S* -SEPARATION AXIOMS

Sabiha I.Mahmood*, Afrah M. Ibraheem**
*Department of Mathematics, College of Science, University of Al Mustansiriyah. Baghdad-Iraq.
**Department of Mathematics, College of Education, University of Al-Mustansiriyah. Baghdad-Iraq.

Abstract

In this paper we introduce a new type of separation axioms which we call s*-separation axioms. We obtain the definition from standard separation properties by replacing open set by s*-open set in their definitions. Moreover, we study the relation between this type of separation axioms and each of generalized separation axioms (g-separation axioms) and standard separation axioms.

بديهيات الفصل – S*

صبيحة إبراهيم محمود*، أفراح محمد إبراهيم**
*قسم الرياضيات، كلية العلوم، الجامعة المستنصرية. بغداد–العراق.
**قسم الرياضيات، كلية التربية، الجامعة المستنصرية. بغداد–العراق.

الخلاصة

في هذا البحث قدمنا نوع جديد من بديهيات الفصل أسميناها ببديهيات الفصل من النمط -S* (S*-separation axioms).
حصلنا على التعريف من خواص الفصل الاعتياديه باتخاذ المجموعه المفتوحة من النمط –S* بدلا من المجموعه المفتوحه بالاضافه إلى ذلك درسنا العلاقة بين هذا النوع من بديهيات الفصل وكل من بديهيات الفصل المعممة (generalized separation axioms) وبديهيات الفصل الاعتياديه.

1.Introduction

Levene,N. (2) generalized the concept of closed sets to the generalized closed sets (g-closed sets). Al-Meklafi, S. (3) generalized the concept of closed sets to the s*-closed sets. The complement of a generalized closed (resp.s*-closed) set is called a generalized open (resp.s*-open) set. In this paper we derive different properties of s*-closed sets and s*-open sets also we introduce a new type of separation axioms namely, S*-separation axioms, which is properly placed in between the standard separation axioms and the generalized separation axioms (g-separation axioms). We obtain the definition from standard separation properties by replacing open set by s*-open set in their definitions. Moreover, we study the relation between this type of separation axioms and each of generalized separation axioms and standard separation axioms.

2. Preliminaries

Throughout this paper (X,τ) and (Y,σ) repressent nonempty topological spaces. If $A \subseteq Y \subseteq X$ then ,$cl(A)$, $int(A)$ and $X-A$ denote the closure of A, the interior of A and the complement of A in X respectively also, $cl_y(A)$ and $int_y(A)$ denote the closure of A and the interior of A in Y respectively.

First we recall the following definitions.

(2.1)Definition(1):

A subset A of a topological space (X,τ) is called **a semi-open** (s-open) set if there exists an open subset U of X such that $U \subseteq A \subseteq cl(U)$. The complement of a semi-open set is defined to be **semi-closed** (s-closed).

(2.2)Definition(2):

A subset A of a topological space (X,τ) is called **a generalized closed** (g-closed) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X,τ). The complement of a g-closed set is defined to be **generalized open** (g-open).
The class of all g-open subsets of (X,τ) is denoted by $GO(X,\tau)$.

(2.3)Definition(3):

A topological space (X,τ) is called **a g-T_0 space** if for any two distinct points x and y of X there is a g-open set of X containing one of them, but not the other.

(2.4)Definition(3):

A topological space (X,τ) is called **a g-T_1-space** if for any two distinct points x and y of X there is a g-open set of x which dose not contain y and a g-open set of y which dose not contain x.

(2.5)Definition(3):

A topological space (X,τ) is called **a g-T_2-space** if for any two distinct points x and y of X there are two g-open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \phi$.

(2.6)Remarks(3):

i)Every T_i- space is g-T_i- space (i= 0,1,2) but the converse may not be true in general.
ii) Every g-T_i- space is g-T_{i-1}- space (i=1,2) but the converse may not be true in general.

3. Properties of s*-open sets and s*-closed sets

(3.1)Definition(3):
A subset A of a topological space (X,τ) is called **an s*-closed set** if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is s-open in (X,τ).
The complement of an s*-closed set is defined to be **s*-open.**
The class of all s*-open subsets of (X,τ) is denoted by $S^*O(X,\tau)$.

