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Purely Maximal Sub-modules

Halah Qahtan Hamdi*, ***N*uhad Salim Al-Mothafar**
epartment of Mathematics, College of Science, University of Baghdad, Baghdad, IraqD

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Abstract
In this paper, we present the notion of a purely maximal sub-module, also we consider some properties of this concept, and we Tresearch some relationships between purely maximal sub-modules and some other related ideas. Furthermore, we give purely-radical modules and purely local modules.

Keywords: Maximal sub-modules, Pure sub-modules, Purely-maximal sub-modules, Purely-local modules, Fully purely-maximal, Purely–radical modules.

المقاسات الجزئية العظمى النقية

هالة قحطان حمدي*, **نهاد سالم المظفر**
قسم الرياضيات، كلية العلوم، جامعة بغداد، بغداد– العراق

الخلاصة
في هذا البحث، نقدم فكرة المقاسات الجزئية العظمى الصرفة، وننظر أيضًا في بعض خصائص هذا المفهوم، ونبحث في بعض العلاقات بين المقاسات الجزئية العظمى الصرفة وبعض الأفكار الأخرى ذات الصلة. علاوة على ذلك، فإننا نقدم مقاسات جذرية صرفة ومقاسات محلية صرفة.

1. Introduction

In this paper, R has an identity and it is commutative. The set M is a unitary left R-module. It is established that, for each ideal I of R, a sub-module N of M is considered pure if $IM \cap N = IN$ (for short $N \leq_{\mathrm{P}} M$) [1]. If N is a proper submodule, of M then N is named a small. In short ($N \ll M$) if a submodule K of a module M with $N + K = M$ implies that K equals M, [2]. A new sub-module was created by Muna Abbas and et.al.,[3] and it is a generalization of a small, which is called a purely small submodule (Pu-small) sub-module, as: The submodule N of M is named a purely small submodule (for short $N \ll_{Pu} M$), if $N + K \neq M$, for every pure proper sub-module K of M. Equivalently; $N \ll_{Pu} M$, if when a pure sub-module K of a module M with $N + K = M$ then $K = M$.

It is well-known that N is a proper submodule of M is maximal, see[1], if whenever $K \leq M$ with N is a proper submodule of K then $K = M$. Many authors generalized the maximal sub-module, see [4-8]. Ahmed and Al-Mothafer in [5] offer the notion of near maximal (N-maximal) submodules as a generality of maximal submodules. In this paper, we study the

*Email: hala.hamdi@sc.uobaghdad.edu.iq

properties of this concept and we call it a purely-maximal sub-module, wherever a proper submodule N of M is named a purely-maximal submodule (for short Pu-maximal), if for any K pure submodule of M, also $N \lneq K \leq M$ implies that K=M. An ideal I is called a Pu-maximal ideal if for any pure ideal J of R with $I \lneq J \leq R$ implies $J = R$,[1]. We study that if M is an F-regular module then every Pu- maximal submodule is a maximal sub-module. If the module M is an F-regular then every sub-module of M is pure, [9,10].

A non-zero R-module M is called a pure-local, (for short Pu-local module) if M has one Pu-maximal sub-module which has all proper sub-module of M. A ring R is called Pu-local, if it is Pu-local as R-module.

Since every small submodule is a purely small submodule this controlled us to study, the ideas of purely radical for modules: If M is an R-module then the sum of all purely small submodules of M is named purely radical (for short, Pu-radical module) of a module M represented by $Rad_{Pu}(M) = \{\sum(N: N \ll_{Pu} M)\}$. It is clear that every radical of R-module M contained in Pu-radical of M i.e. $Rad(M) \leq Rad_{Pu}M$. Where $Rad(M)$= {The sum of all small submodules of M},[11]. Details in ,[12,13].

Some elementary properties, remarks, and propositions of the above concepts have been given in this work.

2. Purely Maximal Sub-modules

This section provides a definition and illustrations of the idea of a purely maximal sub-module as a generalization of a maximal sub-module and we illustrate it with examples. We also provide some of its properties.

