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Square root of fuzzy QS-ideals on QS-algebras

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Abstract.

In real-life problems, we use square roots in natural distributions such as (the probability density function), distances and lengths in the Pythagorean theorem, and quadratic formulas in (the height of falling objects), radius of circles, harmonic movements (pendulum and springs), and standard deviation in statistics. We have observed that using fuzzy sets in real-life problems is more convenient than ordinary sets. Therefore, they are important in algebraic structures. As a result, more effort has been made to study square root structures in fuzzy sets.

This paper introduces the notion of square roots fuzzy of QS-ideals on QS-algebras and some important characteristics. Some illustrative examples have been provided which prove that every SRF-BCK-ideal is an SRF QS-ideal. Also, the image and the inverse image of SRF-QS-ideals are discussed. Finally, the product of SRF-QS-ideals on QS-algebra is defined and some important properties have been proved.

Keywords: QS-algebras, QS-ideal, fuzzy QS-ideal, the image of SRF-QS-ideals and the product of SRF-QS-ideals.

الجذر التربيعي للمثاليات الضبابية – QS على الجبريات-QS

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الخلاصة

في المسائل الواقعية نستخدم الجذور التربيعية في التوزيعات الطبيعية مثل (دالة الكثافة الاحتمالية)، والمسافات والأطوال في نظرية فيثاغورس، والصيغة التربيعية في (ارتفاع الأجسام الساقطة)، ونصف قطر الدوائر، و الحركة التوافقية البسيطة (البندولات والزنبركات) والانحراف المعياري في الاحصاء. لقد تبين أن استخدام المجموعات الضبابية أكثر ملاءمة في مشاكل الحياة الواقعية من المجموعات العادية، لذلك فهي مهمه في حالة الهياكل الجبرية. ونتيجة لذلك، بُذلت جهود كبيرة لدراسة الجذور التربيعية في البنى الجبرية الغامضة.

في هذا البحث، نقدم فكرة الجذر التربيعي الضبابي (المشار إليه بواسطة SRF) للمثاليات– QS على جبر– QS وبعض الخصائص المهمة للمثاليات– QS الضبابية للجذر التربيعي. بعض الامثلة التوضيحية تم تقديمها وكذلك تم اثبات ان كل مثالي –SRF–BCK هو مثالي– SRF QS. كما تمت مناقشة الصورة

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والصورة العكسية للمثاليات- SRF-QS. اخيرا، تم تقديم الضرب للمثاليات- SRF-QS في الجبر-QSوتم اثبات بعض الخصائص المهمه.

1. Introduction

The concept of fuzzy sets was introduced by Zadeh [1]. In 1991, Xi [2] applied the idea of fuzzy sets to BCI, BCK, and MV-algebras. Since its inception, the theory of fuzzy sets, ideal theory, and fuzzification have been developed in many directions and applied to various fields. Yager [3] introduced the Pythagorean fuzzy set, an extension of intuitionistic fuzzy sets. Al-shami et al [4] referred to SR-Fuzzy sets. Salih et al [5] introduced a new type of generalized fuzzy set called CR-fuzzy sets and compared them to Pythagorean and Fermatean fuzzy sets. A novel generalized Pythagorean fuzzy set known as (3, 2)-Fuzzy sets were examined in [6]. Senapati and Yager [7] also developed fundamental operations for Fermatean fuzzy sets. The (m; n)-fuzzy set is a subclass of different fuzzy sets, such as the Fermatean fuzzy set, Pythagorean fuzzy set, n-Pythagorean fuzzy set, (3; 2)-fuzzy set and intuitionistic fuzzy set that were proposed and compared by Jun et al [8]. An innovative function on fuzzy real numbers was introduced by Byun et al. [9]. It is defined on the fuzzy real number set R_n and denotes a novel notion of a fuzzy real number. Bashar and Shirin [10] introduced the square root of continuous fuzzy numbers. Also, Chou [11] studied the square root of a triangular fuzzy number. Saeed and Ibrahim [12] studied square roots and their applications in topology. The idea of regularized square-root was introduced by Chu et al [13] and Stucky and Geer [14]. Then, square roots were introduced into computer programs and approximate solutions by Belloni et al. [15] and Jiang [16].

This work introduces the notion of square root fuzzy (Briefly SRF) of QS-ideals on QS-algebras and some important characteristics of square root fuzzy QS-ideals. The image and the inverse image of SRF-QS-ideals are also discussed. Moreover, the product of SRF-QS-ideals in product QS-algebras is introduced.

2. Preliminaries

Definition 2.1 [17]. A non-empty set $\wp$ with a constant 0 and a binary operation $*$ is called a QS-algebra$(\wp,*,0)$ if: for all $\alpha,\eta,\lambda \in \wp$

(1) $(\alpha * \eta) * \lambda = (\alpha * \lambda) * \eta$.

(2) $\alpha * 0 = \alpha$.

(3) $\alpha * \alpha = 0$.

(4) $(\alpha * \lambda) * (\alpha * \eta) = \eta * \lambda$.

Definition 2.2 [17]. A relation $\leq$ on a QS-algebra $(\wp,*,0)$ is defined as $\alpha \leq \lambda$ if and only if $\alpha * \lambda = 0$, then $\wp$ is a partially ordered set.

Proposition 2.3 [17]. Let $(\wp,*,0)$ be a QS-algebra. Then the following properties hold: for all $\alpha,\lambda,\eta \in \wp$

1. $\alpha \leq \lambda \Rightarrow \eta * \alpha \leq \eta * \lambda$.
2. $\alpha \leq \lambda$ and $\eta \leq \lambda$ imply $\alpha \leq \eta$.
3. $\alpha * \lambda \leq \eta \Rightarrow \alpha * \eta \leq \lambda$.
4. $(\alpha * \eta) * (\lambda * \eta) \leq \alpha * \lambda$.
5. $\alpha \leq \lambda \Rightarrow \alpha * \eta \leq \lambda * \eta$.
6. $0 * (0 * (0 * \alpha)) = 0 * \alpha$.

Lemma 2.4 [17]. In a QS-algebra $(\wp,*,0)$. $0 * (\alpha * \lambda) = \lambda * \alpha$ for all $\alpha,\lambda \in \wp$.

Corollary 2.5 [17]. In a QS-algebra $(\wp,*,0)$. $0 * (0 * \alpha) = \alpha$ for all $\alpha \in \wp$.

Lemma 2.6 [17]. In a QS-algebra$(\wp,*,0)$. $\alpha * (0 * \lambda) = \lambda * (0 * \alpha)$ for all $\alpha,\lambda \in \wp$

Proposition 2.7 [17]. In a QS-algebra$(\wp,*,0)$. The following hold for all $\alpha,\lambda,\eta \in \wp$

1. $\alpha * (\alpha * \lambda) = \lambda$.
2. $\alpha * \left(\alpha * (\alpha * \lambda)\right) = \alpha * \lambda$.
3. $\left(\alpha * \ (\alpha * \lambda)\right) * \lambda = 0$.
4. $(\alpha * \eta) * (\lambda * \eta) = v * \eta$.
5. $(\alpha * \lambda) * = 0 * \lambda$.
6. $\alpha * 0 = 0 \Longrightarrow \alpha = 0$.
7. $0 * (\alpha * \lambda) = (0 * \alpha) * (0 * \lambda)$.
8. $\alpha * \lambda = 0 , \lambda * \alpha = 0 \Longrightarrow \alpha = \lambda$.

