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Compactness in Čech Fuzzy Soft Closure Spaces

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Abstract

In this paper, the concepts of compactness, almost compactness, and nearly compactness for Čech fuzzy soft closure spaces are introduced and discussed. Their characterizations in terms of the finite intersection property are established, and their hereditary properties are investigated for compactness and almost compactness types. The relationships between the three types of compactness are investigated and illustrated with examples. The sufficient condition for the equivalence between compactness and almost compactness has been provided. In addition, new types of fuzzy soft mappings on Čech fuzzy soft closure spaces, namely Čech fuzzy soft strongly (respectively Čech fuzzy soft θ, and Čech fuzzy soft almost) continuous mapping, are introduced to study the behavior of the presented compactness types under fuzzy soft mappings.

Keywords: Čech fuzzy soft closure space, Čech fuzzy soft cover, fuzzy soft topological space, Čech fuzzy soft compact space, Čech fuzzy soft continuous (strongly continuous) mapping.

التراص في فضاءات الاغلاق الضبابية الناعمة تشيك

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قسم الرياضيات, كلية التربية للعلوم الصرفة ابن– الهيثم, جامعة بغداد, بغداد, العراق

الخلاصة

في هذا البحث، تم تعريف ومناقشة مفاهيم التراص، التراص التقريبي، وشبه التراص في فضاءات الاغلاق الضبابية الناعمة تشيك. تم دراسة خصائصها من حيث خاصية التقاطع المنتهي، والخاصية الوراثية للنوعين التراص والتراص التقريبي، كما تم دراسة العلاقات بين الانواع الثلاثة من التراص وتوضيحها بالأمثلة. تم توفير الشرط الكافي للتكافؤ بين التراص والتراص التقريبي. بالإضافة إلى ذلك، تم تقديم أنواع جديدة من الدوال المستمرة القوية (الدوال المستمرة θ والدوال المستمرة تقريبا، على التوالي) الضبابية الناعمة تشيك من اجل دراسة الصورة المباشرة لأنواع الفضاءات المرصوصة المقدمة تحت تأثير هذه الانواع من الدوال الضبابية الناعمة.

1. Introduction

Many engineering, medical, economic, and environmental challenges are fraught with ambiguity. Classical mathematical materials are not sufficient to handle the practical aspects of these areas. Zadeh [1] suggested the fuzzy set theory in order to address ambiguity that

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existing methods cannot handle. This theory represents a significant shift in mathematical paradigms. Since its inception, it has helped address several practical concerns and solve real-world problems. However, this hypothesis has faces difficulties, probably due to insufficient parameterization tools, as Molodtsov noted in [2]. Molodtsov established the notion of soft set theory, an entirely novel method to simulate uncertainty. Maji et al. [3] described fuzzy soft sets as fuzzy extensions of soft sets. In 2011, the concept of topological structure based on fuzzy soft sets was introduced by Tanay and Kandemir [4].

In 1966, $\check{C}$ech [5] presented closure spaces (U, C), where C maps the power set of U to itself. This mapping is referred to as a $\check{C}$ech closure operator on U. It is similar to a topological closure operator but does not require idempotency. Mashhour and Ghanim [6] proposed fuzzy closure spaces in 1985. To achieve this, fuzzy sets were used instead of sets in the formulation of $\check{C}$ech closure space. Gowri and Jegadeesan [7] 2014 explored and introduced $\check{C}$ech soft closure spaces using the soft set concept. In the same year, Krishnaveni and Sekar [8] proposed and studied $\check{C}$ech soft closure spaces. Here, the set of all soft sets over U that go to itself is defined as the soft closure operator. Majeed [9] has recently used fuzzy set theory to propose and investigate the concept of $\check{C}$ech fuzzy soft closure space (abbreviated $\check{C}\mathcal{FSCS}$).

Compactness is one of the most fundamental and important concepts of greatest importance according to topologists, and it seems to be the best-known manner of covering the feature. There are many authors have made several contributions to topologies; see, for instance, [10, 11, 12, 13]. Chang [14] proposed the concept of compactness in fuzzy topology. Mashhour and Ghanim [6] described the concept of compactness in fuzzy closure spaces. In soft topological space, compactness was first introduced by Zorlutuna et al. [15]. Then Gain et al. [16] and Osmanoglu and Tokat [17] introduced the notion of compactness in fuzzy soft topology as a generalization of Chang's fuzzy compactness. Later on, soft topology and fuzzy soft topology, compactness has been discussed in [18, 19, 20, 21, 22].

This paper aims to define and investigate the notions of compactness, almost compactness, and nearly compactness in $\check{C}$ech fuzzy soft closure spaces using $\check{C}$ech fuzzy soft cover. In Section 3, we present a concept of $\check{C}$ech fuzzy soft compact space, this is a generalization of the concept of fuzzy soft compact space [17], and study the properties of this type, such as its characterizations in terms of finite intersection property, hereditary property, and $\check{C}$ech fuzzy soft compactness obtained by $\check{C}$ech fuzzy soft strongly continuous mappings. In Section 4, an almost $\check{C}$ech fuzzy soft compact space is introduced as a second type of compactness. Some results and theorems are connected to this notion are examined, along accompanied by some essential examples. The relation between $\check{C}$ech fuzzy soft compact space and almost $\check{C}$ech fuzzy soft compact space is discussed. Moreover, a sufficient condition for the equivalence between compactness and almost compactness is studied. Finally, the third type of compactness, namely nearly $\check{C}$ech fuzzy soft compact space and some of its properties are introduced in Section 5.

2. Preliminaries

We assume that the reader to be acquainted with the basic concepts in fuzzy set theory. In our paper, U refers to the original universe, $\mathbb{I} = [0,1]$, $\mathbb{I}_0 = (0,1]$, $\mathbb{I}^U$ is the family that includes all fuzzy sets of U, and $\mathcal{P}$ is the set of parameters for U. The abbreviation $\mathcal{FS}$- stands for fuzzy soft, and $\mathcal{J}, \mathcal{L}$ for index sets.

An $\mathcal{FS}$-set λ_B over U is a mapping from $\mathcal{P}$ to $\mathbb{I}^U$, where $\lambda_B(k) \neq \bar{0}$ if $k \in B \subseteq \mathcal{P}$ and $\lambda_B(k) = \bar{0}$ if $k \notin B \subseteq \mathcal{P}$, where $\bar{0}$ denotes the empty fuzzy set. The collection of all $\mathcal{FS}$-sets over the U is represented by $\mathcal{FSS}(U,\mathcal{P})$ (see [23, 24]). Let $\lambda_B, \eta_C \in \mathcal{FSS}(U,\mathcal{P})$, then λ_B is called a $\mathcal{FS}$-subset of η_C, represented by $\lambda_B \sqsubseteq \eta_C$, if $\lambda_B(k) \leq \eta_C(k)$, for all $k \in \mathcal{P}$. Also λ_B and η_C are said to be equal, represented by $\lambda_B = \eta_C$ if $\lambda_B \sqsubseteq \eta_C$ and $\eta_C \sqsubseteq \lambda_B$. The union (respectively, intersection) of λ_B and η_C, represented by $\lambda_B \cup \eta_C$ (respectively, $\lambda_B \cap \eta_C$) is the $\mathcal{FS}$-set $\mu_{(B\cup C)}(k)$ defined by $\mu_{(B\cup C)}(k) = \lambda_B(k) \vee \eta_C(k)$ (respectively, is the $\mathcal{FS}$-set $\mu_{(B\cap C)}$ defined by $\mu_{(B\cap C)}(k) = \lambda_B(k) \wedge \eta_C(k)$), for all $k \in \mathcal{P}$. The constant $\mathcal{FS}$-sets taking, values $\bar{0}$ and $\bar{1}$ respectively, at every $k \in \mathcal{P}$ are represented by $\bar{0}_{\mathcal{P}}$ and $\bar{1}_{\mathcal{P}}$, respectively. Two $\mathcal{FS}$-sets λ_B, $\eta_C \in \mathcal{FSS}(U,\mathcal{P})$ are called disjoint, represented by $\lambda_B \cap \eta_C = \bar{0}_K$, if $\lambda_B(k) \wedge \eta_C(k) = \bar{0}$ for all $k \in \mathcal{P}$ (see [25]). For the $\mathcal{FS}$-set λ_B over U, $\bar{1}_{\mathcal{P}} - \lambda_B$ will represent the complement of λ_B, is the $\mathcal{FS}$-set defined as $(\bar{1}_{\mathcal{P}} - \lambda_B)(k) = \bar{1} - \lambda_B(k)$, for each $k \in \mathcal{P}$, Its clear that $\bar{1}_{\mathcal{P}} - (\bar{1}_{\mathcal{P}} - \lambda_B) = \lambda_B$ (see [24]). According to the concept of Atmaca and Zorlutuna [26] a $\mathcal{FS}$-set $\lambda_B \in \mathcal{FSS}(U,\mathcal{P})$ is called $\mathcal{FS}$-point, represented by x_t^k, if there exist $x \in U$ and $k \in \mathcal{P}$ such that $\lambda_B(k)(x) = t$ ($0 < t \leq 1$) and 0 otherwise for all $y \in U - \{x\}$. The $\mathcal{FS}$-point x_t^k is said to belong to the $\mathcal{FS}$-set λ_B, represented by $x_t^k \tilde{\in} \lambda_B$ if for the element $k \in \mathcal{P}, t \leq \lambda_B(k)(x)$ (see [26]). Two $\mathcal{FS}$-points x_t^k and $y_s^{k'}$ are said to be distinct if $x \neq y$ or $k \neq k'$ (see [27]).

