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Invertible Operators on Soft Normed Spaces

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Abstract
Despite ample research on soft linear spaces, there are many other concepts that can be studied. We introduced in this paper several new concepts related to the soft operators, such as the invertible operator. We investigated some properties of this kind of operators and defined the spectrum of soft linear operator along with a number of concepts related with this definition; the concepts of eigenvalue, eigenvector, eigenspace are defined. Finally the spectrum of the soft linear operator was divided into three disjoint parts.

Keywords: Soft linear operator, invertible soft operator, spectrum of soft operator, eigenvalue of soft operator, eigenvector of soft operator.

التحويلات القابلة للعكس على الفضاءات المرنة الخطية

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الخلاصة
على الرغم من البحوث الكثيرة في موضوع الفضاء المرن الخطي , هناك الكثير من المفاهيم تحتاج الى الدراسة . قدمنا في هذ البحث بعض المفاهيم الجديدة المتعلقة بالفضاء المرن مثل تعريف التحويل القابل للعكس ودراسة بعض الخواص لهذا النوع من التحويلات وتعريف طيف تحويل خطي مرن وبعض المفاهيم المتعلقة بهذا التعريف . عرفنا مفاهيم القيم الذاتية والمتجهات الذاتية والفضاء الذاتي واخيرا قسمنا طيف التحويل المرن الخطي الى ثلاثة اجزاء منفصلة .

1. INTRODUCTION

In 1999, Molodtsov [1] started the concept of soft sets as a new mathematical instrument for dealing with uncertainties. He introduced some presentations of this theory for solving several real-world problems in engineering, economy, medical science, community science, etc. Few years later, Maji et al. [2] introduced a number of operations on soft sets to solve decision making problems. Feng et al. [3] described some new operations on soft sets. On the reduction line and addition of parameters of the soft sets, some work was done by Chen [4]. Aktas and Cagman [5] introduced the notion of soft group and discussed various properties. Feng et al. [6] worked on soft ideals, soft semiring and idealistic soft semiring. Shabir and Naz [7] introduced the idea of soft topological spaces. Mappings between soft sets were described by Majumdar and Samanta [8]. Feng et al. [3] worked on soft sets together with fuzzy sets and rough sets. Das and Samanta [8] introduced the notions of soft real sets, soft real numbers, soft complex sets, soft complex numbers and investigated some of their basic properties. They presented some applications of soft real sets and soft real numbers in real life problems. Later, they introduced the concepts of soft metric over an absolute soft set and soft norm, as well as soft inner product over soft linear spaces. Many properties of soft metric spaces, soft linear

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spaces, soft normed linear spaces and soft inner product spaces were investigated with examples and counter examples.

2. PRELIMINARIES

The basic definitions and theorems were introduced in this section that may also be found in earlier studies.

Definition 2.1 [1] Suppose X is a universe set and E is a set of parameters. Let $\wp$(X) symbolizes the power set of X and A $\neq \emptyset$ be a subset of E. A pair (H, A) is named a soft set over X, where H is a mapping given by H: A $\rightarrow \wp$(X). In other words, a soft set over X is a parameterized family of subsets of the universe X. For $w \in$ A, H (w) can be thought as the set of w- approximate elements of the soft set (H, A).

Definition 2.2 [3] For two soft sets (H, A) and (G, D) over a shared universe X, then (H, A) is a soft subset of (G, D) if:

(1) A $\subseteq$ D.

(2) For all e $\in$ A, H (e) $\subseteq$ G (e). We write (H, A) $\tilde{\subseteq}$ (G, D).

(G, D) is said to be a soft superset of (H, A). We write (H, A) $\tilde{\subseteq}$((G, D) if (H, A) is a soft subset of (G, D).

Definition 2.3 [3] Two soft sets (H, A) and (G, D) over a shared universe X are said to be identical, if (H, A) is a soft subset of (G, D) and (G, D) is a soft subset of (H, A).

Definition 2.4 [2] The union of two soft sets (H, A) and (G, D) over the shared universe X is the soft set

(J, C), where C = A $\cup$ D and for all e $\in$ C,

$$J(e) = \begin{cases} H(e) & if\ e \in A - D \\ G(e) & if\ e \in D - A \\ H(e) \cup G(e) & \text{if } e \in A \cap D \end{cases}$$

We express it as (H, A) $\tilde{\cup}$(G, D) = (J, C).

Definition 2.5 [6] The intersection of two soft sets (H , A)and (G , D) over the shared universe X is the soft set (L , C), where C = A $\cap$ D and for all e $\in$C, L (e) = H(e) $\cap$ G(e). We write (H, A)$\tilde{\cap}$(G , D)= (L , C).

Suppose that X is an initial universal set and A is the non-empty set of parameters. In the above definitions, the set of parameters may differ from a soft set to another, but in our considerations through this paper, all soft sets have the same set of parameters A. The above definitions are also useable for these types of soft sets as a particular case of those definitions.

Definition 2.6 [8] The complement of a soft set (F, A) is denoted by $(F, A)^c$= (F^c, A), where F^c: A$\rightarrow \wp$(X) is a mapping given by $F^c(\lambda$) = X $-$ F (λ), for all $\lambda \in$ A.

Definition 2.7 [2] A soft set (F, A) over X is said to be an absolute soft set symbolized by $\tilde{X}$ if F (λ) = X for every $\lambda \in$ A.

Definition 2.8 [2] A soft set (F, A) over X is said to be a null soft set symbolized by $\tilde{\Phi}$ if for every $\lambda \in$ A, F(λ) = ϕ.

Definition 2.9 [7] The difference (H , A) of two soft sets (F , A) and (G , A) over X, denoted by (F , A)\(G,A), is defined by H (λ) = F(λ)\G(λ) for all $\lambda \in$ A.

Proposition 2.10 [7] Let (F, A) and (G, A) be two soft subsets of $\tilde{X}$. Then:

(i) $[(F, A)\ \tilde{\cup}\ ((G, A)]^c = (F, A)^c\ \tilde{\cap}\ (G, A)^c$.

(ii) $[(F, A)\ \tilde{\cap}\ ((G, A)]^c = (F, A)^c\ \tilde{\cup}\ (G, A)^c$.

Definition 2.11 [9] Let X be a non-empty set of elements and A$\neq \emptyset$ is a set of parameters. Then, a function ε: A $\rightarrow$X is called a soft element of X. A soft element ε of X belongs to a soft set B of X, which is symbolized by $\varepsilon\ \tilde{\in}$ B, if ε (λ) $\in$ B(λ) for every $\lambda \in$A. Thus, for a soft set B of X (with respect to the index set A) we have B (λ) = {$\varepsilon(\lambda)$, $\varepsilon\ \tilde{\in}$B}, $\lambda \in$ A.

It is to be well-known that each singleton soft set (a soft set (H, A) for which H (λ) is a singleton set, for every $\lambda \in$ A), can be recognized with a soft element by just identifying the singleton set with the element that it contains for all $\lambda \in$ A.

Definition 2.12 [10] Let $\mathfrak{B}$ (R) be the collection of all non-empty bounded subsets of R (R is real number) and A booked as a parameters set. Then, a mapping H: A $\rightarrow \mathfrak{B}$ (R) is named a soft real set. It is symbolized by (H, A). If specifically (H, A) is a singleton soft set, then when identifying (H, A) with the corresponding soft element, it will be named a soft real number.