Example 2.8[18]. Let $\wp = \{0, l, m\}$ be a set with an operation $*$ is defined as follows:

Table 1: A QS-algebra

$*$	0	$\boldsymbol{l}$	$\boldsymbol{m}$
0	0	m	l
$\boldsymbol{l}$	l	0	m
$\boldsymbol{m}$	m	l	0

Then, $(\wp, * ,0)$ is a QS-algebra.

Definition 2.9 [17]. In a QS-algebra $(\wp, * ,0)$. We denote $\alpha \wedge \lambda = \lambda * (\lambda * \alpha), \forall \alpha , \lambda \in \wp$.

Definition 2.10 [18]. The sub set $\varphi \neq I \subseteq \wp$ is called a sub-algebra of $\wp$ if $\alpha * \lambda \in I$ whenever $\alpha , \lambda \in I$.

Definition 2.11[19]. The sub set $\varphi \neq I \subseteq \wp$ is called a QS-ideal of $\wp$ if: $\forall \alpha, \lambda, \eta \in \wp$

(I_1) $0 \in I$

(I_2) $(\alpha * \eta) \in I , \eta * \lambda \in I \Rightarrow \alpha * \lambda \in I$

Definition 2.12[1]. A fuzzy set A of $\wp$ is of the form $A = \{\left(\alpha, \mu_A(\alpha)\right), \alpha \in \wp\}$, where $\mu_A(\alpha): \wp \to [0,1]$ is called the degree of existence of the element α in the set A and $0 \leq \mu_A(\alpha) \leq 1$

Definition 2.13 [19]. Let $(\wp, * ,0)$ be a QS-algebra. A fuzzy set μ in $(\wp, * ,0)$ is called a fuzzy QS-ideal if **:**

(FQS_1) $\mu(0) \geq \mu(\alpha)$,

(FQS_2) $\mu(\alpha * \lambda) \geq \min\{\mu((\alpha * \eta)), \mu(\alpha * \lambda)\}$ for all α, λ *and* $\eta \in \wp$.

Definition 2.14[20]. An Intuitionistic fuzzy set A of a set $\wp$ is the form $A = \{(\alpha, \mu_A(\alpha), \gamma_A(\alpha)), \alpha \in \wp\}$, where the function $\mu_A(\alpha): \wp \to [0,1]$ is the degree of membership and $\gamma_A(\alpha): \wp \to [0,1]$ is the degree of non-membership and $0 \leq \mu_A(\alpha) + \gamma_A(\alpha) \leq 1$, for all $\alpha \in \wp$. An intuitionistic fuzzy set A in $\wp$ can be identified as an order pair (μ_A, γ_A) in $I^{\wp} \times I^{\wp}$.

Remark 2.15. Let $\mu_A(\alpha): \wp \to [0,1]$ and $\gamma_A(\alpha): \wp \to [0,1]$ be fuzzy sets in $\wp$. Then, the structure

$A = \{(\alpha, \mu_A(\alpha), \gamma_A(\alpha)), \alpha \in \wp\}$ is called:

(1) An intuitionistic fuzzy set if: $0 \leq \mu_A(\alpha) + \gamma_A(\alpha) \leq 1$, (see **[20]**.)

(2) A Pythgorean fuzzy set if: $0 \leq \mu_A^2(\alpha) + \gamma_A^2(\alpha) \leq 1$, (see **[3]**.)

(3) An n-Pythgorean fuzzy set if: $0 \leq \mu_A^n(\alpha) + \gamma_A^n(\alpha) \leq 1$, (see **[3])**.

(4) A Fermatean fuzzy set if**:** $0 \leq \mu_A^3(\alpha) + \gamma_A^3(v) \leq 1$, (see [7])

(5) (3; 2)-fuzzy set if: $0 \leq \mu_A^3(\alpha) + \gamma_A^2(\alpha) \leq 1$, (see [8],[6])

(6) (n; m)-fuzzy set if: $0 \leq \mu_A^n(\alpha) + \gamma_A^m(\alpha) \leq 1$, (see [8])

(7) SR-Fuzzy set fuzzy set if: $0 \leq \mu_A^2(\alpha) + \sqrt{\gamma_A(\alpha)} \leq 1$, (see [8]).

3. SRF-QS-(sub-algebra) ideals on QS-algebras

Definition3.1. Let $\wp$ be a non-empty set. Then a pair $Y = \sqrt{\mathcal{L}(\alpha)} = (\sqrt{\mathcal{L}(\alpha)}^{(+)}, \sqrt{\mathcal{L}(\alpha)}^{(-)} : \alpha \in \wp)$ is named a square root fuzzy set (briefly, SRFS) of $\wp$, if $\sqrt{\mathcal{L}}^{(+)} : \wp \to [0,1]$ and $\sqrt{\mathcal{L}}^{(-)} : \wp \to [-1,0]$ are mappings, such that $0 \le \sqrt{\mathcal{L}}^{(+)} \le 1$, $-1 \le \sqrt{\mathcal{L}}^{(-)} \le 0$.

Example3.2. Define $Y = \{\sqrt{\mathcal{L}(\alpha)} = \left(\sqrt{\mathcal{L}(\alpha)}^{(+)}, \sqrt{\mathcal{L}(\alpha)}^{(-)} : \alpha \in \wp\right)\}$ in example 2.8 as follows:

$\mathcal{L}(0) = \{0.06\}$, $\mathcal{L}(l) = \{0.05\}, \mathcal{L}(m) = \{0.04\}$, since $\sqrt{0.06} = (0.2449489742783178, -0.2449489742783178)$

$\sqrt{0.05} = (0.223606797749979, -0.223606797749979)$

$\sqrt{0.04} = (0.2, -0.2)$.

It is easy to check that $\sqrt{\mathcal{L}(\alpha)} = (\sqrt{\mathcal{L}(\alpha)}^{(+)}, \sqrt{\mathcal{L}(\alpha)}^{(-)} : \alpha \in \wp)$ is a SRF-set.

Definition 3.3. Let $\sqrt{\mathcal{L}(\alpha)} = (\sqrt{\mathcal{L}(\alpha)}^{(+)}, \sqrt{\mathcal{L}(\alpha)}^{(-)} : \alpha \in \wp)$ and $\sqrt{\Psi(\alpha)} = (\sqrt{\Psi(\alpha)}^{(+)}, \sqrt{\Psi(\alpha)}^{(-)} : \alpha \in \wp)$ be two *S*RF- sets of $\wp$, then we say:

$\mathcal{L} \subseteq \Psi$ if and only if $\sqrt{\mathcal{L}(\alpha)}^{(+)} \le \sqrt{\Psi(\alpha)}^{(+)}, \sqrt{\mathcal{L}(\alpha)}^{(-)} \ge \sqrt{\Psi(\alpha)}^{(-)}$.