(3.2)Remarks:

i)Every open (closed) set is an s*-open (s*-closed) set respectively.
ii) Every s*-open (s*-closed) set is a g-open (g-closed) set respectively.
The converse of (i) and (ii) may not be true in general.
iii) s-open sets and s*-open sets are independent.

(3.3)Theorem:

A subset A of a topological space (X,τ) is an s*-closed set iff $cl(A)-A$ contains no non-empty s-closed set.

Proof:

Necessity, Let F be an s-closed subset of (X,τ) such that $F \subseteq cl(A)-A$. Then $A \subseteq X-F$. Since A is s*-closed and $X-F$ is s-open , then $cl(A) \subseteq X-F$. This implies $F \subseteq X-cl(A)$ So $F \subseteq X-cl(A) \cap cl(A) = \phi$. Therefore $F=\phi$.
Sufficiency, suppose A is a subset of (X,τ) such that $cl(A)-A$ does not contain any non-empty s-closed set .Let O be an s-open set of (X,τ) such that $A \subseteq O$ .If $cl(A) \not\subset O$, then $cl(A) \cap (X-O)$ is a non-empty s-closed subset of $cl(A)-A$. This is a contradiction .Therefore A is an s*-closed set .

(3.4)Theorem:

If A and B are s*-closed sets, then $A \cup B$ is also an s*-closed set.

Proof:

If $A \cup B \subseteq O$ and if O is s-open , then $cl(A \cup B) = cl(A) \cup cl(B) \subseteq O$ .Thus $A \cup B$ is an s*-closed set.

(3.5)Theorem:

If A is an s*-closed set of (X,τ) and $A \subseteq B \subseteq cl(A)$,then B is also an s*-closed set of (X,τ).

Proof:

Let O be an s-open set of (X,τ) such that $B \subseteq O$ .Then $A \subseteq O$ .Since A is s*-closed, then $cl(A) \subseteq O$. Now, $cl(B) \subseteq cl(cl(A)) = cl(A) \subseteq O$. Therefore B is also an s*-closed set of (X, τ).

(3.6)Theorem:

A subset A of a topological space (X, τ) is s*-open iff $F \subseteq int(A)$ whenever F is an s-closed subset of (X, τ) and $F \subseteq A$.

Proof:

Suppose that A is s*-open and $F \subseteq A$, where F is s-cloced, then $X - A \subseteq X - F$. Since $X - F$ is s-open and $X - A$ is s*-closed, then $cl(X - A) \subseteq X - F$. Hence $X - int(A) \subseteq X - F$. Therefore $F \subseteq int(A)$.

Conversely, suppose that $F \subseteq int(A)$ whenever F is s-closed and $F \subseteq A$. To prove that A is s*-open.

Let $X - A \subseteq U$, where U is s-open in (X, τ). Then $X - U \subseteq A$. Since $X - U$ is s-closed, then $X - U \subseteq int(A)$, hence $X - int(A) \subseteq U$. Therefore $cl(X - A) \subseteq U$. Thus $X - A$ is an s*-closed set .i.e. A is an s*-open set in (X, τ).

(3.7)Theorem:

If A and B are s*-open sets, then $A \cap B$ is also an s*-open set.

Proof

The proof follows immediately from (3.4) by showing that $X - (A \cap B)$ is s*-closed.

(3.8)Theorem:

If A and B are separated s*-open sets, then $A \cup B$ is s*-open.

Proof:

Let F be an s-closed subset of $A \cup B$. Then $F \cap cl(A) \subseteq A$. By (1) $F \cap cl(A)$ is s-closed and hence by (3.6) $F \cap cl(A) \subseteq int(A)$. Similarly, $F \cap cl(B) \subseteq int(B)$.

Now,

$F = F \cap (A \cup B) \subseteq$
$(F \cap cl(A)) \cup (F \cap cl(B)) \subseteq$
$int(A) \cup int(B) \subseteq int(A \cup B)$.

Hence $F \subseteq int(A \cup B)$ and by (3.6) $A \cup B$ is s*-open.