The collection of each soft real numbers is symbolized by R(A) and the collection of all non-negative soft real numbers is symbolized by $R(A)^*$.

Definition 2.13 [11] Let $\wp$ ($\mathbb{C}$) be the set of all non-empty bounded subsets of the set of complex numbers $\mathbb{C}$, and let A be a set of parameters. Then, a mapping H: A → $\wp$ ($\mathbb{C}$) is named a soft complex set symbolized by (H, A). If, in particular, (H, A) is a singleton soft set, then by identifying (H, A) with the agreeing soft element, it will be named a soft complex number.

The set of all soft complex numbers is denoted by $\mathbb{C}$(A).

Definition 2.14 [11] Let (H, A) be a soft complex set. The complex conjugate of (H, A) is symbolized by ($\overline{H}$, A) and is defined by $\overline{H}(\lambda)$ = {$\bar{z}$: z ∈ H (λ)}, for every λ ∈ A, where $\bar{z}$ is a complex conjugate of the ordinary complex number z. The complex conjugate of a soft complex number (H, A) is $\overline{H}(\lambda)$ = $\bar{z}$: z = H(λ), for every λ ∈ A.

Definition 2.15 [11] Let (F, A), (G, A) $\tilde{\in}$ $\mathbb{C}$(A). Then, the sum, difference, product and division are defined by

(F + G) (λ) = z + w, z ∈ F (λ), w ∈ G (λ), for all λ ∈ A.

(F − G) (λ) = z − w; z ∈ F (λ), w ∈ G (λ), for all λ ∈ A.

(FG) (λ) = zw, z ∈ F (λ), w ∈ G (λ), for all λ ∈ A.

(F/G) (λ) = z/w, z ∈ F (λ), w ∈ G (λ), provided G (λ) $\neq 0$, for all λ ∈ A.

Definition 2.16 [11] Let (F, A) be a soft complex number. Then, the modulus of (F, A) is denoted by ($|F|$, A) and is defined by $|F|(\lambda)$ =$|z|$; z ∈ F (λ), for all λ ∈ A, where z is an ordinary complex number.

Since the modulus of each ordinary complex number and ordinary real number are a non-negative real number, and by definition of soft real numbers, it follows that ($|F|$, A) is a non-negative soft real number for every soft complex number (F, A) or soft real number (F , A) .

Let X be a non-empty set and $\tilde{X}$ be the absolute soft set, i.e. V (λ) = X, for each λ ∈ A, where (V, A) = $\tilde{X}$. Suppose S ($\tilde{X}$) be the collection of all soft sets (H, A) over X for which H (λ) $\neq \phi$, for all λ ∈ A, together with the null soft set $\tilde{\Phi}$. Let (H, A) ($\neq \Phi$) ∈ S($\tilde{X}$), then the collection of all soft elements of (H , A) will be denoted by SE (H , A). For a collection $\mathfrak{B}$ of soft elements of $\tilde{X}$, the soft set generated by $\mathfrak{B}$ is denoted by SS($\mathfrak{B}$).

Definition 2.17 [12] A mapping d: SE($\tilde{X}$) × SE($\tilde{X}$) → $R(A)^*$ is called a soft metric on the soft set $\tilde{X}$ if d satisfies the following conditions:

(1) d ($\tilde{x}$; $\tilde{y}$) $\tilde{\geq}$ $\bar{0}$, for all $\tilde{x}$, $\tilde{y}$ $\tilde{\in}$ $\tilde{X}$.

(2) d ($\tilde{x}$, $\tilde{y}$) = $\bar{0}$, if and only if $\tilde{x}$ = $\tilde{y}$.

(3) d ($\tilde{x}$, $\tilde{y}$) = d ($\tilde{y}$, $\tilde{x}$) for all $\tilde{x}$, $\tilde{y}$ $\tilde{\in}$ $\tilde{X}$.

(4) For all $\tilde{x}, \tilde{y}, \tilde{z}$ $\tilde{\in}$ $\tilde{X}$, d ($\tilde{x}$, $\tilde{z}$) $\tilde{\leq}$ d ($\tilde{x}$, $\tilde{y}$) + d ($\tilde{y}$, $\tilde{z}$).

The soft set $\tilde{X}$ with a soft metric d on $\tilde{X}$ is said to be a soft metric space and is denoted by ($\tilde{X}$, d, A) or ($\tilde{X}$, d).

Definition 2.18 [13] Let V be a vector space over K (in which K is a field) and A is a set of parameters. Let G be a soft set over (V, A). If for all λ ∈ A, G (λ) is a vector subspace of V, then G is called a soft vector space of V over K.

Definition 2.19 [14] Suppose that H is a soft vector space of V over K. Let G: A → $\wp$ (V) be a soft set over (V, A). If for each λ ∈ A, G (λ) is a vector subspace of V over K and H (λ) ⊇ G (λ), then G is called a soft vector subspace of H.

Definition 2.20 [13] Suppose that G is a soft vector space of V over K, then a soft element of G is called a soft vector of G. In a similar manner, a soft element of the soft set (K, A) is said to be a soft scalar, with K being the scalar field.

Definition 2.21 [13] Let $\tilde{x}$, $\tilde{y}$ be soft vectors of G and $\tilde{k}$ be a soft scalar. The addition $\tilde{x}$ + $\tilde{y}$ of $\tilde{x}$, $\tilde{y}$ and the scalar multiplication $\tilde{k}.\tilde{x}$ of $\tilde{k}$ and $\tilde{x}$ are defined by ($\tilde{x}$ + $\tilde{y}$) (λ) = $\tilde{x}$ (λ) + $\tilde{y}$ (λ) , $\tilde{k}.\tilde{x}$ (λ) = $\tilde{k}$ (λ) . $\tilde{x}$ (λ) for all λ ∈ A. Obviously, $\tilde{x}$ + $\tilde{y}$, $\tilde{k}$. $\tilde{x}$ are soft vectors of G.

Definition 2.22 [15] Let $\tilde{X}$ be the absolute soft vector space, i.e. $\tilde{X}$ (λ) = X, for all λ ∈ A. Then a mapping ‖ . ‖ : SE($\tilde{X}$) → $R(A)^*$is said to be a soft norm on the soft vector space $\tilde{X}$ if ‖ . ‖ satisfies the following situations:

(1). ‖ . ‖ $\tilde{\geq}$ $\bar{0}$ for every $\tilde{x}$ $\tilde{\in}$ $\tilde{X}$.

(2). ‖$\tilde{x}$‖ = $\bar{0}$ if and only if $\tilde{x}$ =Θ.

(3). $\|\tilde{\alpha}\,.\,\tilde{x}\| = |\tilde{\alpha}|\ \|\tilde{x}\|$ for each $\tilde{x} \tilde{\in} \tilde{X}$ as well as for each soft scalar $\tilde{\alpha}$.
(4). For each $\tilde{x}, \tilde{y} \tilde{\in} \tilde{X}$, $\|\tilde{x} + \tilde{y}\| \tilde{\leq} \|\tilde{x}\| + \|\tilde{y}\|$

The soft vector space $\tilde{X}$ with a soft norm $\|\,.\,\|$ on $\tilde{X}$ is said to be a soft normed linear space and is symbolized by $(\tilde{X}, \|\,.\,\|, A)$ or $(\tilde{X}, \|\,.\,\|)$. The exceeding conditions are called soft norm axioms.