Remark 3.4.

1. Define $\sqrt{\mathcal{L}(\alpha)} = \sqrt{0.2}$ and $\sqrt{\Psi(\alpha)} = \sqrt{0.3}$ be two SRF-sets of $\wp$, then $\sqrt{\mathcal{L}} \le \sqrt{\Psi}$, since

$\sqrt{\mathcal{L}}^{(+)} = 0.4472135954999579 < \sqrt{\Psi}^{+} = 0.5477225575051661$,

$\sqrt{\mathcal{L}}^{(-)} = 0.4472135954999579 > \sqrt{\Psi}^{-} = -0.5477225575051661$.

2. $\sqrt{\mathcal{L}} = \sqrt{\Psi}$ if and only if $\sqrt{\mathcal{L}}^{\mp} = \sqrt{\Psi}^{\mp}$.

3. The complement of the fuzzy square root set is denoted by $(\sqrt{\mathcal{L}(\alpha)})^c = (\alpha \in \wp; (\sqrt{\mathcal{L}(\alpha)}^{(+)})^C, (\sqrt{\mathcal{L}(\alpha)}^{(-)})^C)$ and it is a square root fuzzy set in $\wp$ defined by: $(\sqrt{\mathcal{L}(\alpha)})^c = ((\sqrt{\mathcal{L}(\alpha)}^{(-)})^C, (\sqrt{\mathcal{L}(\alpha)}^{(+)})^C ; \alpha \in \wp\ ) = \left(-1 - \sqrt{\mathcal{L}(\alpha)}^{(-)}, 1 - \sqrt{\mathcal{L}(\alpha)}^{(+)}\right)$.

For example. Define $\sqrt{\mathcal{L}(\alpha)} = \sqrt{0.2}$, then

$\sqrt{\mathcal{L}(\alpha)})^c = (-1 - \sqrt{\mathcal{L}(\alpha)}^{(-)}, 1 - \sqrt{\mathcal{L}(\alpha)}^{(+)}) = (-1 + 0.4472135954999579, 1 - 0.4472135954999579) \approx (-0.55278640450011\ , 0.5527864045001\ )$.

4. The intersection of two square root fuzzy sets $\sqrt{\mathcal{L}(\alpha)}$ and $\sqrt{\Psi(\alpha)}$ denoted by $\sqrt{\mathcal{L}(\alpha)} \cap \sqrt{\Psi(\alpha)}$, is a square root fuzzy set in $\wp$ defined as: for each $\alpha \in \wp$,

$(\sqrt{\mathcal{L}(\alpha)} \cap \sqrt{\Psi(\alpha)}) = \{ \sqrt{\mathcal{L}(\alpha)}^{(+)} \wedge \sqrt{\Psi(\alpha)}^{(+)}, \sqrt{\mathcal{L}(\alpha)}^{(-)} \vee \sqrt{\Psi(\alpha)}^{(-)} \}$.

For example. Define $\sqrt{\mathcal{L}(\alpha)} = \sqrt{0.2}$ and $\sqrt{\Psi(\alpha)} = \sqrt{0.3}$ be two SRF- sets of $\wp$, then

$\left(\sqrt{\mathcal{L}(\alpha)} \cap \sqrt{\Psi(\alpha)}\right) = \left\{ \sqrt{\mathcal{L}(\alpha)}^{(+)} \wedge \sqrt{\Psi(\alpha)}^{(+)}, \sqrt{\mathcal{L}(\alpha)}^{(-)} \vee \sqrt{\Psi(\alpha)}^{(-)} \right\} = \{0.4472135954999579, -0.4472135954999579\}$.

5. The union of two square root fuzzy sets $\sqrt{\mathcal{L}(\alpha)}$ and $\sqrt{\Psi(\alpha)}$ denoted by $\sqrt{\mathcal{L}(\alpha)} \cup \sqrt{\Psi(\alpha)}$ is a square root fuzzy set in $\wp$ defined as: for all $\alpha \in \wp$,

$(\sqrt{\mathcal{L}(\alpha)} \cup \sqrt{\Psi(\alpha)}) = \left\{ \sqrt{\mathcal{L}(\alpha)}^{(+)} \vee \sqrt{\Psi(\alpha)}^{(+)}, \sqrt{\mathcal{L}(\alpha)}^{(-)} \wedge \sqrt{\Psi(\alpha)}^{(-)} \right\}$.

6. Results. Let $\sqrt{\mathcal{L}}, \sqrt{\Psi}, \sqrt{\Omega}$ be square root fuzzy sets in a set $\wp$. Then
(1) $\left(\sqrt{\mathcal{L}} \cap \sqrt{\mathcal{L}}\right) = \sqrt{\mathcal{L}},\ \left(\sqrt{\mathcal{L}} \cup \sqrt{\mathcal{L}}\right) = \sqrt{\mathcal{L}}$.
(2) $\left(\sqrt{\mathcal{L}} \cap \sqrt{\Psi}\right) = \sqrt{\Psi} \cap \sqrt{\mathcal{L}}, \left(\sqrt{\mathcal{L}} \cup \sqrt{\Psi}\right) = \sqrt{\Psi} \cup \sqrt{\mathcal{L}}$.
(3) $\left(\sqrt{\mathcal{L}} \cap \sqrt{\Psi}\right) \cap \sqrt{\Omega} = \sqrt{\mathcal{L}} \cap \left(\sqrt{\Psi} \cap \sqrt{\Omega}\right),\ \left(\sqrt{\mathcal{L}} \cup \sqrt{\Psi}\right) \cup \sqrt{\Omega} = \sqrt{\mathcal{L}} \cup (\sqrt{\Psi} \cup \sqrt{\Omega})$.

Definition 3.5. Let $\wp \neq \varphi$ be a QS-algebra, and then a SRF-set $\sqrt{\mathcal{L}(\alpha)}$ over a set $\wp$ is called SRF-QS-sub algebras if the following are satisfies
(***S***RF-S$_1$) $\sqrt{\mathcal{L}(\alpha * \eta)}^{(+)} \geq min\{\sqrt{\mathcal{L}(\alpha)}^{(+)}, \sqrt{\mathcal{L}(\eta)}^{(+)}\}$ and
(***S***RF-S$_2$) $\sqrt{\mathcal{L}(\alpha * \eta)}^{(-)} \leq max\{\sqrt{\mathcal{L}(\alpha)}^{(-)}, \sqrt{\mathcal{L}(\eta)}^{(-)}\}$, where
$\sqrt{\mathcal{L}(\alpha)}^{(+)} : \wp \to [0,1]$, $\sqrt{\mathcal{L}(\alpha)}^{(-)} : \wp \to [-1,0]$ such that $0 \leq \sqrt{\mathcal{L}(\alpha)}^{(+)} \leq 1$,
$-1 \leq \sqrt{\mathcal{L}(\alpha)}^{(-)} \leq 0$.