We start with this definition:

Definition 2.1 [1]: A proper submodule N of M is named a purely-maximal (for short, Pu-maximal), if for every pure submodule K of M with $N \lneq K \leq M$ implies that $K = M$.
An ideal $I$of a ring R is named Pu-maximal if for any pure ideal J of R such that $I \lneq J \leq R$ implies $J = R$.

Examples and Remarks 2.2 :

1. It is clear that each maximal submodule of M is a Pu-maximal.
2. In general, the converse of (**1**) is incorrect. As an illustration: -
Consider the module $M = Z_{12}$ as Z-module, $4Z_{12} = \{\bar{0}, \bar{4}, \bar{8}\}$ is a Pu -maximal sub-module since Z_{12} is the only pure sub-module containing $4Z_{12}$ such that $4Z_{12} \lneq Z_{12} \leq_{P} Z_{12}$ and $Z_{12} = M$. But it is not a maximal sub-module since $4Z_{12} \lneq 2Z_{12} \leq Z_{12}$ then $2Z_{12} \neq Z_{12}$.
3. Consider $M = Z_4 \oplus Z_2$ as Z-module, Let $K = 2Z_4 \oplus (\bar{0})$ and $H = Z_4 \oplus (\bar{0})$ such that $2Z_4 \oplus (\bar{0}) \lneq Z_4 \oplus (\bar{0}) \leq_{P} Z_4 \oplus Z_2$. Since H is a summand, hence by **[14]** H is pure in *M,* which implies K is not a Pu-maximal of M since $H \neq M$ and K is not a maximal sub-module since $Z_4 \oplus (\bar{0}) \neq Z_4 \oplus Z_2$.
4. A sub-module of Pu-maximal sub-module must not be Pu-maximal sub-module as the resulting example: Let $M = Z_{24}$ as a Z-module, let $3Z_{24} \lneq Z_{24} \leq_P Z_{24}$ implies $3Z_{24}$ is Pu-maximal sub-module of Z_{24} since $Z_{24} \leq_P Z_{24}$, but $6Z_{24}$ is not a Pu -maximal sub-module, since $6Z_{24} \lneq 3Z_{24} \leq Z_{24}$, and $3Z_{24}$ is pure in Z_{24} but $3Z_{24} \neq Z_{24}$.
5. The $2Z_4$ is Pu-maximal sub-module in Z_4 as Z- module since $2Z_4$ is maximal sub-module.
6. The $2Z_6, 3Z_6$ are Pu-maximal sub-modules in Z_6, since they are maximal.
7. Let $H \leq M$, if M / H is simple, implies that a submodule H is a Pu-maximal.

Proof:
As M / H is simple then a submodule H is maximal, now by **(1)** H is a Pu - maximal sub-module.

8. If H is a Pu-maximal of K, with $H \leq K \leq M$ and K is a Pu-maximal of M, this is not enough for H to be a Pu- maximal in M. For instance:- Consider $M = Z_{24}$, as a Z-module, $H = 6Z_{24}, K = 3Z_{24}, 6Z_{24} \lneq 3Z_{24} \leq_{\text{P}} 3Z_{24}$, and $3Z_{24} = 3Z_{24}$, implies that $6Z_{24}$ is a Pu-maximal submodule of $3Z_{24}$, and since $3Z_{24}$ is the maximal sub-module in Z_{24} , therefore, it is a Pu-maximal submodule, but $6Z_{24}$ is not a Pu-maximal sub-module of a module M, since we have $6Z_{24} \lneq 3Z_{24} \leq_{\text{P}} Z_{24}$ but $3Z_{24} \neq Z_{24}$.

Proposition 2.3 :
If M is an F-regular module, then each Pu-maximal submodule is a maximal submodule.