Theorem 2.23 [13] Suppose that a soft norm $\|\,.\,\|$ achieves the situation (N5). For $\xi \in X$ and $\lambda \in A$, the set $\{\|\tilde{x}\|(\lambda) : \tilde{x}(\lambda) = \xi\}$ is a one element set. Then for each $\lambda \in A$, the mapping $\|\,.\,\|_{\lambda} : X \to R^{+}$, defined by $\|\xi\|_{\lambda} = \|\tilde{x}\|(\lambda)$, for all $\xi \in X$ and $\tilde{x} \tilde{\in} \tilde{X}$ such that $\tilde{x}(\lambda) = \xi$, can be considered as a norm on X.

Definition 2.24 [14] Consider $(\tilde{X}, \|\,.\,\|, A)$ as a soft normed linear space, $\tilde{r} \tilde{\geq} \bar{0}$ is a soft real number. We define the followings:

$B(\tilde{x}, \tilde{r}) = \{ \tilde{y} \tilde{\in} \tilde{X} : \|\tilde{x} - \tilde{y}\| \tilde{<} \tilde{r} \} \subset SE(\tilde{X})$,

$\bar{B}(\tilde{x}, \tilde{r}) = \{ \tilde{y} \tilde{\in} \tilde{X} : \|\tilde{x} - \tilde{y}\| \tilde{\leq} \tilde{r} \} \subset SE(\tilde{X})$,

$S(\tilde{x}, \tilde{r}) = \{ \tilde{y} \tilde{\in} \tilde{X} : \|\tilde{x} - \tilde{y}\| = \tilde{r} \} \subset SE(\tilde{X})$,

$B(\tilde{x}, \tilde{r})$, $\bar{B}(\tilde{x}, \tilde{r})$, $S(\tilde{x}, \tilde{r})$ are respectively called an open ball, a closed ball and a sphere with a center at $\tilde{x}$ and a radius $\tilde{r}$. $SS(B(\tilde{x}, \tilde{r}))$, $SS(\bar{B}(\tilde{x}, \tilde{r}))$ and $SS(S(\tilde{x}, \tilde{r}))$ are respectively called a soft open ball, a soft closed ball and a soft sphere with a center at $\tilde{x}$ and a radius $\tilde{r}$.

Definition 2.25 [13] A sequence of soft elements $\{\widetilde{x_n}\}$ in a soft normed space $(\tilde{X}, \|\,.\,\|, A)$ is called convergent sequence, if $\|\widetilde{x_n} - \tilde{x}\| \to \bar{0}$ as $n \to \infty$, we say that the sequence converges to a soft element $\tilde{x}$. In other words, for each $\tilde{\epsilon} \tilde{\geq} \bar{0}$, there exists $N \in \mathbb{N}$, $N = N(\tilde{\epsilon})$ and $\bar{0} \tilde{\leq} \|\widetilde{x_n} - \tilde{x}\| \tilde{\leq} \tilde{\epsilon}$ whenever $n > N$.
i.e., $n > N$ implies $\widetilde{x_n} \in B(\tilde{x}, \tilde{\epsilon})$. We symbolize this by $\widetilde{x_n} \to \tilde{x}$ as $n \to \infty$ or by $\lim_{n\to\infty} \widetilde{x_n} = \tilde{x}$. The soft element $\tilde{x}$ is said to be the limit of the sequence $\widetilde{x_n}$ as $n \to \infty$.

Definition 2.26 [13] A sequence $\{\widetilde{x_n}\}$ of soft elements in a soft normed space $(\tilde{X}, \|\,.\,\|, A)$ is said to be a Cauchy sequence in $\tilde{X}$, if corresponding to each $\tilde{\epsilon} \tilde{>} \bar{0}$, there exists $m > N$ such that $\|\tilde{x_\iota} - \tilde{x_\jmath}\| \tilde{\leq} \tilde{\epsilon}$, for all $i, j \geq m$, i.e., $\|\tilde{x_\iota} - \tilde{x_\jmath}\| \to \bar{0}$ as $i; j \to \infty$.

Definition 2.27 [13] Let $(\tilde{X}, \|\,.\,\|, A)$ be a soft normed space. Then, $\tilde{X}$ is said to be complete if every Cauchy sequence in $\tilde{X}$ converges to a soft element of $\tilde{X}$. The complete soft normed space is said to be a soft Banach Space.

Theorem 2.28 [13] Every Cauchy sequence in R(A), where A is a finite set of parameters, is convergent, i.e. the set of all soft real numbers together with its usual modulus soft norm, with respect to finite set of parameters, is a soft Banach space.

Definition 2.29[14] A series $\sum_{k=1}^{\infty} \widetilde{x_k}$ of soft elements is called soft convergent, if the partial sum of the series $\widetilde{S_n} = \sum_{k=1}^{n} \widetilde{x_k}$ is soft convergent.

Let $\tilde{X}$, $\tilde{Y}$ be the corresponding absolute soft normed spaces, i.e., $\tilde{X}(\lambda) = X$, $\tilde{Y}(\lambda) = Y$, for all $\lambda \in A$. We use the notation $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ to represent soft vectors of a soft vector space.

Definition 2.30[13] Suppose that $T: SE(\tilde{X}) \to SE(\tilde{Y})$ is an operator. T is called soft linear, if
(L1). T is additive, i.e., $T(\tilde{x_1} + \tilde{x_2}) = T(\tilde{x_1}) + T(\tilde{x_2})$ for all soft elements $\tilde{x_1}, \tilde{x_2} \tilde{\epsilon} \tilde{X}$.
(L2). T is homogeneous, i.e., for all soft scalars $\tilde{k}$, $T(\tilde{k}.\,\tilde{x}) = \tilde{k}\,T(\tilde{x})$, for all soft elements $\tilde{x} \tilde{\epsilon} \tilde{X}$.
The properties (L1) and (L2) can be put in a combined form $T(\widetilde{k_1}.\,\tilde{x_1} + \widetilde{k_2}.\,\tilde{x_2}) = \widetilde{k_1}\,T(\tilde{x_1}) + \widetilde{k_2}\,T(\tilde{x_2})$ for every soft element $\tilde{x_1}, \tilde{x_2} \tilde{\epsilon} \tilde{X}$ and every soft scalar $\widetilde{k_1}$, $\widetilde{k_2}$.

Definition 2.31[13] The operator $T : SE(\tilde{X}) \to SE(\tilde{Y})$ is said to be continuous at $\widetilde{x_0} \tilde{\epsilon} \tilde{X}$, if for every sequence $\{\widetilde{x_n}\}$ of soft elements of $\tilde{X}$ with $\widetilde{x_n} \to \widetilde{x_0}$ as $n \to \infty$, we have $T(\widetilde{x_n}) \to T(\widetilde{x_0})$ as $n \to \infty$, i.e., $\|\widetilde{x_n} - \widetilde{x_0}\| \to \bar{0}$ as $n \to \infty$ implies $\|T(\widetilde{x_n}) - T(\widetilde{x_0})\| \to \bar{0}$ as $n \to \infty$. If T is continuous at every soft element of $\tilde{X}$, then T is called a continuous operator.