Definition 2.1 [28] Let $\mathcal{FSS}(U,\mathcal{P})$, and $\mathcal{FSS}(Y,\mathcal{R})$, represent two families of $\mathcal{FS}$-sets over U and Y, respectively. Let $v: U \to Y$ and $s: \mathcal{P} \to \mathcal{R}$ be two mappings. Then, $f_{vs}: \mathcal{FSS}(U,\mathcal{P}) \to \mathcal{FSS}(Y,\mathcal{R})$ is called fuzzy soft mapping ($\mathcal{FS}$-mapping).

(1) If $\lambda_B \in \mathcal{FSS}(U,\mathcal{P})$, then the image of λ_B under the $\mathcal{FS}$-mapping f_{vs} is the $\mathcal{FS}$-set over Y characterized by $f_{vs}(\lambda_B)$, where $\forall\, r \in s(\mathcal{P}), \forall y \in Y$,

$$f_{vs}(\lambda_B)(r)(y) = \begin{cases} \vee_{v(x)=y} (\vee_{s(k)=r} (\lambda_B(k)))(x) & if\, v^{-1}(y) \neq \emptyset, s^{-1}(r) \cap B \neq \emptyset \\ 0 & otherwise. \end{cases}$$

(2) If $\mu_A \in \mathcal{FSS}(Y,\mathcal{R})$, then the pre-image of μ_A under the $\mathcal{FS}$-mapping f_{vs} is the $\mathcal{FS}$-set over U characterized by $f_{vs}^{-1}(\mu_A)$, where $\forall k \in s^{-1}(\mathcal{R}), \forall x \in U$,

$$f_{vs}^{-1}(\mu_A)(k)(x) = \begin{cases} \mu_A(s(k))(v(x)) & for\;\; s(k) \in A, \\ 0 & otherwise. \end{cases}.$$

If v and s are surjective (or injective, respectively), the $\mathcal{FS}$-mapping f_{vs} is called surjective (or injective), and if v and s are constant, it is said to be constant.

Definition 2.2 [4] Let $\mathcal{T}$ be a collection of $\mathcal{FS}$-sets over U that satisfy the following axioms:

1. $\bar{0}_{\mathcal{P}}, \bar{1}_{\mathcal{P}} \in \mathcal{T}$,
2. If $\lambda_B, \mu_A \in \mathcal{T}$, then $\lambda_B \cap \mu_A \in \mathcal{T}$,
3. If $(\lambda_B)_i \in \mathcal{T}$, then $\cup_{i\in\mathcal{J}}(\lambda_B)_i \in \mathcal{T}$.

Then, $\mathcal{T}$ is called a $\mathcal{FS}$-topology on U and $(U,\mathcal{T},\mathcal{P})$ is called a fuzzy soft topological space ($\mathcal{FSTS}$, in brief). Every member of $\mathcal{T}$ is referred to as an open $\mathcal{FS}$-set. A closed $\mathcal{FS}$- set is the complement of an open $\mathcal{FS}$- set.

Example 2.3 Let $U = \{a,d,e\}$, $\mathcal{P} = \{k_1,k_2\}$ be the set of parameters, and let $(\lambda_B)_1, (\lambda_B)_2, (\lambda_B)_3, (\lambda_B)_4 \in \mathcal{FSS}(U,\mathcal{P})$, where $(\lambda_B)_1 = \{(k_1, a_{0.5})\}$, $(\lambda_B)_2 = \{(k_1, a_{0.3} \vee d_{0.6}), (k_2, d_1 \vee e_1)\}$, $(\lambda_B)_3 = \{(k_1, a_{0.3})\}$, and $(\lambda_B)_4 = \{(k_1, a_{0.5} \vee d_{0.6}), (k_2, d_1 \vee e_1)\}$. Then, $\mathcal{T} = \{\bar{0}_{\mathcal{P}}, \bar{1}_{\mathcal{P}}, (\lambda_B)_1, (\lambda_B)_2, (\lambda_B)_3, (\lambda_B)_4\}$ be a $\mathcal{FS}$-topology on U and $(U,\mathcal{T},\mathcal{P})$ be a $\mathcal{FSTS}$.

Definition 2.4 [17] A family $\Upsilon = \{(\lambda_B)_i : i \in \mathcal{J}\}$ is a cover of a $\mathcal{FS}$-set μ_A if $\mu_A \sqsubseteq \cup\{(\lambda_B)_i : i \in \mathcal{J}\}$. It is called an $\mathcal{FS}$-open cover if every member of Υ is an open $\mathcal{FS}$-set. A subcover of Υ is a subfamily of Υ, which itself is a cover. An $\mathcal{FSTS}$ $(U,\mathcal{T},\mathcal{P})$ it's named $\mathcal{FS}$-compact if each $\mathcal{FS}$-open cover of $\bar{1}_{\mathcal{P}}$ has a finite subcover.

Definition 2.5 [17] A family Υ of $\mathcal{FS}$-sets has the finite intersection property (FIP, in brief), if the intersection of the members of each finite subfamily of Υ is not the null $\mathcal{FS}$-set.

Now we need as follows the definitions and basic results about Č$\mathcal{FSCS}$'s

Definition 2.6 [9] An operator $\mathcal{C}: \mathcal{FSS}(U,\mathcal{P}) \to \mathcal{FSS}(U,\mathcal{P})$ is called Čech fuzzy soft closure operator ($\check{C}$-fsco, in brief) on U, if the following conditions are hold:

(C1) $\mathcal{C}(\bar{0}_{\mathcal{P}}) = \bar{0}_{\mathcal{P}}$,

(C2) $\lambda_B \subseteq \mathcal{C}(\lambda_B)$, for all $\lambda_B \in \mathcal{FSS}(U,\mathcal{P})$,

(C3) $\mathcal{C}(\lambda_B \cup \mu_A) = \mathcal{C}(\lambda_B) \cup \mathcal{C}(\mu_A)$, for all $\lambda_B, \mu_A \in \mathcal{FSS}(U,\mathcal{P})$.

The triple $(U,\mathcal{C},\mathcal{P})$ is called a $\check{C}$ech fuzzy soft closure space (Č$\mathcal{FSCS}$, in brief). A $\mathcal{FS}$-set λ_B is defined as a closed $\mathcal{FS}$-set in $(U,\mathcal{C},\mathcal{P})$ if $\lambda_B = \mathcal{C}(\lambda_B)$. A $\mathcal{FS}$-set λ_B is defined as an open $\mathcal{FS}$-set if $\bar{1}_{\mathcal{P}} - \lambda_B$ is a closed $\mathcal{FS}$-set.

Definition 2.7 [9] Let $(U,\mathcal{C},\mathcal{P})$ be an Č$\mathcal{FSCS}$, and let $\lambda_B \in \mathcal{FSS}(U,\mathcal{P})$. The interior of λ_B, represented by $Int(\lambda_B)$ is defined as $Int(\lambda_B) = \bar{1}_{\mathcal{P}} - \mathcal{C}(\bar{1}_{\mathcal{P}} - \lambda_B)$. A $\mathcal{FS}$-set λ_B is said to be $\mathcal{FS}$ neighborhood of a $\mathcal{FS}$-point x_t^k, if $x_t^k \,\tilde{\in}\, \mathrm{Int}(\lambda_B)$.

Proposition 2.8 [9] Let $\mathcal{FSS}(U,\mathcal{P})$ an Č$\mathcal{FSCS}$, and let $\lambda_B, \mu_A \in \mathcal{FSS}(U,\mathcal{P})$. Then, λ_B is an open $\mathcal{FS}$-set $\Leftrightarrow Int(\lambda_B) = \lambda_B$.

Theorem 2.9 [9] Let $(U,\mathcal{C},\mathcal{P})$ an Č$\mathcal{FSCS}$ and let $\mathcal{T}_{\mathcal{C}} \subseteq \mathcal{FSS}(U,\mathcal{P})$, defined as, $\mathcal{T}_{\mathcal{C}} = \{\bar{1}_{\mathcal{P}} - \lambda_B : \mathcal{C}(\lambda_B) = \lambda_B\}$. Then, $\mathcal{T}_{\mathcal{C}}$ is a $\mathcal{FS}$ topology on U and $(U,\tau_{\mathcal{C}},\mathcal{P})$ is called an associative $\mathcal{FS}$ topological space (associative $\mathcal{FSTS}$, in brief) of $(U,\mathcal{C},\mathcal{P})$.

Theorem 2.10 [9] Let $(U,\mathcal{C},\mathcal{P})$ be Č$\mathcal{FSCS}$ and $(U,\tau_{\mathcal{C}},\mathcal{P})$ be an associative $\mathcal{FSTS}$ of $(U,\mathcal{C},\mathcal{P})$. Then for any $\mathcal{FS}$-set $\lambda_B \in \mathcal{FSS}(U,\mathcal{P})$, we have $\tau_{\mathcal{C}}\text{-}int(\lambda_B) \subseteq Int(\lambda_B) \subseteq \lambda_B \subseteq \mathcal{C}(\lambda_B) \subseteq \tau_{\mathcal{C}}\text{-}cl(\lambda_B)$, where $\tau_{\mathcal{C}}\text{-}int$ (respectively, $\tau_{\mathcal{C}}\text{-}cl$) stand for $\mathcal{FS}$-interior (respectively, $\mathcal{FS}$-closure) for a $\mathcal{FS}$-set λ_B in the associative $\mathcal{FSTS}$ $(U,\mathcal{T}_{\mathcal{C}},\mathcal{P})$.