Proof: Suppose H $,K \leq M$ also H is a Pu-maximal submodule such that $H \lneq K \leq M$.Since a module M is an F-regular, thus $K \leq_{\text{P}} M$, which implies H is a Pu-maximal , therefore K= M. So, a submodule H is a maximal of M.

Corollary 2.4:
If a module M is a semisimple, implies that each Pu-maximal submodule is a maximal submodule.

Proof: Since by **[15]** every semi-simple is an F–regular then we are done.

Recall that every multiplication module contains a maximal sub-module, see **[16].**

Proposition 2.5:
Every multiplication module contains Pu-maximal sub-module.

Proof: Every multiplication module has a Pu-maximal sub-module as each module has a maximal.

Corollary 2.6:
The Pu-maximul sub-module exists in each cyclic R-module.

Proof: By[16]each cyclic module is a multiplication, hence by using Proposition 2.5, we can get the result.

Proposition 2.7:
If N is a Pu- maximal submodule in M , with H is a pure submodule in an R-module M with $H \leq N$ then $\frac{N}{H}$ a Pu-maximal sub-module in $\frac{M}{H}$.

Proof: To prove $\frac{N}{H}$ is a Pu–maximal sub-module in $\frac{M}{H}$. Assume $\frac{L}{H}$ is a pure sub-module in $\frac{M}{H}$ with $\frac{N}{H} \lneq \frac{L}{H} \leq_{\text{P}} \frac{M}{H}$, $N \lneq L \leq M$, since $\frac{L}{H}$ is a pure submodule in $\frac{M}{H}$ with $H \leq_{\text{P}} M$, implies using [**17**, Remark 3.1.5, p.56], L is a pure submodule in a module M. But N is a Pu-maximal sub-module in a module M then $L = M$ implies $\frac{L}{H} = \frac{M}{H}$.Thus, $\frac{N}{H}$ is a Pu–maximal submodule in $\frac{M}{H}$.

Proposition 2.8:
Assume that M, N are two modules, and assume that $f: M \rightarrow N$ be an isomorphism. If A a Pu-maximal of a module M, then f (A) is a Pu-maximal of a module N.

Proof: Assume L is a pure module submodule of a module N such that $f(A) \lneq L \leq_{\text{P}} N$, since f an isomorphism, then by [17, Proposition 3.1.11, p.58] A=$f^{-1}(f(A)) \lneq f^{-1}(L) \leq_{\text{P}}$ $f^{-1}(N) = M$, then $A \lneq f^{-1}(L) \leq_{\text{P}} M$. Now, since A is a Pu-maximal sub-module in a implies $L = N$. Thus, f (A) is a $f^{-1}\left(f(L)\right) = f(M) = N$ module M then $f^{-1}(L) = M$, then Pu-maximal sub-module of N.

Proposition 2.9:
Suppose M, N are two R- modules and assume that $f: M \to N$ be an epimorphism. If A a Pu-maximal sub-module in R-module M and Ker$f \leq A$ with Ker$f$ is pure in M then $f(A)$ is a Pu-maximal sub-module in R-module N.

Proof: To show $f\ (A) \lneq N$. Suppose $f\ (A)= N = f\ (M)$, since $A\ \neq M$, so there exists $m \in M$, so $m \notin A$. Now, f is epimorphism, $\exists$ y $\in M$ such that $y = f(m) \in f(A)$ implies $f(m) = f(a), a \in A$, hence $m - a \in\ Kerf\ \leq A$, then $m - a\ = a_1$, $a_1 \in A$ implies $m = a + a_1\ \in A$ which is a contradiction. By[7] (the First Isomorphism Theorem) $\frac{M}{Kerf} \simeq f(M)$, but $A \leq M$ and Ker$f\ \leq\ A$ then $\frac{A}{Kerf} \simeq f\ (A)$. Since A is a Pu -maximal submodule in R-module M. Then using Proposition 2.7 $\frac{A}{Kerf}$ is a Pu -maximal submodule in $\frac{M}{Kerf}$ then $f(A)\ \simeq \frac{A}{Kerf}$ is a Pu -maximal sub-module in $\frac{M}{Kerf} \simeq f(M) = N$.

