\begin{bmatrix}(B-CE^{-1}C^*)^{-1} & -(B-CE^{-1}C^*)^{-1}CE^{-1}\\ -E^{-1}C^*(B-CE^{-1}C^*)^{-1} & (E-C^*B^{-1}C)^{-1}\end{bmatrix}\leq I$$

,so from proposition1.2.1&3) We have $0\leq(B-CE^{-1}C^*)^{-1}\leq I$ , $0\leq(B-CE-1C^*)^{-1}\leq I$ then from proposition1.1.7) $B-CE^{-1}C^*\geq I$, $E-C^*B^{-1}C\geq I$.

also from proposition 1.2.2) we have that B≥I& E≥I□

**Remark2.2.:** it is easy to check that if B, E, $B-CE^{-1}C^*$, $E-C^*B^{-1}C$ are invertible then $A=\begin{bmatrix}B & C\\ C^* & E\end{bmatrix}$ is invertible and

$$A^{-1}=\begin{bmatrix}(B-CE^{-1}C^*)^{-1} & -(B-CE^{-1}C^*)^{-1}CE^{-1}\\ -E^{-1}C^*(B-CE^{-1}C^*)^{-1} & (E-C^*B^{-1}C)^{-1}\end{bmatrix}.$$

But what is about:

**Question2.3.:**is it true that if B≥I,E≥I, $B-CE^{-1}C^*\geq I$, $E-C^*B^{-1}C\geq I$. then A≥I?

**Remark2.4.:**since $(B-CE^{-1}C^*)^{-1}CE^{-1}=B^{-1}C(E-C^*B^{-1}C)^{-1}$

,and since $A=\begin{bmatrix}B & C\\ C^* & E\end{bmatrix}\geq I$ ,hence A-I≥0 and A≥0,therefore there exists a contraction X and a contraction Y such that

$C=\sqrt{B}X\sqrt{E}=\sqrt{B-I}Y\sqrt{E-I}$

then we have alternative forms of $A^{-1}$ such:

1)

$$A^{-1}=\begin{bmatrix}(B-CE^{-1}C^*)^{-1} & -B^{-1}C(E-C^*B^{-1}C)^{-1}\\ -E^{-1}C^*(B-CE^{-1}C^*)^{-1} & (E-C^*B^{-1}C)^{-1}\end{bmatrix}$$

or

2)

$$A^{-1}=\begin{bmatrix}(\sqrt{B})^{-1}(I-XX^*)^{-1}(\sqrt{B})^{-1} & -(\sqrt{B})^{-1}(I-XX^*)^{-1}X(\sqrt{E})^{-1}\\ -(\sqrt{E})^{-1}X^*(I-XX^*)^{-1}(\sqrt{B})^{-1} & (\sqrt{E})^{-1}(I-X^*X)^{-1}(\sqrt{E})^{-1}\end{bmatrix}$$

…etc.

**Remark2.5.:**the second form of $\mathbf{A}^{-1}$ above show that $I-X^*X$,$I-XX^*$ are invertible and this is easy to check.

**Remark2.6.:**we know that if a ,c , e are complex numbers( the complex number is a special case of an operator) and

$A=\begin{bmatrix}b & c\\ c^* & e\end{bmatrix}$ where $c^*$ is the conjugate of c

then $A^{-1}=\begin{bmatrix}\frac{e}{be-|c|^2} & \frac{-c}{be-|c|^2}\\ \frac{-c^*}{be-|c|^2} & \frac{b}{be-|c|^2}\end{bmatrix}$ but from above:

$$A^{-1}=\begin{bmatrix}(b-ce^{-1}c^*)^{-1} & -(b-ce^{-1}c^*)^{-1}ce^{-1}\\ -e^{-1}c^*(b-ce^{-1}c^*)^{-1} & (e-c^*b^{-1}c)^{-1}\end{bmatrix}=$$

$$\begin{bmatrix}\frac{1}{b-\frac{|c|^2}{e}} & -\frac{1}{b-\frac{|c|^2}{e}}c\frac{1}{e}\\ -\frac{1}{e}c^*\frac{1}{b-\frac{|c|^2}{e}} & \frac{1}{e-\frac{|c|^2}{b}}\end{bmatrix}=$$

$$\begin{bmatrix}\frac{e}{be-|c|^2} & \frac{-c}{be-|c|^2}\\ \frac{-c^*}{be-|c|^2} & \frac{b}{be-|c|^2}\end{bmatrix}$$

**Remark2.7.:** of course we can generalize the $2\times 2$ case to the $n\times n$ case by iteration. For example: if $A=\begin{bmatrix} B & C & D \\ C^* & E & G \\ D^* & G^* & F \end{bmatrix}\geq I$ ,then

$$A=\begin{bmatrix} B & C & D \\ C^* & E & G \\ D^* & G^* & F \end{bmatrix}$$

$$=\begin{bmatrix} \begin{bmatrix} B & C \\ C^* & E \end{bmatrix} & \begin{bmatrix} D \\ G \end{bmatrix} \\ \begin{bmatrix} D^* & G^* \end{bmatrix} & F \end{bmatrix}=\begin{bmatrix} \begin{bmatrix} B & C \\ C^* & E \end{bmatrix} & \begin{bmatrix} D \\ G \end{bmatrix} \\ \begin{bmatrix} D \\ G \end{bmatrix}^* & F \end{bmatrix}$$

and we can first find the inverse of $\begin{bmatrix} B & C \\ C^* & E \end{bmatrix}\geq I$ ,then find the inverse of A.

**Remark2.8.:** there is no general relation between the invertiblity of $A=\begin{bmatrix} B & C \\ D & E \end{bmatrix}$ and the invertiblity of B,C,D,E ,and all the 32 cases can be hold,for example

1) $A=\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is not invertible but B,C,D,E are invertible

2) $A=\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ is invertible and also B,C,D,E are invertible

3)A= $\begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 1 \end{bmatrix}$ is not invertible

[since detA=0] and $B=\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is not invertible,but

$C=\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, D=\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, E=\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ is invertible

And so on.

of course, $A=\begin{bmatrix} B & C \\ D & E \end{bmatrix}$ is invertible if and only if $\begin{bmatrix} C & B \\ E & D \end{bmatrix}$ is invertible if and only if $\begin{bmatrix} D & E \\ B & C \end{bmatrix}$ is invertible if and only if $\begin{bmatrix} E & D \\ C & B \end{bmatrix}$ is invertible is useful here

**Question2.9.:** How can we find the inverse of the general case $A=\begin{bmatrix} B & C \\ D & E \end{bmatrix}\in B(H\oplus K, L\oplus M)$ ?.

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