Theorem 2.11 [9] Let $(U,\mathcal{C},\mathcal{P})$ be an Č$\mathcal{FSCS}$, $V \subseteq U$ and let $\mathcal{C}_V: \mathcal{FSS}(V,\mathcal{P}) \to \mathcal{FSS}(V,\mathcal{P})$ defined as follows: $\mathcal{C}_V(\lambda_B) = \bar{V}_{\mathcal{P}} \sqcap \mathcal{C}(\lambda_B)$. Then $\mathcal{C}_V$ is a $\check{C}$-fsco. The triple $(V,\mathcal{C}_V,\mathcal{P})$ is said to be Čech fuzzy soft closure subspace (Č$\mathcal{FSCS}$-subspace, in brief) of $(U,\mathcal{C},\mathcal{P})$, where $\bar{V}_{\mathcal{P}}$ is a $\mathcal{FS}$-set defined as $\bar{V}_{\mathcal{P}}(k) = \bar{1}_V$ for all $k \in \mathcal{P}$.

Definition 2.12 [9] An Č$\mathcal{FSCS}(U,\mathcal{C},\mathcal{P})$ is said to be T_2- Č$\mathcal{FSCS}$, if for every two distinct $\mathcal{FS}$-points x_t^k and $y_s^{k'}$, there exist disjoint open $\mathcal{FS}$-sets λ_B and μ_A such that $x_t^k \,\tilde{\in}\, \lambda_B$ and $y_s^{k'} \,\tilde{\in}\, \mu_A$.

Definition 2.13 [9] Let $(U,\mathcal{C},\mathcal{P})$ and $(Y,\mathcal{C}^*,\mathcal{R})$ be two Č$\mathcal{FSCS}$'s. A $\mathcal{FS}$-mapping $f_{vs}: (U,\mathcal{C},\mathcal{P}) \longrightarrow (Y,\mathcal{C}^*,\mathcal{R})$ is said to be Čech fuzzy soft continuous ($\check{C}\mathcal{FS}$-continuous, in brief) mapping, if $f_{vs}(\mathcal{C}(\lambda_B)) \subseteq \mathcal{C}^*(f_{vs}(\lambda_B))$, for every $\mathcal{FS}$-set $\lambda_B \in \mathcal{FSS}(U,\mathcal{P})$.

3- Čech fuzzy soft compact closure spaces

This section will provide an introduction to the concept of Čech fuzzy soft compactness, the first type of $\mathcal{FS}$-compactness in Č$\mathcal{FSCS}$'s, and discuss some of its characteristics.

Definition 3.1 A collection $\{(\lambda_B)_i: i \in \mathcal{I}\}$ of $\mathcal{FS}$-sets is said to be a Čech fuzzy soft cover (Č$\mathcal{FS}$-cover, in brief) of $(U, \mathcal{C}, \mathcal{P})$, if $\bar{1}_{\mathcal{P}} = \cup\{Int((\lambda_B)_i): i \in \mathcal{I}\}$.

Definition 3.2 An Č$\mathcal{FSCS}$ $(U, \mathcal{C}, \mathcal{P})$ is said to be Čech fuzzy soft compact (Č$\mathcal{FS}$-compact, in brief), if every Č$\mathcal{FS}$-cover of $\bar{1}_{\mathcal{P}}$ has a finite Č$\mathcal{FS}$-subcover.

In the following, we provide an example to clarify the concept of Č$\mathcal{FS}$-compact space.

Example 3.3 Let $U = \{a, d, e\}, \mathcal{P} = \{k_1, k_2\}$ and $\mu_A \in \mathcal{FSS}(U, \mathcal{P})$ such that $\mu_A = \{(k_1, d_{0.5}), (k_2, d_{0.5})\}$. Define Č-fsco $\mathcal{C}: \mathcal{FSS}(U, \mathcal{P}) \to \mathcal{FSS}(U, \mathcal{P})$ as follows:

$$\mathcal{C}(\lambda_B) = \begin{cases} \bar{0}_{\mathcal{P}} & if\ \lambda_B = \bar{0}_{\mathcal{P}}, \\ \{(k_1, a_{0.5} \vee d_{0.5}), (k_2, a_{0.5} \vee d_{0.5})\} & if\ \lambda_B \subseteq \mu_A, \\ \bar{1}_{\mathcal{P}} & otherwise. \end{cases}$$

Then, $(U, \mathcal{C}, \mathcal{P})$ is Č$\mathcal{FSCS}$, and for any λ_B in $(U, \mathcal{C}, \mathcal{P})$, the interior of λ_B is defines as follows:

$$Int(\lambda_B) = \begin{cases} \bar{0}_{\mathcal{P}} & if\ \lambda_B = \bar{0}_{\mathcal{P}}, \\ \{(k_1, a_{0.5} \vee d_{0.5}), (k_2, a_{0.5} \vee d_{0.5})\} & if\ \lambda_B \in \{\Upsilon_1, \Upsilon_2, \Upsilon_3\}, \\ \bar{1}_{\mathcal{P}} & if\ \lambda_B = \bar{1}_{\mathcal{P}}, \\ \bar{0}_{\mathcal{P}} & otherwise. \end{cases}$$

Where $\Upsilon_1, \Upsilon_2, \Upsilon_3$ are families of $\mathcal{FS}$-sets in $(U, \mathcal{C}, \mathcal{P})$ defined as follows:
$\Upsilon_1 = \{(k_1, a_1 \vee d_{1-t_1}), (k_2, a_1 \vee d_1): 0 < t_1 \le 0.5\}$,
$\Upsilon_2 = \{(k_1, a_1 \vee d_1), (k_2, a_1 \vee d_{1-t_2}): 0 < t_2 \le 0.5\}$, and
$\Upsilon_3 = \{(k_1, a_1 \vee d_{1-t_1}), (k_2, a_1 \vee d_{1-t_2}): 0 < t_1, t_2 \le 0.5\}$.
It is clear that, any Č$\mathcal{FS}$-cover of $\bar{1}_{\mathcal{P}}$ must contains $\bar{1}_{\mathcal{P}}$. So, $(U, \mathcal{C}, \mathcal{P})$ is Č$\mathcal{FS}$-compact space.

Proposition 3.4 Let $(U, \mathcal{C}, \mathcal{P})$ be a Č$\mathcal{FSCS}$. Then, every $\mathcal{FS}$-open cover of $\bar{1}_{\mathcal{P}}$ is a Č$\mathcal{FS}$-cover of $\bar{1}_{\mathcal{P}}$.
The next proposition's proof follows immediately from Proposition 3.4.
Proposition 3.5 If $(U, \mathcal{C}, \mathcal{P})$ is Č$\mathcal{FS}$-compact, then $(U, \tau_{\mathcal{C}}, \mathcal{P})$ is $\mathcal{FS}$-compact.

Theorem 3.6 A Č$\mathcal{FSCS}$ $(U, \mathcal{C}, \mathcal{P})$ is Č$\mathcal{FS}$-compact if and only if each family $\mathcal{S}$ of $\mathcal{FS}$-sets of U such that the family $\mathcal{C}(\mathcal{S})$=$\{\mathcal{C}((\lambda_B)_i): (\lambda_B)_i \in \mathcal{S}\}$ has the FIP, then $\mathcal{C}(\mathcal{S})$ has a non-null $\mathcal{FS}$- intersection.
Proof: Let $\mathcal{S} = \{(\lambda_B)_i; i \in \mathcal{I}\}$ be a family of $\mathcal{FS}$-sets of U such that $\mathcal{C}(\mathcal{S})$=$\{\mathcal{C}((\lambda_B)_i): (\lambda_B)_i \in \mathcal{S}\}$ has the FIP. Suppose $\cap\{\mathcal{C}((\lambda_B)_i): i \in \mathcal{I}\} = \bar{0}_{\mathcal{P}}$. Then, $\cup\{\bar{1}_{\mathcal{P}} - \mathcal{C}((\lambda_B)_i): i \in \mathcal{I}\} = \bar{1}_{\mathcal{P}}$ which equal to $\cup\{Int(\bar{1}_{\mathcal{P}} - (\lambda_B)_i): i \in \mathcal{I}\} = \bar{1}_{\mathcal{P}}$. That means $\{\bar{1}_{\mathcal{P}} - (\lambda_B)_i: i \in \mathcal{I}\}$ is a Č$\mathcal{FS}$-cover of $\bar{1}_{\mathcal{P}}$. As $(U, \mathcal{C}, \mathcal{P})$ is Č$\mathcal{FS}$-compact, thereafter, a finite subset exists. $\mathcal{L} \subset \mathcal{I}$ such that $\{\bar{1}_{\mathcal{P}} - (\lambda_B)_i: i \in \mathcal{L}\}$ is a Č$\mathcal{FS}$-cover of $\bar{1}_{\mathcal{P}}$. This yields $\cup\{Int(\bar{1}_{\mathcal{P}} - (\lambda_B)_i): i \in \mathcal{L}\} = \bar{1}_{\mathcal{P}}$. Therefore, $\bar{1}_K - \cup\{Int(\bar{1}_K - (\lambda_B)_i): i \in \mathcal{L}\} = \cap\{\mathcal{C}((\lambda_B)_i): i \in \mathcal{L}\} = \bar{0}_{\mathcal{P}}$ which contradicts the FIP of $\mathcal{S}$. Therefore, the prerequisite is satisfied.