Theorem 2.32[13] Let $T: SE(\tilde{X}) \to SE(\tilde{Y})$ be a soft linear operator, where $\tilde{X}, \tilde{Y}$ are soft normed linear spaces. If T is continuous at some soft element $\widetilde{x_0} \tilde{\epsilon} \tilde{X}$, then T is continuous at every soft element of $\tilde{X}$.

Definition 2.33[13] Let $T: SE(\tilde{X}) \to SE(\tilde{Y})$ be a soft linear operator, where $\tilde{X}, \tilde{Y}$ are soft normed linear spaces. The operator T is said to be bounded if there exists some positive soft real number $\tilde{M}$ such that for each $\tilde{x} \tilde{\epsilon} \tilde{X}$, $\|T(\tilde{x})\| \tilde{\leq} \tilde{M}\,\|\tilde{x}\|$.

Theorem 2.34[13] Let $T: SE(\tilde{X}) \to SE(\tilde{Y})$ be a soft linear operator, where $\tilde{X}$, $\tilde{Y}$ are soft normed linear spaces. If T is bounded, then T is continuous.

Theorem 2.35[13] **(Decomposition Theorem)** Suppose a soft linear operator T: SE($\tilde{X}$) → SE($\tilde{Y}$), where $\tilde{X}$, $\tilde{Y}$ are soft normed spaces, which fulfills the situation (L3). For $\xi \in$ X, and $\lambda \in$ A the set {T ($\tilde{x}$) (λ): $\tilde{x} \tilde{\epsilon} \tilde{X}$ such that $\tilde{x}$ (λ) = ξ} is a singleton set. Then for each $\lambda \in$ A, the mapping T_λ : X → Y defined by T_λ (ξ) = T ($\tilde{x}$) (λ), for all $\xi \in$ X and $\tilde{x} \tilde{\epsilon} \tilde{X}$ such that $\tilde{x}$ (λ) = ξ, is a linear operator.
Theorem 2.36[13] Let T_λ : X ⟶Y, $\lambda \in$ A be a family of crisp linear operators from the vector space X to the vector space Y, and $\tilde{X}$, $\tilde{Y}$ be the corresponding absolute soft vector spaces. Then, there exists a soft linear operator T : SE($\tilde{X}$) → SE($\tilde{Y}$) defined by T ($\tilde{x}$) (λ) = $T_\lambda(\xi)$ if $\tilde{x}$ (λ) = ξ ¸ $\lambda \in$ A. which satisfies (L3) and T (λ) = T_λ for all $\lambda \in$ A.
Theorem 2.37[13] Let $\tilde{X}$ and $\tilde{Y}$ be soft normed linear spaces which satisfy (N5) and T: SE($\tilde{X}$) → SE($\tilde{X}$) be a soft linear operator satisfying (L3). If T is continuous, then T is bounded.
Theorem 2.38[13] Let $\tilde{X}$ and $\tilde{Y}$ be soft normed linear spaces which satisfy (N5) and T: SE($\tilde{X}$) → SE($\tilde{Y}$) be a soft linear operator satisfying (L3). If $\tilde{X}$ is of finite dimension, then T is bounded and hence continuous.
Definition 2.39[13] Let T be a bounded soft linear operator from SE($\tilde{X}$) into SE($\tilde{Y}$). Then, the norm of the operator T denoted by ‖T‖, is a soft real number defined as the following:
For each λ ∈ A, $\|T\|(\lambda) = \inf\{t \in R; \|T(\tilde{x})\| \le t.\|\tilde{x}\|(\lambda)$, for each $\tilde{x} \tilde{\in} \tilde{X}\}$.
Theorem 2.40[13] Let $\tilde{X}, \tilde{Y}$ be soft normed linear spaces which satisfy (N5) and T satisfies (L3). Then for each λ ∈ A, $\|T\|(\lambda) = \|T_\lambda\|_\lambda, where\ \|T_\lambda\|_\lambda$ is the norm of the linear operator T_λ: X → Y.
Theorem 2.41[13] $\|T(\tilde{x})\| \tilde{\leqq} \|T\|\|\tilde{x}\|$, for all $\tilde{x} \tilde{\in} \tilde{X}$.
Theorem 2.42[13] Let $\tilde{X}$and $\tilde{Y}$ be soft normed linear spaces which satisfy (N5) and T: SE($\tilde{X}$) → SE($\tilde{Y}$) be a soft linear operator satisfying (L3). Then:
(i) $\|T\|(\lambda) = \sup\{\|T(\tilde{x})\|(\lambda): \|\tilde{x}\| \tilde{\leqq} \bar{1}\} = \|T_\lambda\|_\lambda, for\ each$ λ ∈ A.
(ii) $\|T\|(\lambda) = \sup\{\|T(\tilde{x})\|(\lambda): \|\tilde{x}\| = \bar{1}\} = \|T_\lambda\|_\lambda, for\ each$ λ ∈ A.
(iii) $\|T\|(\lambda) = \sup\left\{\frac{\|T(\tilde{x})\|}{\|\tilde{x}\|}(\lambda): \|\tilde{x}\|(\mu) \neq 0, for\ all\ \mu \in A\right\} = \|T_\lambda\|_\lambda$, for each λ ∈ A.

Theorem 2.43[13] Let $\tilde{X}$ and $\tilde{Y}$ be soft normed linear spaces which satisfy (N5). Let T: SE($\tilde{X}$) → SE($\tilde{Y}$) be a continuous soft linear operator satisfying (L3).Then, T_λ is continuous on X for each λ ∈ A.
Theorem 2.44[14] Let $\tilde{X}$and $\tilde{Y}$ be soft normed linear spaces which satisfy (N5). Let { T_λ; λ ∈ A} be a family of continuous linear operators such that T_λ: X → Y for each λ. Then, the soft linear operator T: SE($\tilde{X}$) → SE($\tilde{Y}$) defined by $(T(\tilde{x}))(\lambda) = T_\lambda(\tilde{x}(\lambda))$, for all λ ∈ A, is a continuous soft linear operator satisfying (L3).
Definition 2.45[14] **(Soft linear space of operators)** Let $\tilde{X}, \tilde{Y}$ be soft normed linear spaces satisfying (N5). Consider the set W of all continuous soft linear operators S, T,.. etc. which satisfy (L3) each mapping SE($\tilde{X}$) into SE($\tilde{Y}$), then using Theorem 2.43, it follows that for each λ ∈ A; S_λ,T_λ, …. etc. are continuous linear
operators from X to Y. Let $W(\lambda) = \{T_\lambda(= T(\lambda)); T \in W\}$, for all λ ∈ A. Also using Theorem 2.43 and Theorem 2.44, it follows that $W(\lambda)$ is the collection of all continuous linear operators from X to Y. By the property of crisp linear operators, it follows that $W(\lambda)$ forms a vector space for each λ ∈ A with respect to the usual operations
of addition and scalar multiplication of linear operators. It also follows that $W(\lambda)$ is identical with the set of all continuous linear operators from X to Y for all λ ∈ A. Thus, the absolute soft set generated by $W(\lambda)$ forms an absolute soft vector space. Hence, W can be interpreted to form an absolute soft vector space. We shall denote this absolute soft linear (vector) space by L ($\tilde{X}, \tilde{Y}$).
Proposition 2.46[14] Each element of $SE(L(\tilde{X}, \tilde{Y}))$ can be identified uniquely with a member of W (i.e., to a continuous soft linear operator T: SE($\tilde{X}$) → SE($\tilde{Y}$)).
Theorem 2.47[14] Let L ($\tilde{X}, \tilde{Y}$) be a soft normed linear space, then for $\hat{f} \in SE(L(\tilde{X}, \tilde{Y}))$, we can identify $\hat{f}$ to be a unique T ∈ W and $\|\hat{f}\|$ is defined by $\|\hat{f}\|(\lambda) = \|T\|(\lambda) = \sup\{\|T(\tilde{x})\|(\lambda): \|\tilde{x}\| \tilde{\leqq} \bar{1}\}$, for each λ ∈ A.
Definition 2.48[13] Suppose that T: SE($\tilde{X}$) ⟶ SE($\tilde{Y}$) is a soft linear operator where $\tilde{X}, \tilde{Y}$ are soft normed spaces. Then, T is called injective or one-to-one if $T(\tilde{x_1})(\lambda) = T(\tilde{x_2})(\lambda)$ implies $(\tilde{x_1})(\lambda) =$