(3.9)Corollary:

let A and B be two s*-closed sets and suppose that $X - A$ and $X - B$ are separated. Then $A \cap B$ is s*-closed.

Proof:

The proof follows immediately from (3.8) by showing that $X - (A \cap B)$ is s*-open.

(3.10)Theorem:

If A is an s*-open set of (X, τ) and $int(A) \subseteq B \subseteq A$, then B is also an s*-open set of (X, τ).

Proof:

Since $X - A \subseteq X - B \subseteq X - int(A) = cl(X - A)$ and $X - A$ is s*-closed, then by (3.5) $X - B$ is s*-closed. Thus B is s*-open.

(3.11)Theorem:

A subset A of a topological space (X, τ) is s*-closed iff $cl(A) - A$ is s*-open.

Proof:

Necessity, suppose that A is s*-closed and that $F \subseteq cl(A) - A$, F being s-closed. Then by (3.3) $F = \phi$ and hence $F \subseteq int(cl(A) - A)$. Therefore by (3.6) $cl(A) - A$ is s*-open.

Sufficiency, suppose that $cl(A) - A$ is s*-open and $A \subseteq O$, where O is an s-open set.

Now, $cl(A) \cap (X - O) \subseteq cl(A) \cap (X - A) = cl(A) - A$ and since $cl(A) \cap (X - O)$ is s-closed and $cl(A) - A$ is s*-open, it follows that $cl(A) \cap (X - O) \subseteq int(cl(A) - A) = \phi$. Therefore $cl(A) \cap (X - O) = \phi$ or $cl(A) \subseteq O$. Thus A is s*-closed.

(3.12)Theorem:

Let (X, τ) be a topological space and (Y, τ') be a closed subspace of (X, τ). If A is s*-open in (X, τ), then $A \cap Y$ is s*-open in (Y, τ').

Proof:

Let F be an s-closed subset of Y such that $F \subseteq A \cap Y$, then $F \subseteq A$. Since F is s-closed in Y and Y is s-closed in X, then by (4) F is s-closed in X, hence by (3.6) $F \subseteq int(A)$. Since $F = F \cap Y \subseteq int(A) \cap Y \subseteq int_y(A \cap Y)$ then

$F \subseteq int_y(A \cap Y)$. Thus $A \cap Y$ is an s*-open set in Y.

§ 4. S*-Separation axioms

As an application of s*-open sets, we introduce five new spaces namely, s*-T_0-spaces , s* -T_1-spaces , s*-T_2-spaces , s* -T_3-spaces and s*-$T_{3\frac{1}{2}}$-spaces.

(4.1)Definition:

A topological space (X,τ) is called **an s*-T_0-space** if for any two distinct points x and y of X there is an s*-open set of X containing one of them ,but not the other .

Since every open set is an s*-open set,then we have the following theorem:-

(4.2)Theorem:

Every T_0-space is an s*-T_0-space.

Proof:

It is obvious .

(4.3)Remark:

The converse of (4.2) may not be true in general.Consider the following example:-

Example:

Let X={a,b,c} & τ={ϕ,X,{a,b}}.

Since $S^*O(X,\tau)$ ={ϕ,X,{a},{b},{a,b}}.

Then (X,τ) is an s*-T_0-space ,but not a T_0-space.

Since every s*-open set is a g-open set, then we have the following theorem:-

(4.4)Theorem:

Every s*-T_0-space is a g-T_0-space.

Proof:

It is obvious .

(4.5)Remark:

The converse of (4.4) may not be true in general .Consider the following example:-

Example:

Let X={a,b,c} & τ={ϕ,X,{a}}.

Since $GO(X,\tau)$={ϕ, X, {a}, {b}, {c}{a,b}, {a,c}} and $S^*O(X,\tau)$ ={ϕ,X,{a}} .Then (X,τ) is a g-T_0-space,but not s*-T_0.

(4.6)Theorem:

s*-T_0 property is a closed-hereditary property

Proof:

Let (Y,τ') be a closed subspace of an s*-T_0-space (X,τ).To verify that (Y,τ') is an s*-T_0-space .Let x and y be two distinct points of Y , then x and y be two distinct points of X . Since (X,τ) is an s*-T_0-space, then there exists an s*-open set U in (X,τ) containing x or y ,say x but not y .Now , by (3.12) $U \cap Y$ is an s*-open set in (Y,τ') containing x , but not y . Hence (Y,τ') is an s*-T_0-space.