Conversely, let $\{(\lambda_B)_i: i \in \mathcal{I}\}$ be a Č$\mathcal{FS}$-cover of $\bar{1}_{\mathcal{P}}$. That means $\cup\{Int((\lambda_B)_i): i \in \mathcal{I}\} = \bar{1}_{\mathcal{P}}$. Then $\cap\{\bar{1}_{\mathcal{P}} - Int((\lambda_B)_i): i \in \mathcal{I}\} = \cap\{\mathcal{C}(\bar{1}_{\mathcal{P}} - (\lambda_B)_i): i \in \mathcal{I}\} = \bar{0}_{\mathcal{P}}$. Therefore, there is a finite subset $\mathcal{L} \subset I$ such that $\cap\{\mathcal{C}(\bar{1}_{\mathcal{P}} - (\lambda_B)_i): i \in \mathcal{L})\} = \bar{0}_{\mathcal{P}}$. By taking the relative complement we have $\cup\{Int((\lambda_B)_i); i \in \mathcal{L}\} = \bar{1}_{\mathcal{P}}$. Therefore, $\{(\lambda_B)_i; i \in \mathcal{L}\}$ is a finite Č$\mathcal{FS}$-cover of $\bar{1}_{\mathcal{P}}$. Hence, $(U, \mathcal{C}, \mathcal{P})$ is Č$\mathcal{FS}$-compact.

Theorem 3.7 A closed Č$\mathcal{FS}$-closure subspace of an Č$\mathcal{FS}$-compact is Č$\mathcal{FS}$-compact.

Proof: Let $(V, \mathcal{C}_V, \mathcal{P})$ be a closed $\check{C}\mathcal{FS}$-closure subspace of an $\check{C}\mathcal{FS}$-compact space $(U, \mathcal{C}, \mathcal{P})$, and let $\mathcal{S}=\{(\lambda_B)_i : i \in \mathcal{I}\}$ be a collection of $\mathcal{FS}$-subsets of $(V, \mathcal{C}_V, \mathcal{P})$ such that $\mathcal{C}_V(\mathcal{S})=\{\mathcal{C}_V((\lambda_B)_i) : (\lambda_B)_i \in \mathcal{S}\}$ has the FIP. Since $\mathcal{C}_V((\lambda_B)_i) = \bar{V}_\mathcal{P} \cap \mathcal{C}((\lambda_B)_i)$ and $\bigcap_{i=1}^{n} \mathcal{C}_V((\lambda_B)_i) = \bigcap_{i=1}^{n}(\bar{V}_\mathcal{P} \cap \mathcal{C}((\lambda_B)_i)) \neq \bar{0}_\mathcal{P}$, then $\{\mathcal{C}(\bar{V}_\mathcal{P}), \mathcal{C}((\lambda_B)_i) : (\lambda_B)_i \in \mathcal{S}\}$ has the FIP in U. By $\check{C}\mathcal{FS}$-compactness of $(U, \mathcal{C}, \mathcal{P})$ and Theorem 3.6, $\{\mathcal{C}(\bar{V}_\mathcal{P}), \mathcal{C}((\lambda_B)_i) : (\lambda_B)_i \in \mathcal{S}\}$ has a non-null $\mathcal{FS}$-intersection. Therefore, $\{\mathcal{C}_V((\lambda_B)_i) : i \in \mathcal{I}\}$ has a non-null $\mathcal{FS}$-intersection. Thus,$(V, \mathcal{C}_V, \mathcal{P})$ is $\check{C}\mathcal{FS}$-compact.

The next example shows that the above theorem's converse is not always true.

Example 3.8 In Example 3.3, let us take $\bar{V}_\mathcal{P} = \{(k_1, a_1 \vee d_1), (k_2, a_1 \vee d_1)\}$. Then $\bar{V}_\mathcal{P}$ is $\check{C}\mathcal{FS}$-compact. But $\bar{V}_\mathcal{P}$ is not closed $\mathcal{FS}$-set in $(U, \mathcal{C}, \mathcal{P})$, since $\mathcal{C}(\bar{V}_\mathcal{P}) = \bar{1}_\mathcal{P}$.

Theorem 3.9 Let $\bar{V}_\mathcal{P}$ be a $\check{C}\mathcal{FS}$-compact $\mathcal{FS}$-set in a T_2-$\check{C}\mathcal{F}$SCS $(U, \mathcal{C}, \mathcal{P})$. Then $\bar{V}_\mathcal{P}$is a closed $\mathcal{FS}$-set in $(U, \mathcal{C}, \mathcal{P})$.

Proof: Let $\tilde{p} = x_t^k \tilde{\in} \bar{1}_\mathcal{P} - \bar{V}_\mathcal{P}$. For each $\tilde{q} = y_s^{k'} \tilde{\in} \bar{V}_\mathcal{P}$, we have $x \neq y$ which implies x_t^k and $y_s^{k'}$ are distinct $\mathcal{FS}$-points in a T_2-$\check{C}\mathcal{F}$SCS $(U, \mathcal{C}, \mathcal{P})$. So, there are disjoint open $\mathcal{FS}$-sets $(\lambda_B)_{\tilde{q}}$ and $(\mu_A)_{\tilde{q}}$ such that $\tilde{p} = x_t^k \tilde{\in} (\lambda_B)_{\tilde{q}}$ and $\tilde{q} = y_s^{k'} \tilde{\in} (\mu_A)_{\tilde{q}}$. Then, $\{(\mu_A)_{\tilde{q}} : \tilde{q} \tilde{\in} \bar{V}_\mathcal{P}\}$ is an $\mathcal{FS}$-open cover of $\bar{V}_\mathcal{P}$. This yields $\{(\mu_A)_{\tilde{q}} : \tilde{q} \tilde{\in} \bar{V}_\mathcal{P}\}$ is a $\check{C}\mathcal{FS}$-cover of $\bar{V}_\mathcal{P}$. Since $\bar{V}_\mathcal{P}$ is $\check{C}\mathcal{FS}$-compact, then there is a finite $\check{C}\mathcal{FS}$-subcover $\{(\mu_A)_{\tilde{q}_1}, (\mu_A)_{\tilde{q}_2}, \dots, (\mu_A)_{\tilde{q}_n}\}$. Then, $\bigcap_{i=1}^{n}(\lambda_B)_{\tilde{q}_i}$ is an open $\mathcal{FS}$-set such that $\tilde{p} = x_t^k \tilde{\in} \bigcap_{i=1}^{n}(\lambda_B)_{\tilde{q}_i} \subseteq \bar{1}_\mathcal{P} - \bar{V}_\mathcal{P}$. Thus, $\tilde{p} \tilde{\in} \mathrm{Int}(\bigcap_{i=1}^{n}(\lambda_B)_{\tilde{q}_i}) \subseteq Int(\bar{1}_\mathcal{P} - \bar{V}_\mathcal{P})$. Therefore, $\bar{1}_\mathcal{P} - \bar{V}_\mathcal{P}$ is a nbhd of $\tilde{p}$. Hence, for all $x_t^k \tilde{\in} \bar{1}_\mathcal{P} - \bar{V}_\mathcal{P}$, we have $x_t^k \tilde{\in} Int(\bar{1}_\mathcal{P} - \bar{V}_\mathcal{P})$ which means $\bar{1}_\mathcal{P} - \bar{V}_\mathcal{P} \subseteq Int(\bar{1}_\mathcal{P} - \bar{V}_\mathcal{P})$. On the other hand, $Int(\bar{1}_\mathcal{P} - \bar{V}_\mathcal{P}) \subseteq \bar{1}_\mathcal{P} - \bar{V}_\mathcal{P}$. Then $\bar{1}_\mathcal{P} - \bar{V}_\mathcal{P} = Int(\bar{1}_\mathcal{P} - \bar{V}_\mathcal{P})$ and this indicates to $\bar{V}_\mathcal{P}$ is a closed $\mathcal{FS}$-set in $(U, \mathcal{C}, \mathcal{P})$.

In order to study the behavior of $\check{C}\mathcal{FS}$-compactness through the $\mathcal{FS}$-mapping we need to introduce the notion of $\check{C}\mathcal{FS}$-strongly continuous between $\check{C}\mathcal{FSCS}$'s.

Definition 3.10 An $\mathcal{FS}$-mapping $f_{vs}:(U, \mathcal{C}, \mathcal{P}) \to (Y, \mathcal{C}^*, \mathcal{R})$ is called $\check{C}\mathcal{FS}$-strongly continuous, if $f_{vs}(\mathcal{C}(\lambda_B)) \subseteq f_{vs}(\lambda_B)$ for every $\mathcal{FS}$-set $\lambda_B \in \mathcal{FSS}(U, \mathcal{P})$.

Clearly, $\check{C}\mathcal{FS}$-strongly continuous mapping is $\check{C}\mathcal{FS}$-continuous but not conversely.

Theorem 3.11 Let $f_{vs}:(U, \mathcal{C}, \mathcal{P}) \to (Y, \mathcal{C}^*, \mathcal{R})$ be $\check{C}\mathcal{FS}$-strongly continuous mapping and $(U, \mathcal{C}, \mathcal{P})$ be an $\check{C}\mathcal{FS}$-compact, then$(Y, \mathcal{C}^*, \mathcal{R})$ is $\check{C}\mathcal{FS}$-compact.