Example3.6. Define $\sqrt{\mathcal{L}}$ in example 2.9 as follows:
$\mathcal{L}(0) = \{0.06\},\ \mathcal{L}(l) = \{0.05\}, \mathcal{L}(m) = \{0.04\}$, then $\sqrt{\mathcal{L}}$ is SRF-QS-sub algebras.

Definition 3.7. Let $\wp \neq \varphi$ be QS-algebras, then the SRF set $\sqrt{\mathcal{L}}$ over a set $\wp$ is called SRF-BCK-ideal if: for all $\alpha, \lambda \in \wp$, where BCK-ideal is an ideal of BCK-algebra.
(SRF-bck (a)) $\sqrt{\mathcal{L}(0)}^{(+)} \geq \sqrt{\mathcal{L}(\alpha)}^{(+)}, \sqrt{\mathcal{L}(0)}^{(-)} \leq \sqrt{\mathcal{L}(\alpha)}^{(-)}$,
(SRF-bck (b)) $\sqrt{\mathcal{L}(\alpha)}^{(+)} \geq min\{\sqrt{\mathcal{L}(\alpha * \lambda)}^{(+)}, \sqrt{\mathcal{L}(\lambda)}^{(+)}\}$,
$\sqrt{\mathcal{L}(\alpha)}^{(-)} \leq max\{\sqrt{\mathcal{L}(\alpha * \lambda)}^{(-)}, \sqrt{\mathcal{L}(\lambda)}^{(-)}\}$, such that, $0 \leq \sqrt{\mathcal{L}(\alpha)}^{(+)} \leq 1$,
$-1 \leq \sqrt{\mathcal{L}(\alpha)}^{(-)} \leq 0$.

Definition 3.8. Let $\wp \neq \varphi$ be QS - algebras, then the SRF-fuzzy set $\sqrt{\mathcal{L}}$ over a set $\wp$ is called SRF-QS-ideal if: for all $\alpha, \lambda, \eta \in \wp$
(SRF-QSI$_0$): $\sqrt{\mathcal{L}(0)}^{(+)} \geq \sqrt{\mathcal{L}(\alpha)}^{(+)}, \sqrt{\mathcal{L}(0)}^{(-)} \leq \sqrt{\mathcal{L}(\alpha)}^{(-)}$,
(SRF-QS I$_1$): $\sqrt{\mathcal{L}(\alpha * \lambda)}^{(+)} \geq \min\{\sqrt{\mathcal{L}(\alpha * \eta)}^{(+)}, \sqrt{\mathcal{L}(\eta * \lambda)}^{(+)}\}$,
$\sqrt{\mathcal{L}(\alpha * \lambda)}^{(-)} \leq \max\{\sqrt{\mathcal{L}(\alpha * \eta)}^{(-)}, \sqrt{\mathcal{L}(\eta * \lambda)}^{(-)}\}$, such that, $0 \leq \sqrt{\mathcal{L}(\alpha)}^{(+)} \leq 1$,
$-1 \leq \sqrt{\mathcal{L}(\alpha)}^{(-)} \leq 0$.

Example 3.9. Let $\wp = \{0, h, p, u\}$ be a set and $*$ defined in table 2:

Table 2: SRF-QS-ideal

$*$	0	h	p	u
0	0	h	p	u
h	h	0	u	p
p	p	u	0	h
u	u	p	h	0

Then $(\wp, *\ 0)$ is a QS-algebra. Define $\sqrt{\mathcal{L}}$ as follows:
$\mathcal{L}(0) = \{0.04\},\ \mathcal{L}(h) = \{0.03\}, \mathcal{L}(p) = \{0.02\}, \mathcal{L}(u) = \{0.01\}$.
We can prove that $\sqrt{\mathcal{L}}$ is SRF-QS-ideal on QS-algebra $\wp$.

Lemma 3.10. If $\sqrt{\mathcal{L}}$ is SRF -QS-sub algebra on QS- algebra $\wp$, then
$\sqrt{\mathcal{L}(0)}^{(+)} \geq \sqrt{\mathcal{L}(\alpha)}^{(+)}$ and $\sqrt{\mathcal{L}(0)}^{(-)} \leq \sqrt{\mathcal{L}(\alpha)}^{(-)}$

Proof. In (Definition 3.5) put $= \eta$,
(***S***RF-S$_1$) $\sqrt{\mathcal{L}(\alpha * \alpha)}^{(+)} = \sqrt{\mathcal{L}(0)}^{(+)} \geq min\sqrt{\mathcal{L}(\alpha)}^{(+)}, \sqrt{\mathcal{L}(\alpha)}^{(+)}\} = \sqrt{\mathcal{L}(\alpha)}^{(+)}$, and

(**S**RF-S$_2$) $\sqrt{\mathcal{L}(\alpha*\alpha)}^{-}=\sqrt{\mathcal{L}(0)}^{(-)}\leq max\sqrt{\mathcal{L}(\alpha)}^{(-)},\sqrt{\mathcal{L}(\alpha)}^{(-)}\}=\sqrt{\mathcal{L}(\alpha)}^{(-)}$.

Lemma 3.11. Let $\sqrt{\mathcal{L}}$ be SRF-QS-ideal of $\wp$. If $\alpha\leq\eta$ in $\wp$, then

$\sqrt{\mathcal{L}(\alpha)}^{(+)}\geq\sqrt{\mathcal{L}(\eta)}^{(+)}$ and $\sqrt{\mathcal{L}(\alpha)}^{(-)}\leq\sqrt{\mathcal{L}(\eta)}^{(-)}$, $\forall\alpha,\eta\in\wp$.

Proof. Let $\alpha,\lambda\in\wp$ be such that $\alpha\leq\eta$, then $\alpha*\eta=0$. Put $\lambda=0$, we have from (SRF-QSI$_0$):

$\sqrt{\mathcal{L}(\alpha)}^{(+)}\geq\sqrt{\mathcal{L}(\eta)}^{(+)}$, $\sqrt{\mathcal{L}(\alpha)}^{(-)}\leq\sqrt{\mathcal{L}(\eta)}^{(-)}$, and

(SRF-QS I$_1$): $\sqrt{\mathcal{L}(\alpha*0)}^{(+)}=\sqrt{\mathcal{L}(\alpha)}^{(+)}\geq\min\left\{\sqrt{\mathcal{L}(0)}^{(+)},\sqrt{\mathcal{L}(\alpha)}^{(+)}\right\}=\sqrt{\mathcal{L}(\alpha)}^{(+)}$,

$\sqrt{\mathcal{L}(\alpha*0)}^{(-)}=\sqrt{\mathcal{L}(\alpha)}^{(-)}\leq\max\left\{\sqrt{\mathcal{L}(0)}^{(-)},\sqrt{\mathcal{L}(\alpha)}^{(-)}\right\}=\sqrt{\mathcal{L}(\alpha)}^{(-)}$.