(4.7)Definition:

A topological space (X,τ) is called **an s*-T_1-space** if for any two distinct points x and y of X there is an s*-open set of x which dose not contain y and an s*-open set of y which dose not contain x

(4.8)Theorem:

Every T_1-space is an s*-T_1-space.

Proof:

It is obvious .

(4.9)Remark:

The converse of (4.8) may not be true in general .Consider the following example:-

Example:

Let X={a,b} & τ={ϕ,X}.

Since $S^*O(X,\tau)$={ϕ,X,{a},{b}}.

Then (X,τ) is an s*-T_1-space,but not a T_1-space.

(4.10)Theorem:

Every s*-T_1-space is a g-T_1-space.

Proof:

It is obvious .

(4.11)Remark:

The converse of (4.10) may not be true in general .We observe that the space of (4.5) is a g-T_1-space ,but not an s*-T_1-space.

(4.12)Theorem:

Every s*-T_1-space is an s*-T_0-space.

Proof:

Let (X,τ) be an s*-T_1-space. To prove that (X,τ) is an s*-T_0-space. Let x and y be any two distinct points of (X,τ). Since (X,τ) is an s*-T_1-space, then there exists two s*-open sets U and V in (X,τ) such that $x \in U, y \notin U$ and $y \in V, x \notin V$. Thus (X,τ) is an s*-T_0-space.

(4.13)Remark:

The converse of (4.12) may not be true in general .We observe that the space of (4.3) is an s*-T_0-space,but not an s*-T_1-space.

(4.14)Theorem:

A topological space (X,τ) is an s*-T_1-space if every singleten is s*-closed.

Proof:

Suppose that every singleten is s*-closed in (X,τ). To verify that (X,τ) is an s*-T_1-space. Let x and y be any two distinct points of (X,τ). Put $U = X - \{x\}$ and $V = X - \{y\}$. Since $\{x\}$ and $\{y\}$ are s*-closed sets in (X,τ), then U and V are s*-open sets in (X,τ). Since $y \in U, x \notin U$ and $x \in V, y \notin V$. Thus (X,τ) is an s*-T_1-space.

(4.15)Remarks:

i)By (4.9),we observe that the points of an s*-T_1-space need not be closed.

ii) Not every finite s*-T_1-space is discrete. The space in example (4.9) is finite and s*-T_1-space ,but not discrete.

iii) T_0-space and s*-T_1-space are independent. The space of example (4.9) is an s*-T_1-space , but not a T_0-space.The following example show that T_0-space may not be an s*-T_1-space in general.

Example:

Let X={a,b,c} & τ={ ϕ ,X,{a},{a,b},{a,c}}. Since $S^*O(X,\tau)$={ ϕ ,X,{a},{a,b},{a,c}}.Then (X,τ) is a T_0-space ,but not an s*-T_1-space.

(4.16)Theorem:

s*-T_1 property is a closed-hereditary property

Proof:

Let (Y,τ') be a closed subspace of an s*-T_1-space (X,τ).To verify that (Y,τ') is an s*-T_1-space.Let x and y be two distinct points of Y ,then x and y be two distinct points of X . Since (X,τ) is an s*-T_1-space,then there exists two s*-open sets U and V in (X,τ) such that $x \in U, y \notin U$ and $y \in V$, $x \notin V$.
Now ,by (3.12) $U \cap Y$ and $V \cap Y$ are s*-open sets in (Y,τ') such that $x \in U \cap Y, y \notin U \cap Y$ and $y \in V \cap Y$, $x \notin V \cap Y$ .Thus (Y,τ') is an s*-T_1-space.

(4.17)Definition:

A topological space (X,τ) is called **an s*-T_2-space** if for any two distinct points x and y of X there are two s*-open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \phi$.

(4.18)Theorem:

Every T_2-space is an s*-T_2-space.

Proof:

It is obvious .

(4.19)Remark:

The converse of (4.18) may not be true in general.