Proof: Let $\mathcal{F} = \{(\lambda_F)_i : i \in \mathcal{I}\}$ be a collection of $\mathcal{FS}$-sets in Y such that $\{\mathcal{C}^*((\lambda_F)_i) : i \in \mathcal{I}\}$ has the FIP. Then, $\{f_{vs}^{-1}(\mathcal{C}^*((\lambda_F)_i)) : i \in \mathcal{I}\}$ has the FIP in $(U, \mathcal{C}, \mathcal{P})$. Since $f_{vs}^{-1}(\mathcal{C}^*((\lambda_F)_i)) \subseteq \mathcal{C}(f_{vs}^{-1}(\mathcal{C}^*((\lambda_F)_i))$, then $\bar{0}_\mathcal{P} \neq \bigcap_{i=1}^{n} f_{vs}^{-1}(\mathcal{C}^*((\lambda_F)_i)) \subseteq \bigcap_{i=1}^{n} \mathcal{C}(f_{vs}^{-1}(\mathcal{C}^*((\lambda_F)_i)))$. Hence, $\{\mathcal{C}(f_{vs}^{-1}(\mathcal{C}^*((\lambda_F)_i))) : i \in \mathcal{I}\}$ has the FIP in $(U, \mathcal{C}, \mathcal{P})$. But $(U, \mathcal{C}, \mathcal{P})$ is $\check{C}\mathcal{FS}$-compact. Then $\cap \{\mathcal{C}(f_{vs}^{-1}(\mathcal{C}^*((\lambda_F)_i))) : i \in \mathcal{I}\} \neq \bar{0}_\mathcal{P}$. That means, there exists a $\mathcal{FS}$-point $x_t^k \tilde{\in} \mathcal{C}(f_{vs}^{-1}(\mathcal{C}^*((\lambda_F)_i)))$ for all $i \in \mathcal{I}$. Then $f_{vs}(x_t^k) \tilde{\in} f_{vs}(\mathcal{C}(f_{vs}^{-1}(\mathcal{C}^*((\lambda_F)_i))))$. By the definition of $\mathcal{FS}$-point and f_{vs} is $\check{C}\mathcal{FS}$-strongly continuous mapping we get, $f_{vs}(x_t^k) = v(x)_t^{s(k)} \tilde{\in} f_{vs}(f_{vs}^{-1}(\mathcal{C}^*((\lambda_F)_i)) \subseteq \mathcal{C}^*((\lambda_F)_i)$. Thus, there exist a $\mathcal{FS}$-point $v(x)_t^{s(k)}$ belongs to $\mathcal{C}^*((\lambda_F)_i)$ for each $i \in \mathcal{I}$. Therefore, $\{\mathcal{C}^*((\lambda_F)_i) : i \in \mathcal{I}\}$ has non-null fuzzy soft intersection. Hence, $(Y, \mathcal{C}^*, \mathcal{R})$ is $\check{C}\mathcal{FS}$-compact.

Theorem 3.12 Let $(U, \mathcal{C}, \mathcal{P})$ be an $\check{C}\mathcal{FS}$-compact and $(Y, \mathcal{C}^*, \mathcal{R})$ be a T_2-$\check{C}\mathcal{F}$SCS. Then the image of every $\check{C}\mathcal{FS}$-closed subspace is $\check{C}\mathcal{FS}$-closed subspace in Y if $f_{vs}:(U, \mathcal{C}, \mathcal{P}) \to (Y, \mathcal{C}^*, \mathcal{R})$ is $\check{C}\mathcal{FS}$-strongly continuous.

Proof: Let $\bar{V}_\mathcal{P}$ be a closed $\check{C}\mathcal{FSC}$-subspace of $(U, \mathcal{C}, \mathcal{P})$. From Theorem 3.7, $\bar{V}_\mathcal{P}$ is $\check{C}\mathcal{FS}$-compact subspace of $(U, \mathcal{C}, \mathcal{P})$. Since f_{vs} is $\check{C}\mathcal{FS}$- strongly continuous, then from Theorem

3.11, we have $f_{vs}(\bar{V}_{\mathcal{P}})$ is Č$\mathcal{FS}$-compact. But $(Y, \mathcal{C}^*, \mathcal{R})$ is T_2-Č$\mathcal{F}$SCS, so by Theorem 3.9, $f_{vs}(\bar{V}_{\mathcal{P}})$ is Č$\mathcal{FS}$-closed subspace in $(Y, \mathcal{C}^*, \mathcal{R})$.

4-Almost Čech fuzzy soft compact closure spaces

Definition 4.1 An Č$\mathcal{F}$SCS $(U, \mathcal{C}, \mathcal{P})$ is said to be almost Č$\mathcal{FS}$-compact, if for every Č$\mathcal{FS}$-cover $\{(\lambda_B)_i : i \in \mathcal{I}\}$ of $\bar{1}_{\mathcal{P}}$, there is a finite set $\mathcal{L} \subset \mathcal{I}$ such that $\{\mathcal{C}((\lambda_B)_i): i \in \mathcal{L}\}$ is $\mathcal{FS}$-cover of $\bar{1}_{\mathcal{P}}$.
Proposition 4.2 Every Č$\mathcal{FS}$-compact is an almost Č$\mathcal{FS}$-compact.
Proof: This follows from Definition 4.1 and Theorem 2.10.

The following example illustrates that the converse is not true in general.
Example 4.3 Let $U = [0,1]$, $\mathcal{P} = \{k\}$, Define Č-fsco $\mathcal{C}: \mathcal{FSS}(U, \mathcal{P}) \to \mathcal{FSS}(U, \mathcal{P})$ as follows:

$$\mathcal{C}(\lambda_B) = \begin{cases} \bar{0}_{\mathcal{P}} & if\ \lambda_B = \bar{0}_{\mathcal{P}}, \\ (\lambda_B^*)_1 & if\ \lambda_B \subseteq (\lambda_B)_1, \\ (\lambda_B^*)_i & if\ \lambda_B \subseteq (\lambda_B)_i, \\ \bar{1}_{\mathcal{P}} & otherwise. \end{cases}$$

Where, $(\lambda_B)_1 = \{(k, x_1): x \in (0, \frac{1}{2}]\}$, $(\lambda_B^*)_1 = \{(k, x_1): x \in (0,1)\}$
$(\lambda_B)_i = \{\left(k, x_{\frac{1}{i}}\right): x \in (0, \frac{1}{2i}]\}$, $(\lambda_B^*)_i = \{\left(k, x_{\frac{1}{i}}\right): x \in (0, \frac{1}{i}]\}$, $i = 2,3,4, \dots$.
It is clear that for any $\lambda_B \in \mathcal{FSS}(U, \mathcal{P})$ the $Int(\lambda_B)$ is defined as following:

$$Int(\lambda_B) = \begin{cases} \bar{0}_{\mathcal{P}} & if\ \lambda_B = \bar{0}_{\mathcal{P}}, \\ \bar{1}_{\mathcal{P}} - (\lambda_B^*)_1 & if\ \lambda_B \subseteq \bar{1}_{\mathcal{P}} - (\lambda_B)_1, \\ \bar{1}_{\mathcal{P}} - (\lambda_B^*)_i & if\ \lambda_B \subseteq \bar{1}_{\mathcal{P}} - (\lambda_B)_i, \\ \bar{0}_{\mathcal{P}} & otherwise. \end{cases}$$

Then, for any Č$\mathcal{FS}$-cover of $\bar{1}_{\mathcal{P}}$ this cover must contain the family $\{\bar{1}_{\mathcal{P}} - (\lambda_B)_i: for\ some\ i\}$ and since $\mathcal{C}(\bar{1}_{\mathcal{P}} - (\lambda_B)_i) = \bar{1}_{\mathcal{P}}$ for any i. Then, $(U, \mathcal{C}, \mathcal{P})$ is an almost Č$\mathcal{FS}$-compact. But it is not $(U, \mathcal{C}, \mathcal{P})$ is Č$\mathcal{FS}$-compact because there exists $\{\bar{1}_{\mathcal{P}} - (\lambda_B)_i: i = 1,2,3, \dots\}$ is a Č$\mathcal{FS}$-cover of $\bar{1}_{\mathcal{P}}$ but it has no finite Č$\mathcal{FS}$-cover.

To investigate the property of almost $\mathcal{FS}$-compactness between Č$\mathcal{FSCS}$ and the corresponding $\mathcal{FSTS}$, we need to introduce the following definition:

Definition 4.4 An $\mathcal{FSTS}$ $(U, \mathcal{T}, \mathcal{P})$ is said to be an almost $\mathcal{FS}$-compact if each $\mathcal{FS}$-open cover of $\bar{1}_{\mathcal{P}}$ has a finite subcollection that covers $\bar{1}_{\mathcal{P}}$ by its closure.

Proposition 4.5 If $(U, \mathcal{C}, \mathcal{P})$ is an almost Č$\mathcal{FS}$-compact, then $(U, \tau_{\mathcal{C}}, \mathcal{P})$ is an almost $\mathcal{FS}$-compact.
Proof: The proof obtained directly from Proposition 3.4 and Theorem 2.10.

To give the characterizations of almost $\mathcal{FS}$-compact in terms of the finite intersection property, we first need the following definition:
Definition 4.6 A family $\Upsilon = \{(\lambda_B)_i: i \in \mathcal{I}\}$ of $\mathcal{FS}$-sets over U is said to have the first type of FIP if $\bigcap_{i=1}^{n} Int((\lambda_B)_i) \neq \bar{0}_{\mathcal{P}}$. If Υ satisfies the first type of FIP, then Υ satisfies the FIP.

Theorem 4.7 An Č$\mathcal{FSCS}$ $(U, \mathcal{C}, \mathcal{P})$ is an almost Č$\mathcal{FS}$-compact if and only if each family $\Upsilon = \{(\lambda_B)_i: i \in \mathcal{I}\}$ of $\mathcal{FS}$-sets of $(U, \mathcal{C}, \mathcal{P})$ satisfying the first type of FIP, then the family $\{\mathcal{C}((\lambda_B)_i): i \in \mathcal{I}\}$ has a non-null $\mathcal{FS}$-intersection.
Proof: The proof is analogous to Theorem 3.6.

To obtain the reverse direction of Proposition 4.2, it is important first to provide the following definition:

Definition 4.8 An $\check{C}\mathcal{FSCS}$ $(U,\mathcal{C},\mathcal{P})$ is said to be $\check{C}\mathcal{FS}$-regular if for each $\mathcal{FS}$-point x_t^k and a $\mathcal{FS}$-set λ_B in X such that $x_t^k \mathrel{\tilde{\in}} Int(\lambda_B)$, there is a $\mathcal{FS}$-set μ_A in X such that $x_t^k \mathrel{\tilde{\in}} Int(\mu_A) \subseteq \mathcal{C}(\mu_A) \subseteq Int(\lambda_B)$.