Lemma 3.12. Let $\sqrt{\mathcal{L}}$ be SRF-QS-ideal of $\wp$, if $\alpha*\lambda\leq\eta$, then $\sqrt{\mathcal{L}(\alpha)}^{(+)}\geq\min\left\{\sqrt{\mathcal{L}(\lambda)}^{(+)},\sqrt{\mathcal{L}(\eta)}^{(+)}\right\}$, $\sqrt{\mathcal{L}(\alpha)}^{(-)}\leq\max\left\{\sqrt{\mathcal{L}(\lambda)}^{(-)},\sqrt{\mathcal{L}(\eta)}^{(-)}\right\}$.

Proof. Assume that $\alpha*\lambda\leq\eta$ then $\alpha*\eta\leq\lambda$ holds in $\wp$. By (Lemma 3. 11), we have

$\sqrt{\mathcal{L}(\alpha*\eta)}^{(+)}\geq\sqrt{\mathcal{L}(\lambda)}^{(+)}$ and $\sqrt{\mathcal{L}(\alpha*\eta)}^{(-)}\leq\sqrt{\mathcal{L}(\lambda)}^{(-)}$.

Thus, put $\lambda=0$ in (definition 3.7.) and using lemma 3.10, then we get from

(SRF-QSI$_0$) : $\sqrt{\mathcal{L}(0)}^{(+)}\geq\sqrt{\mathcal{L}(\alpha)}^{(+)}$ and $\sqrt{\mathcal{L}(0)}^{(-)}\leq\sqrt{\mathcal{L}(\alpha)}^{(-)}$

(SRF-QS I$_1$):

$\sqrt{\mathcal{L}(\alpha*0)}^{(+)}=\sqrt{\mathcal{L}(\alpha)}^{(+)}\geq\min\{\sqrt{\mathcal{L}(\alpha*\eta)}^{(+)},\sqrt{\mathcal{L}(\eta*0)}^{(+)}\}\geq$
$min\{\sqrt{\mathcal{L}(\lambda)}^{(+)}\sqrt{\mathcal{L}(\eta)}^{(+)}\}$, $\sqrt{\mathcal{L}(\alpha*0)}^{(+)}=\sqrt{\mathcal{L}(\alpha)}^{(+)}\geq$
$\min\{\sqrt{\mathcal{L}(\alpha*\eta)}^{(+)},\sqrt{\mathcal{L}(\eta*0)}^{(+)}\}\geq min\{\sqrt{\mathcal{L}(\lambda)}^{(+)}\sqrt{\mathcal{L}(\eta)}^{(+)}\}$.

Theorem 3.13. In BCK- algebra, every SRF-BCK-ideal of $\wp$ is an SRF QS- ideal of $\wp$.

Proof. Let $\sqrt{\mathcal{L}}$ be SRF-BCK-ideal of $\wp$ and $\alpha,\lambda,\eta\in\wp$, then $\sqrt{\mathcal{L}(0)}^{(+)}\geq\sqrt{\mathcal{L}(\alpha)}^{(+)}$ and $\sqrt{\mathcal{L}(0)}^{(-)}\leq\sqrt{\mathcal{L}(\alpha)}^{(-)}$, it is (SRF-bck (a))

(SRF-bck (b)) $\sqrt{\mathcal{L}(\alpha)}^{(+)}\geq\min\{\sqrt{\mathcal{L}(\alpha*\lambda)}^{(+)},\sqrt{\mathcal{L}(\lambda)}^{(+)}\}$,

$\sqrt{\mathcal{L}(\alpha)}^{(-)}\leq\max\{\sqrt{\mathcal{L}(\alpha*\lambda)}^{(-)},\sqrt{\mathcal{L}(\lambda)}^{(-)}\}$. Now, put $\alpha*\lambda$ instate of α and $\eta*\lambda$ instate of λ in (SRF-bck) (a) and (SRF-bck)(b) we have (SRF-bck (b))

$$\sqrt{\mathcal{L}(\alpha*\lambda)}^{(+)}\geq\min\left\{\sqrt{\mathcal{L}\left(\overbrace{(\alpha*\lambda)*(\eta*\lambda)}^{(\alpha*\lambda)*(\eta*\lambda)\leq\alpha*\eta\quad by\ proposition 2.3(4)}\right)}^{(+)},\right.$$

$$\sqrt{\mathcal{L}(\eta*\lambda)}^{(+)}\geq\{\sqrt{\mathcal{L}\overbrace{(\alpha*\eta)}^{lamma\ 3.10}}^{(+)},\sqrt{\mathcal{L}(\eta*\lambda)}^{(+)}\}.$$

Similarly, we can prove that

$\sqrt{\mathcal{L}(\eta*\lambda)}^{(+)}\geq\{\sqrt{\mathcal{L}\overbrace{(\alpha*\eta)}^{lamma\ 3.10}}^{(+)},\sqrt{\mathcal{L}(\eta*\lambda)}^{(+)}\}$, then $\sqrt{\mathcal{L}}$ is SRF-QS-ideal of $\wp$.

Theorem 3.14. If $\sqrt{\mathcal{L}}$ is SRF- QS-ideal of $\wp$, then the set

$\wp=\{(v,\sqrt{\mathcal{L}(0)}^{(+)}=\sqrt{\mathcal{L}(v)}^{(+)},\sqrt{\mathcal{L}(0)}^{(-)}=\sqrt{\mathcal{L}(v)}^{(-)};v\in\wp\}$ is a QS-ideal of $\wp$.

Proof. Let $v*\eta$, $\eta*\lambda\in\wp$. Then

$\sqrt{\mathcal{L}(v*\eta)}^{(+)}=\sqrt{\mathcal{L}(0)}^{(+)}=\sqrt{\mathcal{L}(\eta*\lambda)}^{(+)}$ *and* $\sqrt{\mathcal{L}(v*\eta)}^{(-)}=\sqrt{\mathcal{L}(0)}^{(-)}=\sqrt{\mathcal{L}(\eta*\lambda)}^{(-)}$,

we have (SRF-QS I$_1$):

$\sqrt{\mathcal{L}(v*\lambda)}^{(+)} \geq min\left\{\sqrt{\mathcal{L}(v*\eta)}^{(+)}, \sqrt{\mathcal{L}(\eta*\lambda)}^{(+)}\right\}$
$= min\,\{\sqrt{\mathcal{L}(0)}^{(+)}, \sqrt{\mathcal{L}(0)}^{(+)}\} = \sqrt{\mathcal{L}(0)}^{(+)}$, and
$\sqrt{\mathcal{L}(v*\lambda)}^{(-)} \leq max\left\{\sqrt{\mathcal{L}(v*\eta)}^{(-)}, \sqrt{\mathcal{L}(\eta*\lambda)}^{(-)}\right\}$
$= max\,\{\sqrt{\mathcal{L}(0)}^{(-)}, \sqrt{\mathcal{L}(0)}^{(-)}\} = \sqrt{\mathcal{L}(0)}^{(-)}$.
By combining above with definition 3.7, we get
$\sqrt{\mathcal{L}(0)}^{(+)} \geq \sqrt{\mathcal{L}(v)}^{(+)}$ and $\sqrt{\mathcal{L}(0)}^{(-)} \leq \sqrt{\mathcal{L}(v)}^{(-)}$. So $\sqrt{\mathcal{L}(v*\eta)}^{(+)} = \sqrt{\mathcal{L}(0)}^{(+)}$ and $\sqrt{\mathcal{L}(v*\eta)}^{(-)} = \sqrt{\mathcal{L}(0)}^{(-)}$, then $v*\eta \in \wp$. Hence, $\wp$ is a QS-ideal of $\wp$.