Proposition 2.10:
If N, K are proper submodules of M and $N\ \leq\ K$ if N is a Pu-maximal submodule in M, implies that a submodule K is a Pu -maximal in a module M.

Proof: Assume $H \leq_P M$ with $K \lneq H \leq M$, since $N \leq K$ and N is a Pu- maximal submodule with $N \lneq H \leq_P M$ implies that $H = M$, so a submodule K is a Pu -maximal of M.

Corollary 2.11:
If N, K are pure submodules of M with $N \cap K$ of M is a Pu-maximal, implies that both N and K are Pu –maximal sub-modules of M.

Proof: Since a submodule $N\ \cap K$ of K. Also, $N\ \cap\ K$ in M is a Pu-maximal, hence using Proposition 2.10 N is a Pu–maximal in M. Similarly, we can prove K is a Pu -maximal submodule in M.

Corollary 2.12 :
If the proper submodules N,K of a module M and if N or K are Pu–maximal sub-module implies that $K + N$ is a Pu-maximal submodule.

Proof: Since N is Pu-maximal submodule implies $N \leq_P K + N \leq M$ implies that $K + N$ is a Pu-maximal submodule in M by Proposition 2.10, Similarity if a submodule K is a Pu-maximal.

Corollary 2.13 :
If N is a Pu -maximal submodule of M and I is an ideal of R, implies that $[N:_M I]$ is a Pu -maximal submodule of a module M.

Proof: Since $N \lneq\ [N:_M I]\ \leq M$, hence using Proposition (2.10) $[N:_M I]$ is a Pu -maximal sub-module.

Remark 2.14 :
The opposite of Corollary 2.13 in general is not correct. For instance:- Suppose $M = Z_{12}$ as Z-module, let $I\ =\ 2Z$ of Z and $N = 6Z_{12}$, thus, $2Z_{12} = [N:_M\ I] = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}\}$ is a Pu -maximal sub-module, since in maximal in Z_{12} , but $N = 6Z_{12}$ is not a Pu-maximal sub-module of Z_{12}, since $(6Z_{12} < 3Z_{12} \leq Z_{12})$ and $3Z_{12}$ is a pure sub-module in Z_{12} but $3Z_{12} \neq Z_{12}$.

Theorem 2.15:
If M is a multiplication, faithful, and finitely generated module and $H \leq M$, then the next statement are equivalent.

1- The H is a Pu-maximal sub-module of M;
2- The $[H:_R M]$ is a Pu-maximal ideal of R;
3- The $H = IM$ for some Pu-maximal ideal of R.

Proof: $1 \to 2$) Assume $[H :_R M] < J \leq_P R$ with J pure ideal of R . To prove $J = R$, since a module M is a multiplication, implies $H = [H :_R M]M \lneq JM \leq RM = M$, by [**18**, Remark 2.1.34] we have JM is pure of M, since H is a Pu-maximal of M implies that $JM = M = RM$. But M is multiplication finitely generated faithful hence by [18, Remark 2.1.34] $J = R$. Thus $[H : M]$ is a Pu -maximal.
$2 \to 3$) Since M is multiplication implies $H = [H_R : M]M$ by[1, Proposition. 2.1.33] and by **(2)** we have $[H_R : M]$ is an ideal of R, thus $IM= H$ is a Pu -maximal ideal of R.
$3 \to 1$) The $H = IM$ for some Pu-maximal ideal I of R. Let $H \lneq K \leq_P M$. Since M is a multiplication, now $JM = K$, $IM = H$ for some ideal J of R.Then a submodule H with $H \leq IM \lneq JM \leq_P RM = M$. By [18, Remark 2.1.34] then by **(3),** H is a Pu–maximal sub-module of M.