$(\widetilde{x_2})(\lambda)$ $\forall \lambda \in$ A. It is called surjective or onto, if Rang(T) = SE($\tilde{Y}$). The operator T is said to be bijective, if T is both one-to-one and onto.

3. INVERTIBLE LINEAR OPERATOR

Let X,Y *be* two vector spaces over a same field K, $A \neq \emptyset$ is a set of parameters, and $\tilde{X}$, $\tilde{Y}$ be the corresponding absolute soft normed spaces, i.e., $\tilde{X}(\lambda) = X$, $\tilde{Y}(\lambda) = Y$ for all $\lambda \in A$. Let $\tilde{x}, \tilde{y}, \tilde{z}$ be soft vectors of a soft normed space.

Theorem 3.1: Let *T:* $SE(\tilde{X}) \longrightarrow SE(\tilde{Y})$ be a linear operator satisfying (L_3). Then T_λ is one-to-one and onto if and only if T is one to one and onto.

Proof: Suppose that T is one to one, then T $(\widetilde{x_1})(\lambda)$ = T$(\widetilde{x_2})(\lambda)$ implies $(\widetilde{x_1})(\lambda) = (\widetilde{x_2})(\lambda)$ $\forall \lambda \in$ A.
Since T satisfies L_3, so T $(\widetilde{x_1})(\lambda) = T_\lambda(\xi)$ with $(\widetilde{x_1})(\lambda) = \xi$ and T$(\widetilde{x_2})(\lambda) = T_\lambda(\eta)$ with $(\widetilde{x_2})(\lambda) = \eta$ and $\xi, \eta \in$ X.
Now let $T_\lambda(\xi) = T_\lambda(\eta)$. Then, T $(\widetilde{x_1})(\lambda)$ = T$(\widetilde{x_2})(\lambda)$. This yields $(\widetilde{x_1})(\lambda) = (\widetilde{x_2})(\lambda)$. So $\xi = \eta$.Then T_λ is one to one.
Suppose that T is onto $\Rightarrow \forall \tilde{y} \in$ SE ($\tilde{Y}$), there exists $\tilde{x} \in SE(\tilde{X})$ such that T $(\tilde{x})(\lambda) = (\tilde{y})(\lambda)$.
Let $\beta \in$ Y $\Rightarrow (\tilde{y})(\lambda) = \beta$ for some $\tilde{y} \in$ SE(($\tilde{Y}$)
$\beta = (\tilde{y})(\lambda)$ = T$(\tilde{x})(\lambda) = T_\lambda(\xi)$, since β is arbitrary $\Rightarrow \forall \beta \in Y$, there exists $\xi \in$ X such that $\beta = T_\lambda(\xi)$, so T_λ is onto.
Conversely, suppose T_λ is one-to-one and onto and let $\lambda \in$ A, $\xi \in$ X, $\eta \in$ Y such that $(\widetilde{x_1})(\lambda) = \xi$, $(\widetilde{x_2})(\lambda) = \eta$.
Let T$(\widetilde{x_1})(\lambda)$ = T$(\widetilde{x_2})(\lambda) \Rightarrow T_\lambda(\xi) = T_\lambda(\eta) \Rightarrow \xi = \eta$ (since T_λ is one-to-one) $\Rightarrow (\widetilde{x_1})(\lambda) = (\widetilde{x_2})(\lambda)$, so T is one-to-one.
Let $\tilde{y} \in$ SE($\tilde{Y}$) $\Rightarrow (\tilde{y})(\lambda) \in$ Y , and let $(\tilde{y})(\lambda) = \beta$. Then there exists $\xi \in$ X such that $T_\lambda(\xi) = \beta$, since T_λ is onto.
So T$(\tilde{x})(\lambda) = T_\lambda(\xi) = \beta = (\tilde{y})$. So T is onto.

Definition 3.2: Let *T:* $SE(\tilde{X}) \longrightarrow SE(\tilde{Y})$ be a soft operator. Then T is said to be invertible, if there exists a soft operator S $\in$ B($\tilde{Y}, \tilde{X}$) such that : TS$(\tilde{y}) = I_{\tilde{Y}}$ $\forall \tilde{y} \in \tilde{Y}$ and ST$(\tilde{x}) = I_{\tilde{X}}$ $\forall \tilde{x} \in \tilde{X}$, i.e. , TS$(\tilde{y})(\lambda) = \tilde{y}(\lambda)$ and ST$(\tilde{x})(\lambda) = \tilde{x}(\lambda)$ $\forall \lambda \in A$. We write S = T^{-1}.

Remark 3.3: Let $\tilde{X}$, $\tilde{Y}$ be two soft normed spaces and $T : SE(\tilde{X}) \longrightarrow SE(\tilde{Y})$ be soft linear operator. One can show that if T is invertible then T is one-to-one and onto.

Proposition 3.4: Let T*:* $SE(\tilde{X}) \longrightarrow SE(\tilde{Y})$ be a soft linear operator. If T is invertible then $T^{-1}$is soft linear operator.

Proof: Suppose that T is invertible, then T^{-1}: SE($\tilde{Y}$) $\longrightarrow$SE($\tilde{X}$) is a soft operator in which, for $\widetilde{x_1}, \widetilde{x_2}$ $\tilde{\in}$ SE($\tilde{X}$), if T$(\widetilde{x_1}) = \widetilde{y_1}$ and T$(\widetilde{x_2}) = \widetilde{y_2}$ $\widetilde{y_1}, \widetilde{y_2} \tilde{\in} SE(\tilde{Y})$, then $\widetilde{x_1} = T^{-1}(\widetilde{y_1})$, $\widetilde{x_2} = T^{-1}(\widetilde{y_2})$.
For any soft scalar $\tilde{\alpha}$, $\tilde{\beta}$ we have $\tilde{\alpha}\widetilde{y_1} + \tilde{\beta}\widetilde{y_2} = \tilde{\alpha}T(\widetilde{x_1}) + \tilde{\beta}T(\widetilde{x_2})$ = T $(\tilde{\alpha}\widetilde{x_1} + \tilde{\beta}\widetilde{x_2})$ (since T is linear).
$T^{-1}(\tilde{\alpha}\widetilde{y_1} + \tilde{\beta}\widetilde{y_2}) = T^{-1}$ T $(\tilde{\alpha}\widetilde{x_1} + \tilde{\beta}\widetilde{x_2}) = \tilde{\alpha}\widetilde{x_1} + \tilde{\beta}\widetilde{x_2} = \tilde{\alpha}T^{-1}(\widetilde{y_1}) + \tilde{\beta}T^{-1}(\widetilde{y_2})$
So T^{-1} is linear.