Example:

Let X={a,b,c} & τ={ ϕ ,X,{a},{b,c}}. Since $S^*O(X,\tau)$={ ϕ , X, {a}, {b}, {c}, {a,b}, {a,c}, {b,c}}.
Then (X,τ) is an s*-T_2-space ,but not a T_2-space.

(4.20)Theorem:

Every s*-T_2-space is a g-T_2-space.

Proof:

It is obvious .

(4.21)Remark:

The converse of (4.20) may not be true in general .We observe that the space of (4.5) is a g-T_2-space ,but not an s*-T_2-space.

(4.22)Theorem:

Every s*-T_2-space is an s*-T_1-space.

Proof:

Let (X,τ) be an s*-T_2-space .To prove that (X,τ) is an s*-T_1-space. Let x and y be two distinct points of (X,τ). Since (X,τ) is an s*-T_2-space , then there exists two s*-open sets U and V in (X,τ) such that $x \in U$, $y \in V$ and $U \cap V = \phi$. Since $y \notin U$ and $x \notin V$. Thus (X,τ) is an s*-T_1-space.

(4.23)Remark:

The converse of (4.22) may not be true in general . Consider the following example:-

Example:

Let X be any infinte set and let $\tau = \{U \subseteq X : U^c \text{ is finite}\} \cup \{\phi\}$. Then (X,τ) is an s*-T_1-space ,but not an s*-T_2-space, Since in (X,τ) any two non-empty open sets and hence any two non-empty s*-open sets intersect.

(4.24)Remarks:

i) T_0-space and s*-T_2-space are independent. The space of example (4.15)(iii) is a T_0-space, but not an s*-T_2-space. While the space of example (4.9) is an s*-T_2-space,but not a T_0-space.

ii) T_1-space and s*-T_2-space are independent. The space of example (4.23) is a T_1-space ,but not an s*-T_2-space. While the space of example (4.9) is an s*-T_2-space ,but not a T_1-space.

(4.25)Theorem:

s*-T_2 property is a closed-hereditary property.

Proof:

Let (Y,τ') be a closed subspace of an s*-T_2-space (X,τ) .To verify that (Y,τ') is an s*-T_2-space.Let x and y be two distinct points of Y, then x and y be two distinct points of X . Since (X,τ) is an s*-T_2-space, then there exists two s*-open sets U and V of (X,τ) such that $x \in U$, $y \in V$ and $U \cap V = \phi$. Now by (3.12) $U \cap Y$ and $V \cap Y$ are s*-open sets in (Y,τ') such that $x \in U \cap Y$, $y \in V \cap Y$ and $(U \cap Y) \cap (V \cap Y) = (U \cap V) \cap Y = \phi \cap Y = \phi$

Thus (Y,τ') is an s*-T_2-space.

(4.26)Theorem:

s*-T_i property is a topological property (i=0,1,2).

Proof:

Let $f:(X,\tau) \to (Y,\sigma)$ be a homeomorphism and (X,τ) be an s*-T_2-space. To prove that (Y,σ) is an s*-T_2-space .Let $y_1, y_2 \in Y$ such that $y_1 \neq y_2$. Since f is onto , then $\exists\, x_1, x_2 \in X$ such that $f(x_1) = y_1$, $f(x_2) = y_2$ & $x_1 \neq x_2$ .Since (X,τ) is an s*-T_2-space, then there exists two s*-open sets U and V of (X,τ) such that $x_1 \in U$, $x_2 \in V$ and $U \cap V = \phi$. Since f is a homeomorphism ,then $f(U)$ and $f(V)$ are two s*-open sets in Y such that $y_1 = f(x_1) \in f(U)$ and $y_2 = f(x_2) \in f(V)$. Since f is one-to-one, then $f(U) \cap f(V) = f(U \cap V) = f(\phi) = \phi$.

Thus (Y,σ) is an s*-T_2-space.

By the same way we can prove the theorem when i=0,1.

(4.27)Definition:

A topological space (X,τ) is called **an s*-regular space** if for any closed subset F of X and any point x of X which is not in F, there are two s*-open sets U and V such that $x \in U$, $F \subseteq V$ and $U \cap V = \phi$.