Remark 2.16:
If N and K are Pu -maximal of M, implies that $N + K$ is not necessarily a Pu-maximal submodule as the next example illustrations:
Now, Z_6 as Z-module $N = 2Z_6$, $K = 3Z_6$ then $2Z_6$ is a Pu -maximal sub-module and $3Z_6$ is a Pu -maximal but $Z_6 = 2Z_6 + 3Z_6$ not Pu -maximal sub-module since $2Z_6 + 3Z_6$ not proper sub-module of Z_6.

3. Purely Local Modules

Here we introduce the concept of purely local modules. We prove that if M is a non-zero multiplication module and purely–local module also if K is a submodule purely maximal of a multiplication M, implies that a submodule K is maximal.

Recall that, if there is just one maximal, a module M is supposed to be local, also local ring means a ring with a unique maximal ideal, [16], [19].

Definition 3.1:
Let M be named a pure- local (for short, Pu-local module) if a non-zero R-module M has only one Pu-maximal sub-module which contains all proper sub-module of a module M.
The ring R is named (Pu-local) if R is a Pu-local.

Examples and Remarks 3.2:
1. The Z-module Z_4 has only one Pu-maximal sub-module, so is Pu-local. But Z_6 as Z-module is not Pu-local, since $2Z_6$ and $3Z_6$ are Pu-maximal sub-modules.
2. In Z_{12} as Z- module, the Z- module $(\bar{3})$ is a Pu-local, since it has only one Pu-maximal sub-module say $(\bar{6})$.

Proposition 3.3 :
If $M \neq 0$ is a Pu–local and if $K \neq 0$ is a Pu-maximal implies that K is the maximal of a module M.
Proof: Assume that $K \lneq N \leq M$, as M is a Pu-local, thus M has one Pu -maximal submodule since a submodule K is a Pu-maximal implies that $K \lneq M \leq M$, thus K is maximal sub-module.

Corollary 3.4 :
Let R be Pu -local ring. If J is a Pu-maximal ideal of R , $J \neq 0$, implies that it is a maximal ideal of R .
Proof: Since R be Pu-local and if J is a Pu–maximal implies $J \lneq R \leq R$, thus J is maximal.
We provide the definition:

Definition 3.5 :
A module M is named a fully purely-maximal module (for short, fully Pu-maximal module) if each non-zero sub-module of M is Pu-maximal. The ring R is named fully Pu-maximal if every non-zero ideal of R is Pu-maximal ideal.

Example 3.6:
1. The Z_6 as Z-module is fully Pu-maximal , since every sub-module is a Pu-maximal.
2. The Z_{12} is not fully Pu-maximal, since $6Z_{12}$ is not a Pu-maximal.
Theorem 3.7 :
Suppose M be a module such that $M = M_1 \oplus M_2$ where $M_1$and M_2 R-modules and R=$ann_R(M_1)$+ $ann_R(M_2)$, then if $M_1$and M_2 are fully Pu-maximal modules implies that M is a fully Pu-maximal module.
Proof: Assume N is a proper submodule of a module M also H is a submodule of a module M such that $N \subsetneqq H \leq_P M$, since $R = ann_R(M_1) \oplus ann_R(M_2)$, then $N = N_1 \oplus N_2$ for some module $N_1$of M_1 and $N_2$of M_2 also $H = H_1 \oplus H_2$ for some sub-modules H_1 of M_1 and H_2 of M_2 , so $N_1 \oplus N_2 \subsetneq H_1 \oplus H_2 \leq_P M_1 \oplus M_2$, since $M_1$and M_2 are fully Pu-maximal sub-modules, then N_1 and N_2 are Pu -maximal sub-modules of $M_1$and M_2 with H_1 is a pure of $M_1$and H_2 is a pure of M_2, since (H_1 and H_2 are summand of H) $N_1 \subsetneq H_1 \leq_P M_1$, $N_2 \subsetneq H_2 \leq_P M_2$ implies that $H_1 = M_1$, $H_2 = M_2$ thus $H_1 + H_2$= $M_1 + M_2$, therefore N is a Pu-maximal sub-module, hence M is a fully Pu-maximal module.