Proposition 3.5: Let T $\in$ B($\tilde{X}$,$\tilde{Y}$) , S$\in B(\tilde{Y}, \tilde{Z})$ be invertible operators, were $\tilde{X}$, $\tilde{Y}$and $\tilde{Z}$ are soft normed spaces. Then (ST) is invertible with $(ST)^{-1}(\tilde{z})(\lambda) = T^{-1}S^{-1}(\tilde{z})(\lambda)$ $\forall \tilde{z} \in \tilde{Z}$.

Proof: ST: SE $(\tilde{X}) \longrightarrow SE(\tilde{Z})$ and $(ST)^{-1}: SE(\tilde{Z}) \longrightarrow SE(\tilde{X})$

Since T and S are invertible, hence bijective, then ST is bjective (in fact if ST $(\widetilde{x_1})$ = ST $(\widetilde{x_2})$ then S $(\widetilde{x_1})$ = S $(\widetilde{x_2})$ since T is one-to-one, and this implies $\widetilde{x_1} = \widetilde{x_2}$ since S is one-to-one. If $\tilde{z} \tilde{\in} \tilde{Z}$, then there exists $\tilde{y} \tilde{\in} \tilde{Y}$ such that S $(\tilde{y}) = \tilde{z}$. Since S is onto and there exists $\tilde{x} \tilde{\in} \tilde{X}$ such that T $(\tilde{x}) = \tilde{y}$, since T is onto. i.e., S $(\tilde{y})$ = S (T$(\tilde{x})) = \tilde{z}$. So ST is onto.
$TT^{-1}(\tilde{y}) = I_{\tilde{Y}}(\tilde{y})$. Consequntly, $STT^{-1}(\tilde{y}) = S(\tilde{y})$
Since S is onto, then there exists $\tilde{z} \tilde{\in} \tilde{Z}$ such that $\tilde{y} = s^{-1}(\tilde{z})$. Hence $(ST)T^{-1}S^{-1}(\tilde{z}) = SS^{-1}(\tilde{z}) = \tilde{z}$.
By the same way, we can show that $T^{-1}S^{-1}ST(\tilde{x}) = \tilde{x}$. i.e., the operator ST is invertible.
$(S\mathrm{T})(S\mathrm{T})^{-1} = I$ and $(ST)T^{-1}S^{-1} = I$. Hence $(ST)^{-1} = T^{-1}s^{-1}$.

Definition 3.6: Let $\tilde{X}$, $\tilde{Y}$ be two soft normed spaces and *T:* $SE(\tilde{X}) \longrightarrow SE(\tilde{Y})$ be soft bounded linear operator. Then T is said to be soft bounded below if there exists a soft real number $\tilde{\beta}$ such that $\tilde{\beta}\|\tilde{x}\| \tilde{\leq} \|T(\tilde{x})\|$.

Theorem 3.7: Let $\tilde{X}$, $\tilde{Y}$ be two soft normed spaces and *T:* $SE(\tilde{X}) \longrightarrow SE(\tilde{Y})$ be soft bounded linear operator. Then T is invertible if and only if T is bounded below and onto .

Proof: **the "only if" direction.**
Suppose that T is invertible, i.e., T^{-1} exists.
$\|\tilde{x}\|(\lambda) = \|T^{-1}\, T\,(\tilde{x})\|(\lambda) \leq [\|T^{-1}\|\|T(\tilde{x})\|\,](\lambda) = \|T^{-1}\|(\lambda) \,.\, \|T(\tilde{x})\|(\lambda)$
So $\frac{1}{\|T^{-1}\|(\lambda)}\, \|\tilde{x}\|(\lambda) \leq \|T(\tilde{x})\|(\lambda)$
$[\, \frac{\bar{1}}{\|T^{-1}\|}\, \|\tilde{x}\|](\lambda) \leq \|T(\tilde{x})\|(\lambda)$
Bu putting $\frac{\bar{1}}{\|T^{-1}\|} = \tilde{c} \Longrightarrow [\tilde{c}\|\tilde{x}\|](\lambda) \leq \|T(\tilde{x})\|(\lambda) \qquad \forall\, \lambda \in A$
So T is bounded below.
Since T^{-1} exists, then T is onto.
(***if direction***) let T be onto and there exist $\tilde{\beta} \mathrel{\tilde{>}} \bar{0}$ such that: $[\,\tilde{\beta}\|\tilde{x}\|](\lambda) \leq \|T(\tilde{x})\|(\lambda) \quad \forall\, \tilde{x} \mathrel{\tilde{\in}} \tilde{X}$, $\forall\, \lambda \in$ A.
Suppose that T is not one-to-one, then there exists $\tilde{x} \mathrel{\tilde{\in}}$ SE $(\tilde{X})$ such that $\tilde{x} \neq \Theta$ and $T(\tilde{x}) = \Theta$. So $\|T(\tilde{x})\| = \bar{0}$.
By the bounded below, there exists $\tilde{\beta} \mathrel{\tilde{>}} \bar{0}$ such that : $[\,\tilde{\beta}\|\tilde{x}\|](\lambda) \leq \|T(\tilde{x})\|(\lambda) = 0 \quad \forall\lambda \in A$.
But this is a contradiction, since $[\,\tilde{\beta}\|\tilde{x}\|](\lambda) > 0 \quad \forall\lambda \in A$.
So, T is one-to-one
Let $T(\tilde{x})(\lambda) = \tilde{y}(\lambda)$.Then $\tilde{x}(\lambda) = T^{-1}(\tilde{y})(\lambda)$
$[\,\tilde{\beta}\, \|T^{-1}(\tilde{y})\|\,](\lambda) \leq \|T(T^{-1})(\tilde{y})\|(\lambda) = \|\tilde{y}\|(\lambda)$.
Hence, $\|T^{-1}\|(\lambda) \leq (\frac{\bar{1}}{\tilde{\beta}})\,(\lambda) \;\; \forall\lambda \in A$ and $T^{-1} \in$ B $(\tilde{Y}, \tilde{X})$.

Remark 3.8: Let $\tilde{X}$, $\tilde{Y}$ be two soft Banach spaces and *T: SE* $(\tilde{X}) \longrightarrow$ *SE* $(\tilde{Y})$ be a soft linear operator. Then T is invertible if and only if T is bijective .

Theorem 3.9: Let $\tilde{X}$, $\tilde{Y}$ be two soft Banach spaces and *T: SE*$(\tilde{X}) \longrightarrow$ *SE*$(\tilde{Y})$ be soft bounded linear operator. Then T is invertible if and only if T is bounded below and the range of T is dense in $\tilde{Y}$.