(4.28)Definition:

An s*-regular T_1-space is called **an s*-T_3-space.**

(4.29)Theorem:

Every T_3-space is an s*-T_3-space.

Proof:

It is obvious .

(4.30)Theorem:

Every s*-T_3-space is an s*-T_2-space.

Proof:

Let (X,τ) be an s*- T_3-space .To prove that (X,τ) is an s*- T_2-space. Let x and y be two distinct points of (X,τ). Since (X,τ) is a T_1-space, then by (5) $\{y\}$ is closed in X and $x \notin \{y\}$. Since (X,τ) is s*-regular ,then there exists two s*-open sets U and V of (X,τ) such that $x \in U$, $\{y\} \subseteq V$ and $U \cap V = \phi$. Hence $x \in U$ and $y \in V$. Thus (X,τ) is an s*- T_2-space.

(4.31)Remark:

The converse of (4.30) may not be true in general.We observe that the space of (4.9) is an s*- T_2-space,but not an s*- T_3-space**.**

(4.32)Theorem:

Every s*- T_3-space is a g- T_2-space

Proof:

It is obvious .

(4.33)Remark:

The converse of (4.32) may not be true in general .We observe that the space of (4.5) is a g- T_2-space,but not an s*- T_3-space**.**

(4.34)Theorem:

A topological space (X,τ) is s*-regular if for any $x \in X$ and any open set U of x , there is an s*-open set V of x such that $x \in V \subseteq cl(V) \subseteq U$.

Proof:

Suppose that (X,τ) is s*-regular , U is open in X such that $x \in U$. Since U is open , then $X - U$ is closed in X and $x \notin X - U$, since (X,τ) is s*-regular, then there exists two s*-open sets V and W of (X,τ) such that $x \in V$, $X - U \subseteq W$ and $V \cap W = \phi$. Hence $X - W \subseteq U$, Since $X - W$ is s*-closed and U is s-open ,then $cl(X - W) \subseteq U$.

Since $V \cap W = \phi$, then $V \subseteq X - W$, hence $cl(V) \subseteq cl(X - W)$. Therefore $x \in V \subseteq cl(V) \subseteq cl(X - W) \subseteq U$ Thus $x \in V \subseteq cl(V) \subseteq U$.

Conversely, To prove that (X,τ) is s*-regular. Let $x \in X$ and F be a closed subset of X such that $x \notin F$. Then $x \in X - F$, hence by hypothesis there is an s*-open set V of x such that $x \in V \subseteq cl(V) \subseteq X - F$, Since $cl(V) \subseteq X - F$ then $F \subseteq X - cl(V)$ and $X - cl(V)$ is an s*-open set of X .Therefore $x \in V$, $F \subseteq X - cl(V)$ and $V \cap (X - cl(V)) = \phi$ .Thus (X,τ) is an s*-regular space.

(4.35)Remarks:

i) T_0-space and s*-regular space are independent (see (4.15) (iii) and (4.9)).

ii) T_1-space and s*-regular space are independent (see (4.23) and (4.9)).

iii) T_2-space and s*-regular space are independent. The space of example (4.9) is an s*-regular space, but not a T_2-space.

The following example show that T_2-space may not be an s*-regular in general .

Example:

Let $\Re$ be the set of all real numbers and k $=\{\frac{1}{n}$: n is a positive integer }.

Let τ be the topology on $\Re$ whose base consists of:

(a) Every open interval (a,b) such that $0 \notin (a,b)$.

(b) Every set (a,b)-k ,where (a,b) open interval containing 0 .

Then $(\Re, \tau)$ is a T_2-space, but not s*-regular.

(4.36)Theorem:

s*- T_3 property is a closed-hereditary property.

Proof:

Let (Y,τ') be a closed subspace of an s*- T_3-space (X,τ) .To verify that (Y,τ') is an s*- T_3-space. Since (X,τ) is an s*- T_3-space, then (X,τ) is an s*-regular T_1-space .Hence by (5) (Y,τ') is a T_1-space . To prove that (Y,τ') is s*-regular.