Theorem 4.9 In an $\check{C}\mathcal{FS}$-regular space, $\check{C}\mathcal{FS}$-compactness and almost $\check{C}\mathcal{FS}$-compactness are equivalent.

Proof: Since every $\check{C}\mathcal{FS}$-compact is an almost $\check{C}\mathcal{FS}$-compact, it is required only to prove that any almost $\check{C}\mathcal{FS}$-compact in an $\check{C}\mathcal{FS}$-regular space is $\check{C}\mathcal{FS}$-compact. Let $(U,\mathcal{C},\mathcal{P})$be an almost $\check{C}\mathcal{FS}$-compact $\check{C}\mathcal{FS}$-regular space and let $\mathcal{S}$ be a $\check{C}\mathcal{FS}$-cover of $\overline{1}_{\mathcal{P}}$. Then, for every $\mathcal{FS}$-point $x_t^k = \tilde{p}$ in $(U,\mathcal{C},\mathcal{P})$, there is $\lambda_{B_{\tilde{p}}} \in \mathcal{S}$ such that $\tilde{p} \subseteq Int(\lambda_B)$. By $\check{C}\mathcal{FS}$-regular, there is a $\mathcal{FS}$-set $\mu_{A_{\tilde{p}}}$ in U such that $x_t^k \mathrel{\tilde{\in}} Int(\mu_{A_{\tilde{p}}}) \subseteq \mathcal{C}(\mu_{A_{\tilde{p}}}) \subseteq Int(\lambda_{B_{\tilde{p}}})$. Thus,$\left\{\mu_{A_{\tilde{p}}}\colon \tilde{p} \in SP(U,\mathcal{P})\right\}$ is a $\check{C}\mathcal{FS}$-cover of $\overline{1}_{\mathcal{P}}$. Since $(U,\mathcal{C},\mathcal{P})$be an almost $\check{C}\mathcal{FS}$-compact, there is a finite number of $\mathcal{FS}$-points $\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \dots, \tilde{p}_n$ in U such that$\left\{\mathcal{C}(\mu_{A_{\tilde{p}_i}}), i = 1,\dots,n\right\}$ is a $\mathcal{FS}$-cover of $\overline{1}_{\mathcal{P}}$. It follows that $\left\{\lambda_{B_{\tilde{p}_i}}, i = 1,\dots,n\right\}$ is a finite $\check{C}\mathcal{FS}$-cover of $\mathcal{S}$. Hence, $(U,\mathcal{C},\mathcal{P})$ is $\check{C}\mathcal{FS}$-compact.

Definition 4.10 An $\check{C}\mathcal{FS}$-closure subspace $(V,\mathcal{C}_V,K)$ of a $\check{C}\mathcal{FSCS}$ $(U,\mathcal{C},\mathcal{P})$ is called a clopen $\check{C}\mathcal{FS}$-subspace, if it is closed and open $\mathcal{FS}$-set in $(U,\mathcal{C},\mathcal{P})$, i.e., $\mathcal{C}(\bar{V}_{\mathcal{P}}) = \bar{V}_{\mathcal{P}}$ and $Int(\bar{V}_{\mathcal{P}}) = \bar{V}_{\mathcal{P}}$.

The next example explains Definition 4.10.

Example 4.11 Let $U = \{a,d,e\}$, $\mathcal{P} = \{k_1,k_2\}$ and $V = \{a,e\} \subseteq U$. Define Č-fsco $\mathcal{C}\colon \mathcal{FSS}(U,\mathcal{P}) \to \mathcal{FSS}(U,\mathcal{P})$ as follows:

$$\mathcal{C}(\lambda_B) = \begin{cases} \overline{0}_{\mathcal{P}} & if\ \lambda_B = \overline{0}_{\mathcal{P}}, \\ \{(k_1, a_1 \vee e_1),(k_2, a_1 \vee e_1)\} & if\ \lambda_B \subseteq \{(k_1, a_1 \vee e_1),(k_2, a_1 \vee e_1)\}, \\ \{(k_1,d_1),(k_2,d_1)\} & if\ \lambda_B \subseteq \{(k_1,d_1),(k_2,d_1)\}, \\ \overline{1}_{\mathcal{P}} & otherwise. \end{cases}$$

Then, $(U,\mathcal{C},\mathcal{P})$ is $\check{C}\mathcal{FSCS}$. Let $\bar{V}_{\mathcal{P}} = \{(k_1, a_1 \vee e_1),(k_2, a_1 \vee e_1)\}$. Then, from Theorem 2.11 the $\check{C}\mathcal{FSCS}$-subspace $(V,\mathcal{C}_V,\mathcal{P})$ defined as $\mathcal{C}_V\colon \mathcal{FSS}(V,\mathcal{P}) \to \mathcal{FSS}(V,\mathcal{P})$ defined as follows:

$$\mathcal{C}_V(\lambda_B) = \begin{cases} \overline{0}_{\mathcal{P}} & if\ \lambda_B = \overline{0}_{\mathcal{P}}, \\ \bar{V}_{\mathcal{P}} & otherwise \end{cases}$$

Also, note that $\mathcal{C}(\bar{V}_{\mathcal{P}}) = \bar{V}_{\mathcal{P}}$ and $Int(\bar{V}_{\mathcal{P}}) = \overline{1}_{\mathcal{P}} - \mathcal{C}(\overline{1}_{\mathcal{P}} - \bar{V}_{\mathcal{P}}) = \overline{1}_{\mathcal{P}} - \mathcal{C}(\{(k_1,d_1),(k_2,d_1)\}) = \overline{1}_{\mathcal{P}} - \{(k_1,d_1),(k_2,d_1)\} = \bar{V}_{\mathcal{P}}$. It follows, $(V,\mathcal{C}_V,\mathcal{P})$ is a clopen $\check{C}\mathcal{FS}$-subspace.

Theorem 4.12 A clopen $\check{C}FS$- closure subspace $(V,\mathcal{C}_V,\mathcal{P})$ of an almost $\check{C}\mathcal{FS}$-compact space $(U,\mathcal{C},\mathcal{P})$ is an almost $\check{C}\mathcal{FS}$-compact.

Proof: Let $\{(\lambda_B)_i : i \in \mathcal{I}\}$ be a collection of $\mathcal{FS}$-sets in $(V,\mathcal{C}_V,\mathcal{P})$ such that $\{Int_V((\lambda_B)_i)\colon i \in \mathcal{I}\}$ has the FIP. Since for all $i \in \mathcal{I}$

$$\begin{aligned} Int_V((\lambda_B)_i) &= \overline{1}_{\mathcal{P}} - \mathcal{C}_V(\overline{1}_{\mathcal{P}} - (\lambda_B)_i) = \overline{1}_{\mathcal{P}} - \{\bar{V}_{\mathcal{P}} \cap \mathcal{C}(\overline{1}_{\mathcal{P}} - (\lambda_B)_i)\} \\ &= \left\{\overline{1}_{\mathcal{P}} - \bar{V}_{\mathcal{P}}\} \cup \{\overline{1}_{\mathcal{P}} - \mathcal{C}(\overline{1}_{\mathcal{P}} - (\lambda_B)_i)\right\} \\ &= Int(\overline{1}_{\mathcal{P}} - \bar{V}_{\mathcal{P}}) \cup Int((\lambda_B)_i) \subseteq Int((\overline{1}_{\mathcal{P}} - \bar{V}_{\mathcal{P}}) \cup (\lambda_B)_i). \end{aligned}$$

Then, $\{(\bar{1}_{\mathcal{P}} - \bar{V}_{\mathcal{P}}) \cup (\lambda_B)_i : i \in \mathcal{I}\}$ has the FIP in $(U, \mathcal{C}, \mathcal{P})$. By almost Č$\mathcal{FS}$-compactness of $(U, \mathcal{C}, \mathcal{P})$ we get $\{\mathcal{C}((\bar{1}_{\mathcal{P}} - \bar{V}_{\mathcal{P}}) \cup (\lambda_B)_i) : i \in \mathcal{I}\}$ has a non-null fuzzy soft intersection. That means $\cap\{\mathcal{C}((\bar{1}_{\mathcal{P}} - \bar{V}_{\mathcal{P}}) \cup (\lambda_B)_i) : i \in \mathcal{I}\} = \{\mathcal{C}(\bar{1}_{\mathcal{P}} - \bar{V}_{\mathcal{P}}) \cup \mathcal{C}((\lambda_B)_i) : i \in \mathcal{I}\} \neq \bar{0}_{\mathcal{P}}$. Since $(\lambda_B)_i \subseteq \bar{V}_{\mathcal{P}}$, then $\mathcal{C}((\lambda_B)_i) \subseteq \mathcal{C}(\bar{V}_{\mathcal{P}}) = \bar{V}_{\mathcal{P}}$. Therefore, for all $i \in \mathcal{I}$ we have $\{\mathcal{C}(\bar{1}_{\mathcal{P}} - \bar{V}_{\mathcal{P}}) \cup \mathcal{C}((\lambda_B)_i)\} \cap \bar{V}_{\mathcal{P}} \neq \bar{0}_{\mathcal{P}}$. This implies $\{(\bar{1}_{\mathcal{P}} - \bar{V}_{\mathcal{P}}) \cap \bar{V}_{\mathcal{P}}\} \cup \{\mathcal{C}((\lambda_B)_i) \cap \bar{V}_{\mathcal{P}}\} = \bar{0}_{\mathcal{P}} \cup \mathcal{C}_V((\lambda_B)_i) \neq \bar{0}_{\mathcal{P}}$. Hence, $(V, \mathcal{C}_V, \mathcal{P})$ is an almost Č$\mathcal{FS}$-compact subspace.