4. Purely Radical Submodules

In this unit, we recall the definitions of small submodules, the Jacobson radical of a module M, and some of their properties that are important to our work. For more details, see [1], [20].

In [21], [22] the Jacobson radical of a module named $Rad\ (M)$ is the sum of all small R-sub- modules of M.

Definition 4.1:
If M is a module the sum of each purely small submodules of M is called a purely radical sub-module (for short, Pu-radical sub-module) of a module M denoted by $Rad_{Pu}(M)$={$\sum N : N \ll_{Pu} M$}.
Remark 4.2 :
It is clear that the $Rad_{Pu}(M) \leq M$**,** by definition of 4.1 and by [7, Lemma 2.4].
For example**:** Let $M = Z_{12}$ as Z-module, $Rad_{Pu}(Z_{12})$= Z_{12}**.**Thus $Rad_{Pu}(M) \leq M$.

Remarks and Examples 4.3:
1. $Rad_{Pu}(Z_4)$= $(\bar{2})$.
2. $Rad_{Pu}(Z_6)$= $(\bar{0})$.
3. Every radical of an R-module M contain in a Pu-radical of the module M i.e., $Rad(M) \leq Rad_{Pu}M$.
Proof: It is clear that $Rad(M) \leq Rad_{Pu}M$, but the opposite in general is not right. For example: Assume M= Z_{12} as Z-module , $Rad_{Pu}(Z_{12})$= Z_{12}, but $Rad(Z_{12})$= $(\bar{6})$.

Lemma 4.4:
Assume M, N are two modules, and φ: $M \to N$ is an isomorphism if $A \ll_{Pu} M$, then $\varphi(A) \ll_{\mathrm{Pu}} N$.
Proof: Suppose a submodule L is a pure of N, now $\varphi\,(A) + L = N$ since φ is an isomorphism, then by [17, Remark 3.1.5, p.56] $\varphi^{-1}\,(L)$ is pure in M. Hence $\varphi^{-1}\,(L) + A = M$. Since $A \ll_{Pu} M$, hence $\varphi^{-1}\,(L) = M$ also $\varphi\,(M) \leq L$ so $\varphi(A) \leq L$. Thus $L = \varphi\,(A) + L = N$ and $\varphi(A) \ll_{Pu} N$.
Theorem 4.5:
Suppose M, N are R-modules and f: $M \to N$ is an isomorphism, then $f(Rad_{Pu}\,(M)) \leq (Rad_{Pu}\,(N))$.

Proof: Assume $B \subseteq Rad_{Pu}(M)$, $B \ll_{Pu} M$, then by (3) in Remark 4.3 and Lemma 4.4, $f(B) \ll_{Pu} N$. Hence, $f(B) \leq (Rad_{Pu}(N))$ for each $B \ll_{Pu} M$ then $f(Rad_{Pu}(N)) \leq (Rad_{Pu}(M))$.
Recall that, where M is a module, the Jacobson Radical of M (Rad (M)) is the intersection of every maximal sub-module of M. If M has no maximal sub-module, we say that Rad(M) = M). For module M, then Rad (M) is the sum of all small sub-modules of a module M, [20-23].
Recall that, condition the intersection of every two pure submodules is also a pure submodule, then an R-module M has PIP, [24].

Lemma 4.6:
Let M be a module with PIP property and suppose $A, B,$ and N are submodules of a module M where $A \leq B \leq N \leq M$ with N is a pure submodule in an R-module M. If $B \ll_{Pu} N$ implies that $A \ll_{Pu} M$.
Proof: Assume $L \leq_P M$ such that $L + A = M$. To show $L = M$. Now, since $A \leq B$ then $L + B = M$. As well as, $N \cap (L + B) = N \cap M$, since $B \leq N$ then using the[Modular law] $B + (L \cap N) = N$. Implies that N, L are pure submodules in a module M, and M has PIP property. Now, $L \cap N$ is a pure submodule in a module M, however $L \cap N$ submodule in N by [17, Remark 3.1.5, P,56]. Implies $N \cap L$ is a pure in N. Now, $B \ll_{Pu} N$ implies $L \cap N = N$, i.e., $N \leq L$ hence $A \leq N \leq L$, then $A \leq L$, since $A + L = M$,then L=M, therefore $A \ll_{Pu} M$.