Proposition 3.15. Let $(M_i)_{i\in\wedge} = \{(v, \left(\sqrt{\mathcal{L}(v)}\right)_{M_i}; v \in \wp\}$ be a family of SRF-QS-ideals of $\wp$, then $\cap M_i$, $i \in\wedge$ is SRF-QS-ideal of a QS-algebra $\wp$.

Proof. Let $(M_i)_{i\in\wedge}$ be a family of SRF-QS -ideals of a QS-algebras, then for any $v, \lambda, \eta \in \wp$ we get

$(\cap\left(\sqrt{\mathcal{L}(0)}^{(+)}\right)_{M_i}) = inf\left(\left(\sqrt{\mathcal{L}(0)}^{(+)}\right)_{M_i}\right) \geq inf\left(\sqrt{\mathcal{L}(v)}^{(+)}\right)_{M_i} = (\cap\left(\sqrt{\mathcal{L}(v)}^{(+)}\right)_{M_i})$
$(\cup\left(\sqrt{\mathcal{L}(0)}^{(-)}\right)_{M_i}) = sup\left(\left(\sqrt{\mathcal{L}(0)}^{(-)}\right)_{M_i}\right) \leq sup\left(\sqrt{\mathcal{L}(v)}^{(-)}\right)_{M_i} = (\cup\left(\sqrt{\mathcal{L}(v)}^{(-)}\right)_{M_i})$.

Now, we have

$(\cap\left(\sqrt{\mathcal{L}(v*\lambda)}^{(+)}\right)_{M_i}) = inf\left(\left(\sqrt{\mathcal{L}(v*\lambda)}^{(+)}\right)_{M_i}\right) \geq$
$inf\ \{min\left\{\left(\left(\sqrt{\mathcal{L}(v*\eta)}^{(+)}\right)_{M_i}, \left(\sqrt{\mathcal{L}(\eta*\lambda)}^{(+)}\right)_{M_i}\right\}\right\} =$
$min\,\{inf\left\{\left(\left(\sqrt{\mathcal{L}(v*\eta)}^{(+)}\right)_{M_i}, \left(\sqrt{\mathcal{L}(\eta*\lambda)}^{(+)}\right)_{M_i}\right\}\right\} =$
$min\,\{(\cap\left(\sqrt{\mathcal{L}(v*\eta)}^{(+)}\right)_{M_i}, \left(\cap\left(\sqrt{\mathcal{L}(\eta*\lambda)}^{(+)}\right)_{M_i}\right\}$. And
$(\cup\left(\sqrt{\mathcal{L}(v*\lambda)}^{(-)}\right)_{M_i} = sup\,(\left(\sqrt{\mathcal{L}(v*\lambda)}^{(-)}\right)_{M_i}) \leq$
$sup\left\{max\left\{\left(\sqrt{\mathcal{L}(v*\eta)}^{(-)}\right)_{M_i}, \left(\sqrt{\mathcal{L}(\eta*\lambda)}^{(-)}\right)_{M_i}\right\}\right\} =$
$max\left\{sup\left\{\left(\sqrt{\mathcal{L}(v*\eta)}^{(-)}\right)_{M_i}, \left(\sqrt{\mathcal{L}(\eta*\lambda)}^{(-)}\right)_{M_i}\right\}\right\}$
$= max\left\{\cup\left(\sqrt{\mathcal{L}(v*\eta)}^{(-)}\right)_{M_i}, (\cup\left(\sqrt{\mathcal{L}(\eta*\lambda)}^{(-)}\right)_{M_i}\right\}$. Hence, $\cap M_i$, $i \in\wedge$ is SRF QS-ideal.

4. The image (the inverse image) of SRF QS-ideal

Definition 4.1. Let $(\wp, *, 0)$ and $(\mathcal{H}, \dot{*}, \dot{0})$ be QS-algebras. A mapping $f: \wp \to \mathcal{H}$ is said to be a QS-homomorphism if $f(v*\lambda) = f(v)\,\dot{*}\,f(\lambda)$, for all $v, \lambda \in \wp$.

For example: using the set $\wp = \{0, h, p, u\}$ in the Example 3.9 and define a set $\mathcal{H} = \{0', a', b', c'\}$ in the following table in table 3:

Table 3: A QS-algebra

$*'$	$0'$	a'	b'	c'
$0'$	$0'$	a'	b'	c'
a'	a'	$0'$	c'	b'
b'	b'	c'	$0'$	a'
c'	c'	b'	a'	$0'$

A mapping $f:\wp \to \mathcal{H}$ is define as follows $f(v) = v'$, then f is a QS-homomorphism since $f(h * p) = f(u) = c' = f(h) * f(p) = a' * b'$ and so on.

Definition 4.2. Let $\wp$, $\mathcal{H}$ be two QS-algebras and ω be a fuzzy subset of $\wp$, and β be a fuzzy subset of $\mathcal{H}$ and $f:\wp \to \mathcal{H}$ a QS-homomorphism. The image of $\mathcal{L}$ under f is defined by

$$f(\mathcal{L})(\lambda) = \begin{cases} \sup_{v \in f^{-1}}^{\mathcal{L}(v)} & if \quad f^{-1}(\lambda) \neq \emptyset \\ 0 & otherwise \end{cases}$$

The inverse image of β under f is defined by:

$f^{-1}(\beta)(v) = \beta\big(f(v)\big)$, for all $v \in \wp$.

Definition 4.3. Let $f:\wp \to \mathcal{H}$ be a homomorphism of QS-algebras for any SRF-QS-ideal $\sqrt{\mathcal{L}} = \{\sqrt{\mathcal{L}(v)} = (\sqrt{\mathcal{L}(v)}^{(-)}, \sqrt{\mathcal{L}(v)}^{(+)} : v \in \wp\}$ of $\mathcal{H}$. Define a new SRFQS-ideal $\sqrt{\mathcal{L}}^{f} = \{v, (\sqrt{\mathcal{L}(v)}^{(-)})^{f}, (\sqrt{\mathcal{L}(v)}^{(+)})^{f} : v \in \wp)\}$ in $\wp$ by

$(\sqrt{\mathcal{L}(v)}^{(\pm)})^{f} = \sqrt{\mathcal{L}(f(v))}^{(\pm)}, for\ all\ v \in \wp.$

Theorem 4.4. Let $f:\wp \to \mathcal{H}$ be a QS-homomorphism. If

$\sqrt{\mathcal{L}} = \{\sqrt{\mathcal{L}(v)} = (\sqrt{\mathcal{L}(v)}^{(-)}, \sqrt{\mathcal{L}(v)}^{(+)} : v \in \wp\}$ is SRF-QS-ideal of $\mathcal{H}$, then

$\sqrt{\mathcal{L}}^{f} = \{v, (\sqrt{\mathcal{L}(v)}^{(-)})^{f}, (\sqrt{\mathcal{L}(v)}^{(+)})^{f} : v \in \wp\}$ is SRF-QS-ideal of $\wp$.