Proof: The "only if" direction is omitted since it is easy, then we prove the (if direction).
To prove that T is invertible, it is enough to sow that T is onto.
Let $\tilde{x} \mathrel{\tilde{\in}}$ SE $(\tilde{Y})$. Since R (T) is dense in $\tilde{Y}$, there is a sequence $\{T\widetilde{x_n}\}$in R(T) such that : $T\widetilde{x_n} \to \tilde{y}$ in $\tilde{Y}$.
T is bounded below, then there exists $\tilde{\beta} \mathrel{\tilde{>}} \bar{0}$ such that: $[\,\tilde{\beta}\|\tilde{x}\|](\lambda) \leq \|T(\tilde{x})\|(\lambda)$, $\forall\, \tilde{x} \mathrel{\tilde{\in}} \tilde{X}$, $\forall\, \lambda \in$ A.
$[\,\tilde{\beta}\|\widetilde{x_n} - \widetilde{x_m}\|](\lambda) \leq \|T(\widetilde{x_n} - \widetilde{x_m}\|(\lambda) = \|T\widetilde{x_n} - T\widetilde{x_m}\|(\lambda) \qquad \forall\, \lambda \in A \;, \forall\, n = 1,2,3,\ldots$
Hence, $\{\widetilde{x_n}\}$ is Cauchy sequence in $\tilde{X}$. So $\widetilde{x_n} \to \tilde{x}$ in $\tilde{X}$.
$T(\widetilde{x_n}) \to T(\tilde{x})$ by the continuity of T (since if T is bounded then T is continuous).
$\tilde{y} = T(\tilde{x}) \in R(T)$, then $R(T) = SE(\tilde{Y})$ and T is onto.
So, T is invertible by the previous theorem.

Definition 3.10: Let $\tilde{X}$ be a soft Banach space. The series $\sum_{k=1}^{\infty} \tilde{x}_k$ is absolutely convergent in $\tilde{X}$, if $\sum_{k=1}^{\infty}\|\tilde{x}_k\|(\lambda) < \tilde{M}(\lambda)$ for all $\lambda \in$ A.

Theorem 3.11: The soft normed space ($\tilde{X}$, $\|.\|$) is complete if and only if every absolutely convergent series in $\tilde{X}$ is converge in $\tilde{X}$.

Proof: Suppose that $\tilde{X}$ is complete to show that the series $\sum_{k=1}^{\infty} \tilde{x}_k$ converges for all $\lambda \in$ A.
We prove that the sequence of partial sums of this series is Cauchy. For m > n we have:
$\|S_n - S_m\| = \|\sum_{k=1}^{n} \tilde{x}_k - \sum_{k=1}^{m} \tilde{x}_k\|(\lambda) = \|\sum_{k=n+1}^{m} \tilde{x}_k\|(\lambda) \leq \sum_{k=n+1}^{m}\|\tilde{x}_k\|(\lambda)$ for all $\lambda \in$ A and n , m $\to \infty$.
So, $\{\, S_n \,\}$ is Cauchy and, hence, converges, since $\tilde{X}$ is complete.
Conversely, we use the method of contra positive to prove this part, i.e., suppose that $\tilde{X}$ is incomplete to find the absolutely converged series which diverges.
Since $\tilde{X}$ is incomplete, then there exists a Cauchy sequence$\{\,\widetilde{x_n}\}$in $\tilde{X}$ which diverges .Every subsequence of $\{\,\widetilde{x_n}\,\}$ diverges.
Let $\{\,\widetilde{x_{n_j}}\,\}$ be a subsequence of $\{\,\widetilde{x_n}\,\} \Longrightarrow \{\,\widetilde{x_{n_j}}\,\}$ which diverges. Since $\widetilde{x_n}$ is Cauchy, then for all positive integer j, there is a positive integer N_j such that $\|\,\tilde{x}_n - \tilde{x}_m\| \mathrel{\tilde{\leq}} \overline{(2^{-j})}$ for all n , m $\geq N_j$ for all j .
For $n_{j+1} > n_j > N_j$, we have $\left\|\,\tilde{x}_{n_{j+1}} - \tilde{x}_{n_j}\right\| \mathrel{\tilde{\leq}} \overline{(2^{-j})}$. So, $\sum_{j=1}^{\infty}\left\|\,\tilde{x}_{n_{j+1}} - \tilde{x}_{n_j}\right\| \leq \sum_{j=1}^{\infty}\overline{(2^{-j})}$,

i.e., $\sum_{j=1}^{\infty}\left\|\tilde{x}_{n_{j+1}}-\tilde{x}_{n_j}\right\|(\lambda)\leq\sum_{j=1}^{\infty}\overline{(2^{-j})}(\lambda)$ for all $\lambda\in$ A. Since $\sum_{j=1}^{\infty}2^{-j}=1$, then $\sum_{j=1}^{\infty}\left\|\tilde{x}_{n_{j+1}}-\tilde{x}_{n_j}\right\|(\lambda)\leq\bar{1}(\lambda)$ implies $\sum_{j=1}^{\infty}\left\|\tilde{x}_{n_{j+1}}-\tilde{x}_{n_j}\right\|\tilde{\leq}\bar{1}$, then the series $\sum_{j=1}^{\infty}\left(\tilde{x}_{n_{j+1}}-\tilde{x}_{n_j}\right)$ is absolutely converged but the partial sum of this series, which is
$\sum_{j=1}^{k}\left(\tilde{x}_{n_{j+1}}-\tilde{x}_{n_j}\right)=\tilde{x}_{n_{k+1}}-\tilde{x}_{n_1}$, is diverged. So, the series $\sum_{j=1}^{\infty}\left(\tilde{x}_{n_{j+1}}-\tilde{x}_{n_j}\right)$ is an absolutely converged series which diverges. Hence $\tilde{X}$ is complete.

Theorem 3.12: Let $\tilde{X}$, $\tilde{Y}$ be two soft normed spaces such that $\tilde{Y}$ is a soft Banach space. Let $T: SE(\tilde{X})\longrightarrow SE(\tilde{Y})$ be a soft bounded linear operator and $\|T\|(\lambda)\tilde{<}\bar{1}(\lambda)$ $\forall\lambda\in$ A. Then $(I-T)$ is invertible and $(I-T)^{-1}=\sum_{n=0}^{\infty}T^n$.

Proof: Because $\tilde{Y}$ is a Banach space, then B($\tilde{X}$, $\tilde{Y}$) is a Banach space (12).
Since $\|T\|(\lambda)\tilde{<}\bar{1}(\lambda)$ $\forall\lambda\in$ A, then $\sum_{n=0}^{\infty}\|T\|^n(\lambda)$ converges to $\forall\lambda\in$ A.
But $\|T^n\|(\lambda)\tilde{\leq}\|T\|^n(\lambda)$ $\forall\lambda\in$ A and $\forall n\in$ N $\Longrightarrow\sum_{n=0}^{\infty}\|T^n\|(\lambda)$ converge to $\forall\lambda\in$ A, then $\sum_{n=0}^{\infty}T^n(\lambda)$ is converged, by Theorem **(3.11).**
Let $S(\lambda)=\sum_{n=0}^{\infty}T^n(\lambda)$ $\forall\lambda\in$ A and let $S_k(\lambda)=\sum_{n=0}^{k}T^n(\lambda)$ $\forall\lambda\in$ A. Then $\{S_k(\lambda)\}$ converges to $S(\lambda)$ in B($\tilde{X}$, $\tilde{Y}$).
$\left\|(I-T)S_k-I\right\|(\lambda)=\left\|I-T^{k+1}-I\right\|(\lambda)=\left\|-T^{k+1}\right\|(\lambda)\leq\|T\|^{k+1}(\lambda)$ $\forall\lambda\in$ A, $\|T\|^{k+1}(\lambda)\to 0$ as $k\to\infty$
$[(I-T)S](\lambda)=[(I-T)\lim_{n\to\infty}S_k](\lambda)=[\lim_{n\to\infty}(I-T)S_k](\lambda)=I(\lambda)$ $\forall\lambda\in$ A.
By a similar way, $[S(I-T)](\lambda)$, hence I – T is invertible and $(I-T)^{-1}(\lambda)=S(\lambda)=\sum_{n=0}^{\infty}T^n$.