Let F be a closed subset of Y and $y \in Y$ such that $y \notin F$, then $F = F_1 \cap Y$, where F_1 is closed in X and $y \notin F_1$. Since (X,τ) is s*-regular, then there exists two s*-open sets U and V of (X,τ) such that $y \in U$, $F_1 \subseteq V$ and $U \cap V = \phi$.

Now , by (3.12) $U \bigcap Y$ and $V \bigcap Y$ are s*-open sets in (Y, τ') such that $y \in U \bigcap Y$, $F = F_1 \bigcap Y \subseteq V \bigcap Y$ and $(U \bigcap Y) \bigcap (V \bigcap Y) = (U \bigcap V) \bigcap Y = \phi \bigcap Y = \phi$. Therefore (Y, τ') is an s*-regular space . Thus (Y, τ') is an s*-T_3-space.

(4.37)Definition:

Let (X, τ) and (Y, σ) be two topological spaces. A function $f : (X, \tau) \to (Y, \sigma)$ is called **s*-continuous** if $f^{-1}(V)$ is an s*-open set of (X, τ) for each open subset V of Y.

(4.38)Theorem:

Every continuous function is an s*-continuous.

Proof:

It is obvious .

(4.39)Remark:

The converse of (4.38) may not be true in general .Consider the following example:-

Example:

Let X = Y= {a,b,c} , $\tau = \{\phi, X, \{a,b\}\}$ and $\sigma = \{\phi, Y, \{a\}, \{a,c\}\}$.

Define $f : (X, \tau) \to (Y, \sigma)$ by : f(a)=a, f(b)=c and f(c)=b .f is not continuous, since $\{a\}$ is an open set of (Y, σ), but $f^{-1}(\{a\}) = \{a\}$ is not open in (X, τ).However f is s*-continuous, because , $S^*O(X, \tau) = \{\phi, X, \{a\}, \{b\}, \{a,b\}\}$ and $f^{-1}(\phi) = \phi$, $f^{-1}(Y) = X$, $f^{-1}(\{a\}) = \{a\}$, $f^{-1}(\{a,c\}) = \{a,b\}$ which are s*-open sets in (X, τ).

(4.40)Definition:

A topological space (X, τ) is called **an s*-completely regular space** if for any closed subset F of X and any point x of X which is not in F , there is an s*-continuous function $f : (X, \tau) \to ([0,1], \mu')$ such that $f(x) = 0$ and $f(F) = 1$ (where [0,1] is a subspace of $\Re$ with relative usual topology μ').

(4.41)Definition:

An s*-completely regular T_1-space is called **an s*-$T_{3\frac{1}{2}}$-space.**

(4.42)Theorem:

Every s*-$T_{3\frac{1}{2}}$-space is an s*-T_3-space.

Proof:

Let (X, τ) be an s*-$T_{3\frac{1}{2}}$-space.To prove that (X, τ) is an s*-T_3-space. Since (X, τ) is an s*-$T_{3\frac{1}{2}}$-space, then (X, τ) is an s*-completely regular T_1-space. It is enough to prove that (X, τ) is an s*-regular space. Let F be a closed subset of X and $x \in X$ such that $x \notin F$. Since (X, τ) is s*-completely regular, then there is an s*-continuous function $f : (X, \tau) \to ([0,1], \mu')$ such that $f(x) = 0$ and $f(F) = 1$. Since $[0,1] \subseteq \Re$ and $\Re$ with usual topology is a T_2-space then so is $([0,1], \mu')$.

Since $0,1 \in [0,1]$ and $0 \neq 1$,then there are two open sets U and V in $([0,1], \mu')$ such that $0 \in U$, $1 \in V$ and $U \bigcap V = \phi$. Since f is s*-continuous function , then $f^{-1}(U)$ and $f^{-1}(V)$ are s*-open sets in (X, τ) .

$\because\ 0 \in U \Rightarrow f^{-1}(0) \in f^{-1}(U) \Rightarrow x \in f^{-1}(U)$.

$\because\ 1 \in V \Rightarrow f^{-1}(1) \in f^{-1}(V) \Rightarrow F \subseteq f^{-1}(V)$.