Proof. Notes that: (1) $(\sqrt{\mathcal{L}(v)}^{(+)})^{f} = \sqrt{\mathcal{L}(f(v))}^{(+)} \leq \sqrt{\mathcal{L}(f(0))}^{(+)} = (\sqrt{\mathcal{L}(0)}^{(+)})^{f}$

$(\sqrt{\mathcal{L}(v * \lambda)}^{(+)})^{f} = \sqrt{\mathcal{L}(f(v * \lambda))}^{(+)} \geq min\,\{\sqrt{\mathcal{L}(f(v) * f(\eta))}^{(+)}, \sqrt{\mathcal{L}(f(\eta) * f(\lambda))}^{(+)}\}$

$= min\,\{\sqrt{\mathcal{L}(f(v * \eta)}^{(+)}, \sqrt{\mathcal{L}(f(\eta * \lambda)}^{(+)}\}$

$= min\,\{(\sqrt{\mathcal{L}(v * \eta)}^{(+)})^{f},\ (\sqrt{\mathcal{L}(\eta * \lambda)}^{(+)})^{f}\}$ and

$(\sqrt{\mathcal{L}(v * \lambda)}^{(-)})^{f} = \sqrt{\mathcal{L}(f(v * \lambda))}^{(-)} \leq$

$max\,\{\sqrt{\mathcal{L}(f(v) * f(\eta))}^{(-)}, \sqrt{\mathcal{L}(f(\eta) * f(\lambda))}^{(-)}\} = max\,\{\sqrt{\mathcal{L}(f(v * \eta)}^{(-)}, \sqrt{\mathcal{L}(f(\eta * \lambda)}^{(-)}\}$

$= max\{(\sqrt{\mathcal{L}(v * \eta)}^{(-)})^{f},\ (\sqrt{\mathcal{L}(\eta * \lambda)}^{(-)})^{f}\}$. Hence $\sqrt{\mathcal{L}}^{f}$ is SRF-QS-ideal.

Theorem4.5. Let $f:\wp \to \mathcal{H}$ be an epimorphism of QS-algebras.

If $\sqrt{\mathcal{L}}^{f} = \{v, (\sqrt{\mathcal{L}(v)}^{(-)})^{f}, (\sqrt{\mathcal{L}(v)}^{(+)})^{f} : v \in \wp)\}$ is SRF-QS-ideal of $\wp$, then

$\sqrt{\mathcal{L}} = \{\sqrt{\mathcal{L}(v)} = (\sqrt{\mathcal{L}(v)}^{(-)}, \sqrt{\mathcal{L}(v)}^{(+)} : v \in \wp)\}$ is SRF-QS-ideal of $\mathcal{H}$.

Proof. For any $h \in \mathcal{H}$, there exists $v \in \wp$ such that $f(v) = h$. Then

$$\sqrt{\mathcal{L}(h)}^{(+)} = \sqrt{\mathcal{L}(f(v))}^{(+)} = (\sqrt{\mathcal{L}(v)}^{(+)})^{f} \leq (\sqrt{\mathcal{L}(0)}^{(+)})^{f} = \sqrt{\mathcal{L}\big(f(0)\big)}^{(+)} = (\sqrt{\mathcal{L}(0)}^{(+)})^{f},$$

$$\sqrt{\mathcal{L}(h)}^{(-)} = \sqrt{\mathcal{L}(f(v))}^{(-)} = (\sqrt{\mathcal{L}(v)}^{(-)})^{f} \geq (\sqrt{\mathcal{L}(0)}^{(-)})^{f} = \sqrt{\mathcal{L}\big(f(0)\big)}^{(-)} = (\sqrt{\mathcal{L}(0)}^{(-)})^{f}.$$

Let $h, w, u \in \mathcal{H}$, then $f(v) = h, f(\lambda) = w, f(\eta) = u$, for some $v, \lambda, \eta \in \wp$.

It follows that

$\sqrt{\mathcal{L}(h*w)}^{(+)} = \sqrt{\mathcal{L}(f(v)*f(\lambda))}^{(+)} = (\sqrt{\mathcal{L}(v*\lambda)}^{(+)})^{f} \geq$
$\min\left\{(\sqrt{\mathcal{L}(v*\eta)}^{(+)})^{f}, (\sqrt{\mathcal{L}(\eta*\lambda)}^{(+)})^{f}\right\} = min\{\sqrt{\mathcal{L}(f(v*\eta))}^{(+)}, \sqrt{\mathcal{L}(f(\eta*\lambda))}^{(+)}\} =$
$min\{\sqrt{\mathcal{L}(f(v)*f(\eta))}^{(+)}, \sqrt{\mathcal{L}(f(\eta)*f(\lambda))}^{(+)} = min\{\sqrt{\mathcal{L}(h*u)}^{(+)}, \sqrt{\mathcal{L}(u*w)}^{(+)}\}$, and
$\sqrt{\mathcal{L}(h*w)}^{(-)} = \sqrt{\mathcal{L}(f(v)*f(\lambda))}^{(-)} = (\sqrt{\mathcal{L}(v*\lambda)}^{(-)})^{f} \leq$
$max\left\{(\sqrt{\mathcal{L}(v*\eta)}^{(-)})^{f}, (\sqrt{\mathcal{L}(\eta*\lambda)}^{(-)})^{f}\right\} = max\{\sqrt{\mathcal{L}(f(v*\eta))}^{(-)}, \sqrt{\mathcal{L}(f(\eta*\lambda))}^{(-)}\} =$
$max\{\sqrt{\mathcal{L}(f(v)*f(\eta))}^{(-)}, \sqrt{\mathcal{L}(f(\eta)*f(\lambda))}^{(-)}\} = max\{\sqrt{\mathcal{L}(h*u)}^{(-)}, \sqrt{\mathcal{L}(u*w)}^{(-)}\}$.
Hence $\sqrt{\mathcal{L}}$ is SRF-QS-ideal of $\mathcal{H}$.

Definition 4.6. Let $\sqrt{\mathcal{L}_1} = \{\sqrt{\mathcal{L}_1(v)} = (\sqrt{\mathcal{L}_1(v)}^{(-)}, \sqrt{\mathcal{L}_1(v)}^{(+)} : v \in \wp)\}$ and $\sqrt{\mathcal{L}_2} = \{\sqrt{\mathcal{L}_2(v)} = \left(\sqrt{\mathcal{L}_2(v)}^{(-)}, \sqrt{\mathcal{L}_2(v)}^{(+)} : v \in \wp\right)\}$ be two *S*RF-fuzzy sets of $\wp$, the product $\sqrt{\mathcal{L}_1} \times \sqrt{\mathcal{L}_2}$ is defined by $\sqrt{\mathcal{L}_1 \times \mathcal{L}_2(v,\lambda)}^{(+)} = min\left\{\sqrt{\mathcal{L}_1(v)}^{(+)}, \sqrt{\mathcal{L}_2(\lambda)}^{(+)}\right\}$ and $\sqrt{\mathcal{L}_1 \times \mathcal{L}_2(v,\lambda)}^{(-)} = max\left\{\sqrt{\mathcal{L}_1(v)}^{(-)}, \sqrt{\mathcal{L}_2(\lambda)}^{(-)}\right\}$, for all $v, \lambda \in \wp$.