Theorem 4.13 If $\bar{V}_{\mathcal{P}}$ is an almost Č$\mathcal{FS}$-compact set in a T_2- Č$\mathcal{F}$SCS $(U, \mathcal{C}, \mathcal{P})$, then $\bar{V}_{\mathcal{P}}$ is a closed $\mathcal{FS}$-set.
Proof: Similar to proof of Theorem 3.9.

Definition 4.14 An $\mathcal{FS}$-mapping $f_{vs}: (U, \mathcal{C}, \mathcal{P}) \to (Y, \mathcal{C}^*, \mathcal{R})$ is said to be Č$\mathcal{FS}$-θ continuous, if for each $\mathcal{FS}$-point x_t^k in U and for each $\mathcal{FS}$-set μ_A in Y such that $f_{vs}(x_t^k) \tilde{\in} Int^*(\mu_A)$, there is a $\mathcal{FS}$-set λ_B in U such that $x_t^k \tilde{\in} Int(\lambda_B)$ and $f_{vs}(\mathcal{C}(\lambda_B)) \subseteq \mathcal{C}^*(\mu_A)$.

Example 4.15 Let $U = [0,1]$ and $\mathcal{P} = \{k_1, k_2\}$ be the set of parameters on U. Define Č-fsco $\mathcal{C}^*: \mathcal{FSS}(U, \mathcal{P}) \to \mathcal{FSS}(U, \mathcal{P})$ as follows:

$$\mathcal{C}^*(\lambda_B) = \begin{cases} \bar{0}_{\mathcal{P}} & if \ \lambda_B = \bar{0}_{\mathcal{P}}, \\ \nu_G & if \ \lambda_B \subseteq \nu_G, \\ \bar{1}_{\mathcal{P}} & otherwise. \end{cases}$$

Where $\nu_G = \{(k_1, \nu_G(k_1)), (k_2, \nu_G(k_2)) : \nu_G(k_1)(x) = \nu_G(k_2)(x) = 0.3 \ for \ x = 0.5 \ and \ 0 \ for \ x \neq 0.5\}$. Let $\mathcal{C}$ be any Č-fsco on U. Then $(U, \mathcal{C}, \mathcal{P})$ and $(U, \mathcal{C}^*, \mathcal{P})$ are Č$\mathcal{F}$SCS's. Let $f_{vs}: (U, \mathcal{C}, \mathcal{P}) \to (U, \mathcal{C}^*, \mathcal{P})$ be the identity $\mathcal{FS}$-mapping, i.e., v and s are identity mappings. Let x_t^k be any $\mathcal{FS}$-point x_t^k in U and let μ_A be any $\mathcal{FS}$-set in U such that $x_t^k = f_{vs}(x_t^k) \tilde{\in} Int^*(\mu_A)$. If $\mu_A \subseteq \bar{1}_{\mathcal{P}} - \nu_G$, then $\mathcal{C}^*(\mu_A) = \bar{1}_{\mathcal{P}}$ and the same is true if $\mu_A = \bar{1}_{\mathcal{P}}$. Then $f_{vs}(\mathcal{C}(\lambda_B)) \subseteq \mathcal{C}^*(\mu_A)$ for any λ_B is a $\mathcal{FS}$-set in U such that $x_t^k \tilde{\in} Int(\lambda_B)$. Hence, f_{vs} is a Č$\mathcal{FS}$-θ continuous.

Theorem 4.16 An Č$\mathcal{FS}$-θ continuous image of an almost Č$\mathcal{FS}$-compact is an almost Č$\mathcal{FS}$-compact.
Proof: Let $f_{vs}:(U, \mathcal{C}, \mathcal{P}) \to (Y, \mathcal{C}^*, \mathcal{R})$ be Č$\mathcal{FS}$-θ continuous mapping from an almost Č$\mathcal{FS}$-compact space $(U, \mathcal{C}, \mathcal{P})$ onto an Č$\mathcal{FSC}$ $(Y, \mathcal{C}^*, \mathcal{R})$. Let $\mathcal{S}$ be a Č$\mathcal{FS}$-cover of $\bar{1}_{\mathcal{R}}$. Then, for each $\mathcal{FS}$-point $x_t^k = \tilde{p}$ in U, there is a $\mathcal{FS}$-set $\mu_{A_{\tilde{q}}} \in \mathcal{S}$ such that $\tilde{q} = f_{vs}(\tilde{p}) \tilde{\in} Int^*(\mu_{A_{\tilde{q}}})$. Since f_{vs} is Č$\mathcal{FS}$-θ continuous, there is a $\mathcal{FS}$-set $\lambda_{B_{\tilde{p}}}$ in U such that $\tilde{p} \tilde{\in} Int(\lambda_{B_{\tilde{p}}})$ and $f_{vs}(\mathcal{C}(\lambda_{B_{\tilde{p}}})) \subseteq \mathcal{C}^*(\mu_{A_{\tilde{q}}})$. Now, $\{\lambda_{B_{\tilde{p}}}, \ \tilde{p} \in SP(U, \mathcal{P})\}$ is a Č$\mathcal{FS}$-cover of $\bar{1}_{\mathcal{P}}$ and $(U, \mathcal{C}, \mathcal{P})$ is an almost Č$\mathcal{FS}$-compact. Then, there is a finite number of $\mathcal{FS}$-points $\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \ldots, \tilde{p}_n$ in U such that $\{\mathcal{C}(\lambda_{B_{\tilde{p}_i}}), i = 1, \ldots, n\}$ is a $\mathcal{FS}$-cover of $\bar{1}_{\mathcal{P}}$. Hence, $\{\mathcal{C}^*(\mu_{A_{\tilde{q}_i}}), i = 1, \ldots, n\}$, where $\tilde{q}_i = f_{vs}(\tilde{p}_i)$ is a $\mathcal{FS}$-cover of $\bar{1}_{\mathcal{R}}$. Thus, $(Y, \mathcal{C}^*, \mathcal{R})$ is an almost Č$\mathcal{FS}$-compact.

5-Nearly Čech fuzzy soft compact closure spaces

Definition 5.1 An Č$\mathcal{F}$SCS $(U, \mathcal{C}, \mathcal{P})$ is said to be nearly Č$\mathcal{FS}$-compact if for every Č$\mathcal{FS}$-cover $\{(\lambda_B)_i : i \in \mathcal{I}\}$ of $\bar{1}_{\mathcal{P}}$, there is a finite set $\mathcal{L} \subset \mathcal{I}$ such that $\{\mathcal{C}((\lambda_B)_i) : i \in \mathcal{L}\}$ is Č$\mathcal{FS}$-cover of $\bar{1}_{\mathcal{P}}$.

The relationship between nearly $\check{C}\mathcal{FS}$-compact spaces and the other two types of compactness is described in the following Theorem:

Theorem 5.2 Every $\check{C}\mathcal{FS}$-compact is nearly $\check{C}\mathcal{FS}$-compact and every nearly $\check{C}\mathcal{FS}$-compact is almost $\check{C}\mathcal{FS}$-Compact.

Proof: First to prove if $(U,\mathcal{C},\mathcal{P})$ is $\check{C}\mathcal{FS}$-compact, then it is nearly $\check{C}\mathcal{FS}$-compact. Let $(U,\mathcal{C},\mathcal{P})$ be a $\check{C}\mathcal{FS}$-compact. Let $\{(\lambda_B)_i: i \in \mathcal{I}\}$ be any $\check{C}\mathcal{FS}$-cover of $\overline{1}_{\mathcal{P}}$. Since $(U,\mathcal{C},\mathcal{P})$ is $\check{C}\mathcal{FS}$-compact, then there exists a finite set $\mathcal{L} \subset \mathcal{I}$ such that $\{(\lambda_B)_i: i \in \mathcal{L}\}$ is a $\check{C}\mathcal{FS}$-cover of $\overline{1}_{\mathcal{P}}$. That means $\cup\{Int((\lambda_B)_i): i \in \mathcal{L}\} = \overline{1}_{\mathcal{P}}$. Since $(\lambda_B)_i \subseteq \mathcal{C}((\lambda_B)_i)$, then $Int((\lambda_B)_i) \subseteq Int(\mathcal{C}((\lambda_B)_i))$ for all $i \in \mathcal{I}$. This implies $\cup\{Int((\lambda_B)_i): i \in \mathcal{L}\} \subseteq \cup\{Int\big(\mathcal{C}((\lambda_B)_i)\big): i \in \mathcal{L}\}$. Therefore, $\{\mathcal{C}((\lambda_B)_i): i \in \mathcal{L}\}$ is $\check{C}\mathcal{FS}$-cover of $\overline{1}_{\mathcal{P}}$. Hence, $(U,\mathcal{C},\mathcal{P})$ is nearly $\check{C}\mathcal{FS}$-compact.

Second, we prove if $(U,\mathcal{C},\mathcal{P})$ is nearly $\check{C}\mathcal{FS}$-compact, then it is almost $\check{C}\mathcal{FS}$-compact. Let $(U,\mathcal{C},\mathcal{P})$ be nearly $\check{C}\mathcal{FS}$-compact. Let $\{(\lambda_B)_i: i \in \mathcal{I}\}$ be any $\check{C}\mathcal{FS}$-cover of $\overline{1}_{\mathcal{P}}$. Since $(U,\mathcal{C},\mathcal{P})$ is nearly $\check{C}\mathcal{FS}$-Compact, then there exists a finite set $\mathcal{L} \subset \mathcal{I}$ such that $\{\mathcal{C}((\lambda_B)_i): i \in \mathcal{L}\}$ is a $\check{C}\mathcal{FS}$-cover of $\overline{1}_{\mathcal{P}}$. That means $\cup\{Int(\mathcal{C}((\lambda_B)_i)): i \in \mathcal{L}\} = \overline{1}_{\mathcal{P}}$. Since for any $i \in \mathcal{I}$, $Int(\mathcal{C}((\lambda_B)_i)) \subseteq \mathcal{C}((\lambda_B)_i)$, then $\{\mathcal{C}((\lambda_B)_i): i \in \mathcal{L}\}$ is a $\mathcal{FS}$-cover of $\overline{1}_{\mathcal{P}}$. Hence, $(U,\mathcal{C},\mathcal{P})$ is almost $\check{C}\mathcal{FS}$-compact.