4. Spectrum of soft linear operator

Definition 4.1: Suppose that $\tilde{X}$ is a soft normed space. Let $T: SE(\tilde{X})\longrightarrow SE(\tilde{X})$ be a soft linear operator. The spectrum of T is denoted by $\sigma(T)$ and defined by $\sigma(T)=\{\mu\in\mathbb{C}$, $T-\mu I$ is not invertible$\}$. The complement of σ(T), symbolized by $\rho(T)=\mathbb{C}-\sigma(T)$ in the complex plane ℂ, is said to be the Resolvent of T. The spectrum $\sigma(T)$ is divided into three separate sets, as follows:

1) The point spectrum $\sigma_p(T)$; is the set such that $T-\mu I$ is not one-to-one. A $\mu\in\sigma_p(T)$ is called a soft eigenvalue of T. A soft eigenvalue of a soft linear operator $T: SE(\tilde{X})\longrightarrow SE(\tilde{X})$ is a complex number μ such that $T\tilde{x}=\mu\tilde{x}$ has a solution $\tilde{x}\neq\theta$, and this $\tilde{x}$ is called a soft eigenvector of T corresponding to that soft eigenvalue μ. For a given eigenvalue μ, the set of all $\tilde{x}$, such that $T(\tilde{x})=\mu\tilde{x}$ with zero vector, is called the μ-eigenspace. Each μ-eigenspace is a soft subspace of $\tilde{X}$.

In fact, if $\widetilde{x_1}$, $\widetilde{x_2}$ are two eigenvectors corresponding to the soft eigenvalue μ and $\tilde{\alpha}$, $\tilde{\beta}$ are two soft scalars then:
$T(\tilde{\alpha}\widetilde{x_1}+\tilde{\beta}\widetilde{x_2})=T(\tilde{\alpha}\widetilde{x_1})+T(\tilde{\beta}\widetilde{x_2})=\tilde{\alpha}T(\widetilde{x_1})+\tilde{\beta}T(\widetilde{x_2})=\tilde{\alpha}\mu\widetilde{x_1}+\tilde{\beta}\mu\widetilde{x_2}=(\tilde{\alpha}\widetilde{x_1}+\tilde{\beta}\widetilde{x_2})\mu$ i.e., $\tilde{\alpha}\widetilde{x_1}+\tilde{\beta}\widetilde{x_2}$ is a soft eigenvector of T.

2) The continuous spectrum $\sigma_c(T)$; is the set such that $\mathcal{N}(T-\mu I)=\{0\}$ and the range of $T-\mu I$ is dense in
$\tilde{X}$, but $(T-\mu I)^{-1}$ is not bounded.

3) The residual spectrum $\sigma_r(T)$; is the set such that $\mathcal{N}(T-\mu I)=\{0\}$,but the range of $T-\mu I$ is not dense
in $\tilde{X}$.

Remark 4.2: If $\tilde{X}$ is a soft Banach space, the second set $\sigma_c(T)=\phi$, since every invertible operator has a bounded inverse. Furthermore, if $\tilde{X}$ is a finite dimensional soft normed space, then $\sigma_c(T)=\sigma_r(T)$.

Definition 4.3: Soft vectors $\widetilde{x_1},\widetilde{x_2},\widetilde{x_3},\ldots$ are linearly independent, if $\{\widetilde{x_1},\widetilde{x_2},\ldots\ldots\widetilde{x_n}\}$ is linearly independent for all n ∈ N.

Theorem 4.4: Soft eigenvectors $\widetilde{x_1},\widetilde{x_2},\widetilde{x_3},\ldots$, corresponding to different eigenvalues $\mu_1,\mu_2,\mu_3,\ldots$ of a soft linear operator T on a soft normed space $\tilde{X}$, constitute a linearly independent set.

Proof : It is sufficient to prove that $\{\widetilde{x_1},\widetilde{x_2},\ldots,\widetilde{x_n}\}$ of soft eigenvectors, corresponding to different eigenvalues $\mu_1,\mu_2,\ldots,\mu_n$, is linearly independent for all n ∈ N. Assume that $S=\{\widetilde{x_1},\widetilde{x_2},\ldots,\widetilde{x_n}\}$ is linearly dependent and derive a contradiction. Since S is linearly dependent then $S(\lambda)=\{\widetilde{x_1}(\lambda),\widetilde{x_2}(\lambda),\ldots\widetilde{x_n}(\lambda)\}$ is linearly dependent for all $\lambda\in$A, and $\widetilde{x_1}(\lambda),\widetilde{x_2}(\lambda),\ldots\widetilde{x_n}(\lambda)$ are crisp vectors in X. Let $\widetilde{x_k}(\lambda)$ be the first vector in $S(\lambda)$, which is a linear combination of its predecessors ,say

$\widetilde{x_k}(\lambda) = \widetilde{\alpha_1}\ \widetilde{x_1}(\lambda) + \widetilde{\alpha_2}\ \widetilde{x_2}(\lambda) + \ldots\ldots + \widetilde{\alpha_{k-1}}\ \widetilde{x_{k-1}}(\lambda)$ then $\{\widetilde{x_1}(\lambda), \widetilde{x_2}(\lambda), \ldots\ldots \widetilde{x_{k-1}}(\lambda)\}$ is linearly independent . By applying T-μ_kI on both sides of the previous equation, we have:
(T-μ_kI) $\widetilde{x_k}(\lambda) = \sum_{j=1}^{k-1} \widetilde{\alpha_j}\ (T - \mu_k I)\widetilde{x_j}(\lambda\ ) = \sum_{j=1}^{k-1} \widetilde{\alpha_j}\ \left(\lambda_j - \lambda_k\right)\widetilde{x_j}(\lambda\ )$. Since $\widetilde{x_k}(\lambda)$ is an eigenvector corresponding to μ_k, then the left side is zero . Since the soft vectors on the right side form a linearly independent set, we must have $\widetilde{\alpha_j}\left(\lambda_j - \lambda_k\right) = \overline{0}$, hence $\widetilde{\alpha_j} = 0$ for j = 1,2,….k-1 , since $\left(\lambda_j - \lambda_k\right) \neq 0$, but then $\widetilde{x_k}(\lambda) = \theta$. This contradicts the fact that $\widetilde{x_k}(\lambda) \neq \theta$. Since $\widetilde{x_k}(\lambda)$ is a soft eigenvector, so S = $\{\widetilde{x_1}, \widetilde{x_2}, \ldots\ldots \widetilde{x_n}\}$ is a linearly independent set. The result follows.

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