Remark 4.7. Let $\wp$ and $\mathcal{H}$ be two QS-algebras, we define $*$ on $\wp \times \mathcal{H}$ by the following: $\forall (v,\lambda), (\rho,\tau) \in \wp \times \mathcal{H}$, $(v,\lambda)*(\rho,\tau) = (v*\rho, \lambda*\tau)$. Clearly $(\wp \times \mathcal{H}, *, (0,0))$ is a QS-algebra.

Proposition 4.8. If $\sqrt{\mathcal{L}_1} = \{\sqrt{\mathcal{L}_1(v)} = (\sqrt{\mathcal{L}_1(v)}^{(-)}, \sqrt{\mathcal{L}_1(v)}^{(+)} : v \in \wp)\}$ and $\sqrt{\mathcal{L}_2} = \{\sqrt{\mathcal{L}_2(v)} = \left(\sqrt{\mathcal{L}_2(v)}^{(-)}, \sqrt{\mathcal{L}_2(v)}^{(+)} : v \in \wp\right)\}$ be two SRF QS-ideals of $\wp$, then $\sqrt{\mathcal{L}_1} \times \sqrt{\mathcal{L}_2}$ is SRF-QS-ideal of $\wp \times \mathcal{H}$.

Proof. $\sqrt{\mathcal{L}_1 \times \mathcal{L}_2(0,0)}^{(+)} = min\left\{\sqrt{\mathcal{L}_1(0)}^{(+)}, \sqrt{\mathcal{L}_2(0)}^{(+)}\right\} \geq \min\left\{\sqrt{\mathcal{L}_1(v)}^{(+)}, \sqrt{\mathcal{L}_2(\lambda)}^{(+)}\right\} =$
$\sqrt{\mathcal{L}_1 \times \mathcal{L}_2(0,0)(v,\eta)}^{(+)}$.
For all $v, \lambda \in \wp$. $let (v_1,v_2), (\lambda_1,\lambda_2), (\eta_1,\eta_2) \in \wp \times \wp$, then
$min\{\sqrt{(\mathcal{L}_1 \times \mathcal{L}_2)(v_1,v_2)*(\eta_1,\eta_2)}^{(+)}, \sqrt{(\mathcal{L}_1 \times \mathcal{L}_2)(\eta_1,\eta_2)*(\lambda_1,\lambda_2)}^{(+)}\} =$
$min\{\sqrt{(\mathcal{L}_1 \times \mathcal{L}_2)(v_1*\eta_1, v_2*\eta_2)}^{(+)}, \sqrt{(\mathcal{L}_1 \times \mathcal{L}_2)(\eta_1*\lambda_1, \eta_2*\lambda_2)}^{(+)}\}$
$=$
$min\{min\{\sqrt{\mathcal{L}_1(v_1*\eta_1)}^{(+)}, \sqrt{\mathcal{L}_2(v_2*\eta_2)}^{(+)}\}, min\{\sqrt{\mathcal{L}_1(\eta_1*\lambda_1)}^{(+)}, \sqrt{\mathcal{L}_2(\eta_2*\lambda_2)}^{(+)}\}\} =$
$= min\{\min\{\sqrt{\mathcal{L}_1(v_1*\eta_1)}^{(+)}, \sqrt{\mathcal{L}_1(\eta_1*\lambda_1)}^{(+)}\}, \min\{\sqrt{\mathcal{L}_2(v_2*\eta_2)}^{(+)}, \sqrt{\mathcal{L}_2(\eta_2*\lambda_2)}^{(+)}\} \leq$
$min\{\sqrt{\mathcal{L}_1(v_1*\lambda_1)}^{(+)}, \sqrt{\mathcal{L}_2(v_2*\lambda_2)}^{(+)}\} = \sqrt{(\mathcal{L}_1 \times \mathcal{L}_2)(v_1*\lambda_1, v_2*\lambda_2)}^{(+)}$. Also,
$max\{\sqrt{(\mathcal{L}_1 \times \mathcal{L}_2)(v_1,v_2)*(\eta_1,\eta_2)}^{(-)}, \sqrt{(\mathcal{L}_1 \times \mathcal{L}_2)(\eta_1,\eta_2)*(\lambda_1,\lambda_2)}^{(-)}\}$
$= max\{\sqrt{(\mathcal{L}_1 \times \mathcal{L}_2)(v_1*\eta_1, v_2*\eta_2)}^{(-)}, \sqrt{(\mathcal{L}_1 \times \mathcal{L}_2)(\eta_1*\lambda_1, \eta_2*\lambda_2)}^{(-)}\}$
$= max\{max\{\sqrt{\mathcal{L}_1(v_1*\eta_1)}^{(-)}, \sqrt{\mathcal{L}_2(v_2*\eta_2)}^{(-)}\}, max\{\sqrt{\mathcal{L}_1(\eta_1*\lambda_1)}^{(-)}, \sqrt{\mathcal{L}_2(\eta_2*\lambda_2)}^{(-)}\}$

$$= max\{\max\{\sqrt{\mathcal{L}_1(v_1 * \eta_1)}^{(-)}, \sqrt{\mathcal{L}_1(\eta_1 * \lambda_{1,})}^{(-)}\}, max\{\sqrt{\mathcal{L}_2(v_2 * \eta_2)}^{(-)}, \sqrt{\mathcal{L}_2(\eta_2 * \lambda_2)}^{(-)}\} \geq max\left\{\sqrt{\mathcal{L}_1(v_1 * \lambda_1)}^{(-)}, \sqrt{\mathcal{L}_2(v_2 * \lambda_2)}^{(-)}\right\} = \sqrt{(\mathcal{L}_1 \times \mathcal{L}_2)(v_1 * \lambda_1, v_2 * \lambda_2)}^{(-)}.$$

5. Disclosure and conflict of interest

The authors declare that they have no conflicts of interest.

Conclusions

In this work, we have study the notion of a square root fuzzy set of QS -ideals on QS -algebra and investigated its properties. Furthermore, we have study the homomorphic image and the inverse image of square root fuzzy-QS-ideals of a QS-algebra under homomorphism of QS -algebras. The main purpose of our future work is to study the Cartesian product of square root fuzzy-QS-ideals in Cartesian product QS-algebras, the topological space for square root fuzzy-QS-ideals of QS-algebra and we apply the concept of linear Diophantine square root fuzzy sets in QS-algebras. Finally, we study the novel correlation coefficient between two fuzzy ideals square root, for more applications need to be discussed in our further study.

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