The following example demonstrates that the opposite is not true in general.

Example 5.3 Let $U = \{0,1,2n,-2n,2n+1: n \in \mathbb{N}\}$, $\mathcal{P} = \{k\}$, and let $(\lambda_B)_n$, ρ_C, η_D are $\mathcal{FS}$-sets over U defined as:

$(\lambda_B)_n = \{(k,x_1): x \in U - \{2n,-2n,2n+1\}\}$, $n \in \mathbb{N}$, $\rho_C = \{(k,x_1): x \in U - \{0,2n\}, n \in \mathbb{N}\}$, $\eta_D = \{(k,x_1): x \in U - \{1,2n+1\}, n \in \mathbb{N}\}$.

Define Č-fsco $\mathcal{C}: \mathcal{FSS}(U,\mathcal{P}) \to \mathcal{FSS}(U,\mathcal{P})$ as follows:

$$\mathcal{C}(\lambda_B) = \begin{cases} \overline{0}_{\mathcal{P}} & if\ \lambda_B = \overline{0}_{\mathcal{P}}, \\ (\lambda_B)_n & if\ \lambda_B \subseteq (\lambda_B)_n, \\ \rho_C & if\ \lambda_B \subseteq \rho_C, \\ \eta_D & if\ \lambda_B \subseteq \eta_D, \\ \cup_{n\in\mathbb{N}}(\lambda_B)_n & if\ \lambda_B \subseteq \cup_{n\in\mathbb{N}}(\lambda_B)_n \\ \overline{1}_{\mathcal{P}} & otherwise. \end{cases}$$

Then, $(U,\mathcal{C},\mathcal{P})$ is an almost $\check{C}\mathcal{FS}$-compact space since every $\check{C}\mathcal{FS}$-cover $\{(\lambda_B)_i: i \in \mathcal{I}\}$ of $\overline{1}_{\mathcal{P}}$ must contain $\overline{1}_{\mathcal{P}} - \rho_C$ and $\overline{1}_{\mathcal{P}} - \eta_D$ and this implies there exists a finite subset $\{\overline{1}_{\mathcal{P}} - \rho_C, \overline{1}_{\mathcal{P}} - \eta_D\}$ such that $\mathcal{C}(\overline{1}_{\mathcal{P}} - \rho_C) \cup \mathcal{C}(\overline{1}_{\mathcal{P}} - \eta_D) = \eta_D \cup \rho_C = \overline{1}_{\mathcal{P}}$. But $(U,\mathcal{C},\mathcal{P})$ is not nearly $\check{C}\mathcal{FS}$-compact space since there exists an $\check{C}\mathcal{FS}$-cover $\{\overline{1}_{\mathcal{P}} - (\lambda_B)_n, \overline{1}_{\mathcal{P}} - \rho_C, \overline{1}_{\mathcal{P}} - \eta_D: n \in \mathbb{N}\}$ of $\overline{1}_{\mathcal{P}}$ such that every $\mathcal{FS}$-set in this cover equal the interior of the closure in the original $\mathcal{FS}$-set.

Remark 5.4 The diagram below summarizes the relationships between several forms of $\check{C}\mathcal{FS}$-compactness, based on previous results findings in sections 3 and 4.

$\check{C}\mathcal{FS}$-compact $\Longrightarrow$ nearly $\check{C}\mathcal{FS}$-compact $\Longrightarrow$ almost $\check{C}\mathcal{FS}$-compact

Definition 5.5 An $\mathcal{FS}$-mapping f_{vs}: $(U,\mathcal{C},\mathcal{P}) \to (Y,\mathcal{C}^*,\mathcal{R})$ is said to be $\check{C}\mathcal{FS}$-almost continuous if for each $\mathcal{FS}$-point x_t^k in U and each $\mathcal{FS}$-set μ_A in$Y$ with $f_{vs}(x_t^k) \widetilde{\in} Int^*(\mu_A)$ there is a $\mathcal{FS}$-set λ_B in U such that $x_t^k \widetilde{\in} Int(\lambda_B)$ and $f_{vs}(\lambda_B) \subseteq Int^*(\mathcal{C}^*(\mu_A))$.

Clearly, any $\check{C}\mathcal{FS}$-continuous mapping is $\check{C}\mathcal{FS}$- almost continuous but not conversely as the following example shows.

Example 5.6 In Example 4.15, consider $\mathcal{C}$ be the indiscrete Č-fsco on U (i.e., $\mathcal{C}(\lambda_B) = \overline{0}_{\mathcal{P}}$ for $\lambda_B = \overline{0}_{\mathcal{P}}$ and $\mathcal{C}(\lambda_B) = \overline{1}_{\mathcal{P}}$ for $\lambda_B \neq \overline{0}_{\mathcal{P}}$). Then, f_{vs} is an Č$\mathcal{FS}$-almost continuous but does not Č$\mathcal{FS}$-continuous since there exists a $\mathcal{FS}$-set λ_B in U, $\lambda_B = \{(k_1, \lambda_B(k_1)), (k_2, \lambda_B(k_2)): \lambda_B(k_1)(x) = \lambda_B(k_2)(x) = 0.2\ for\ x = 0.5\ and\ 1\ for\ x \neq 0.5\}$ and $f_{vs}(\mathcal{C}(\lambda_B)) = \overline{1}_{\mathcal{P}} \not\subset \mathcal{C}^*(f_{vs}(\lambda_B))$ because $\mathcal{C}^*(f_{vs}(\lambda_B)) = \Big\{\Big(k_1, \mathcal{C}^*(f_{vs}(\lambda_B))(k_1)\Big), \Big(k_2, \mathcal{C}^*(f_{vs}(\lambda_B))(k_2)\Big) : \mathcal{C}^*(f_{vs}(\lambda_B))(k_1)(x) = \mathcal{C}^*(f_{vs}(\lambda_B))(k_2)(x) = 0.3\ for\ x = 0.5\ and\ 1\ for\ x \neq 0.5\Big\}$.

Theorem 5.7 An Č$\mathcal{FS}$- almost continuous image of Č$\mathcal{FS}$-compact is nearly Č$\mathcal{FS}$-compact.
Proof: Let f_{vs}: $(U, \mathcal{C}, \mathcal{P}) \to (Y, \mathcal{C}^*, \mathcal{R})$ be Č$\mathcal{FS}$-almost continuous mapping from an Č$\mathcal{FS}$-compact $(U, \mathcal{C}, \mathcal{P})$ onto an Č$\mathcal{FS}$SCS $(Y, \mathcal{C}^*, \mathcal{R})$ and let δ be a Č$\mathcal{FS}$-cover of $\overline{1}_{\mathcal{R}}$. For each $\mathcal{FS}$-point $\tilde{p} = x_t^k$ in U, there is $\mu_{A_{\tilde{q}}} \in \delta$ such that $\tilde{q} = f_{vs}(\tilde{p}) \tilde{\in} Int^*(\mu_{A_{\tilde{q}}})$. Since f_{vs} is Č$\mathcal{FS}$-almost continuous, then there is a $\mathcal{FS}$-set $\lambda_{B_{\tilde{p}}}$ in U such that $\tilde{p} \tilde{\in} Int(\lambda_{B_{\tilde{p}}})$ and $f_{vs}(\lambda_{B_{\tilde{p}}}) \subseteq Int^*(\mathcal{C}^*(\mu_{A_{\tilde{q}}}))$. Now, $\left\{\lambda_{B_{\tilde{p}}}, \tilde{p} \in FSP(U, \mathcal{P})\right\}$ is a Č$\mathcal{FS}$-cover of $\overline{1}_{\mathcal{P}}$ and $(U, \mathcal{C}, \mathcal{P})$ is an Č$\mathcal{FS}$-compact. Then, there is a finite number of $\mathcal{FS}$-points $\tilde{p}_1, \ldots, \tilde{p}_n$ in U such that $\left\{\lambda_{B_{\tilde{p}_i}}, i = 1, \ldots\ldots, n\right\}$ is a Č$\mathcal{FS}$-cover of $\overline{1}_{\mathcal{P}}$. Consequently, $\{\mathcal{C}^*\left(\mu_{A_{\tilde{q}_i}}\right): i = 1, \ldots, n\}$ where $\tilde{q}_i = f_{vs}(\tilde{p}_i)$ is Č$\mathcal{FS}$-cover of $\overline{1}_{\mathcal{R}}$. Hence, $(Y, \mathcal{C}^*, \mathcal{R})$ is nearly Č$\mathcal{FS}$-compact.

6- Conclusion

Fuzzy soft sets are of great interest to researchers. Compared to fuzzy soft topological spaces, this approach is more general, and there are numerous applications for fuzzy soft sets. In this paper, for *Č*ech fuzzy soft closure spaces, the notions of compactness, almost compactness, and nearly compactness are presented and investigated. Also, their induced fuzzy soft topological spaces have been introduced and studied, as well as the relationships between them. Next, there are opportunities to look into concept of Lindelöf spaces in *Č*ech fuzzy soft closure spaces and studied the relationships between them.

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