Proposition 4. 7 :
If N is a pure sub-module in a module M, then $Rad_{Pu}(N) \leq Rad_{Pu}(M)$.
Proof: Assume $H \leq Rad_{Pu}(N)$. Implies $H \ll_{Pu} N$, however $N \leq_P M$, then by Lemma 4.6 we get $H \ll_{Pu} M$, therefore $H \leq Rad_{Pu}(M)$, then $Rad_{Pu}(N) \leq Rad_{Pu}(M)$.

Proposition 4.8 :
If K is a pure sub-module in a module M, then $Rad_{Pu}(K) \leq Rad_{Pu}(M) \cap K$.
Proof: By Proposition 4.7 $Rad_{Pu}(K) \leq Rad_{Pu}(M)$ and by Remark 4.2 $Rad_{Pu}(K) \leq K$, hence $Rad_{Pu}(K) \leq Rad_{Pu}(M) \cap K$.
The covers of Proposition 4.8 in general is not right, for example: In Z_{12} as a Z-module $Rad_{Pu}(Z_{12})=(\bar{6})$ and $Rad_{Pu}(\bar{3})=(\bar{6})$, thus $Rad_{Pu}(\bar{3}) \leq Rad_{Pu}(Z_{12}) \cap (\bar{3})$ implies $(\bar{6}) \leq (\bar{6}) \cap (\bar{3}))=(\bar{6})$.

Remark 4.9 :
Suppose M is a module then $Rad_{Pu}(Rad_{Pu}(M)) \neq M$. For example: Z_4 as Z-module $Rad_{Pu}(Rad_{Pu}(Z_4)) \neq Z_4$. $Rad_{Pu}(\bar{2})) \neq Z_4$.
Recall that, An R-module M has (PIP) condition the intersection of every two pure submodules is again pure, [24].

Proposition 4.10 :
Assume M is an R-module has (PIP). If H and $N \leq M$, implies $Rad_{Pu}(H)+ Rad_{Pu}(N) \leq Rad_{Pu}(H + N)$.
Proof: To show $Rad_{Pu}(H) \leq Rad_{Pu}(H + N)$, assume a submodule L is a Pu-small of H, using Lemma 4.6 $L \ll_{Pu} H + N$, hence $L \leq Rad_{Pu}(H + N)$. Thus $Rad_{Pu}(H) \leq Rad_{Pu}(H + N)$. Similarly, $Rad_{Pu}(N) \leq Rad_{Pu}(H + N)$, therefore ($Rad_{Pu}(H) + Rad_{Pu}(N)$) $\leq Rad_{Pu}(H + N)$

5. Conclusions:
In this work, purely maximal submodules and purely local modules, are details of maximal sub-modules and local modules respectively. We also show some of the following results:

- If M is an F-regular module then every Pu-maximal sub-module are maximal submodule.

- If N is a Pu- maximal sub-module in a module M , with $H \leq N$, H is a pure sub-module in R-module M then. $\frac{N}{H}$ is a Pu-maximal sub-module in $\frac{M}{H}$.
- If A a Pu-maximal of a module M also let $f: M \rightarrow N$ be an isomorphism then $f\ (A)$ is a Pu-maximal.
- If $M \neq 0$ is a multiplication module and Pu–local and if $K \neq 0$ is a Pu-maximal implies that K is the maximal of M.
- If M is a module also N is a pure submodule in M, now. $Rad_{Pu}(N) \leq Rad_{Pu}(M)$.

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