Since $f^{-1}(U) \bigcap f^{-1}(V) = f^{-1}(U \bigcap V) = f^{-1}(\phi) = \phi$.

Thus (X, τ) is an s*-T_3-space.

(4.43)Theorem:

S*-$T_{3\frac{1}{2}}$ property is a closed-hereditary property.

Proof:

Let (Y, τ') be a closed subspace of an s*-$T_{3\frac{1}{2}}$-space (X, τ). To verify that (Y, τ') is an s*-$T_{3\frac{1}{2}}$-space. Since (X, τ) is an s*-$T_{3\frac{1}{2}}$-space, then (X,τ) is an s*-completely regular T_1-space .

Hence by (5) (Y,τ') is a T_1-space. To prove that (Y,τ') is an s*-completely regular Let F be a closed subset of Y and $y \in Y$ such that $y \notin F$, then $F = F_1 \cap Y$, where F_1 is closed in X and $y \notin F_1$. Since (X,τ) is s*-completely regular, then there is an s*-continuous function $f:(X,\tau) \to ([0,1],\mu')$ such that $f(y)=0$ and $f(F_1)=1$. Let $g = f/Y:(Y,\tau') \to ([0,1],\mu')$ be a function defined by :- g(x) = f(x), $\forall\ x \in Y$.

Since f is s*-continuous and Y be a closed subspace of X ,then g is also an s*-continuous .Since g(y) = f(y)=0 and g(y) = f(y) =1, $\forall\ y \in F$, then g(y)=0 and g(F)=1. Hence (Y,τ') is an s*-completely regular. Thus (Y,τ') is an s*-$T_{3\frac{1}{2}}$-space.

(4.44)Theorem:

i) $s^*\text{-}T_3$ property is a topological property.

ii) $s^*\text{-}T_{3\frac{1}{2}}$ property is a topological property.

Proof:

ii) Let $f:(X,\tau) \to (Y,\sigma)$ be a homeomorphism and (X,τ) be an s*-$T_{3\frac{1}{2}}$-space.To prove that (Y,σ) is an s*-$T_{3\frac{1}{2}}$-space. Since (X,τ) is s*-$T_{3\frac{1}{2}}$-space, then (X,τ) is an s*-completely regular T_1-space. Hence by (5) (Y,σ) is a T_1-space .To prove that (Y,σ) is an s*-completely regular . Let F be a closed subset of Y and $y \in Y$ such that $y \notin F$. Since f is onto , then there is $x \in X$ such that f(x) = y. Since f is continuous, then $f^{-1}(F)$ is closed in X and $x \notin f^{-1}(F) = F_1$. Since (X,τ) is s*-completely regular, then there is an s*-continuous function $h:(X,\tau) \to ([0,1],\mu')$ such that $h(x)=0$ and $h(F_1)=1$ Since $f^{-1}:(Y,\sigma) \to (X,\tau)$ is homeomorphism and $h:(X,\tau) \to ([0,1].\mu')$ is s*-continuous ,then $h \circ f^{-1}:(Y,\sigma) \to ([0,1],\mu')$ is s*-continuous . Since $h \circ f^{-1}(y) = h(f^{-1}(y)) = h(x) = 0$ and $h \circ f^{-1}(F) = h(f^{-1}(F)) = h(F_1) = 1$ Hence (Y,σ) is an s*-completely regular space .Thus (Y,σ) is an s*-$T_{3\frac{1}{2}}$-space. By the same way we can prove that (i).

References

1. Levine,N .**1963**. Semi-open sets and semi-Continuity in topological spaces. *Amer. Math.Monthly*,70:36-41.
2. Levine ,N .**1970**. Generalized closed Set in topology .*Rend. Circ. Math. Palermo*,**19**(2). :89-96.
3. Al-Meklafi, S. **2002**. *On new types of separation axioms*. M. SC. Thesis .Department of Mathematics, college of Education, AL-Mustansiriya University, pp.34-58.
4. Maheshwari , S. N. and Prasad,R .**1975**. Some new separation axioms. *annales .soc. Scien Bruxelles*, **89:**395-402.
5. Willard,S . **1970**.*General Topology*. Addison-wesley Inc.,Mass,pp. 86.